PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 49 (63), 1991, 67–70

SHARP ESTIMATES FOR SOME INTEGRAL OPERATORS OF CONVEX FUNCTIONS OF ORDER ALPHA

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Abstract. For $0 \leq \alpha < 1$, let $C(\alpha)$ be the class of normalised analytic univalent functions, convex of order α. Sharp lower bounds are obtained for certain integral operators in $C(\alpha)$.

Introduction

For $0 \leq \alpha < 1$, denote by $C(\alpha)$ the class of normalised univalent convex functions f of order α, defined in the open unit disc $D = \{z : |z| < 1\}$. Thus $f \in C(\alpha)$, if and only if, $f(0) = 0, f'(0) = 1$ and

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$$

for $z \in D$. The class $C(\alpha)$ has been extensively studied. In [1] Bernardi gave a series of non-sharp lower bounds for the real part of certain weighted integral operators of $f \in C(0)$. The object of this paper is to give sharp versions of some of Bernardi's results for $f \in C(\alpha)$. We also extend a classical result of Strohhäcker [3] to obtain sharp estimates for the real part of some iterated integral operators in $C(\alpha)$. Our methods are quite elementary.

Results

THEOREM 1. *Let $f \in C(\alpha)$ and $z = re^{i\theta} \in D$. For $n \geq 2$, set $n!A_n(\alpha) = \prod_{k=1}^{\infty}(k - 2\alpha)$ and $A_1(\alpha) = 1$. Then*

(i) *For a real and $a \neq -1, -2, \dots,$*

$$\operatorname{Re}\left(z^{-(1+a)}\int_0^z t^{a-1}f(t)dt\right) \geq \sum_{n=1}^{\infty}\frac{(-r)^{n-1}A_n(\alpha)}{(n+a)},$$

AMS Subject Classification (1985): Primary 30C45

(ii) *For* $c_1, c_2 \neq -1, -2, \ldots$ *and* $c_2 > c_1$,

$$\operatorname{Re}\left(z^{-2}\int_0^z f(t)\left[(t/z)^{c_1-1}-(t/z)^{c_2-1}\right]dt\right) \geq (c_2-c_1)\sum_{n=1}^{\infty}\frac{(-r)^{n-1}A_n(\alpha)}{(n+c_1)(n+c_2)},$$

(iii) *For* a, c *real and* $a \neq 0, -1, -2, \ldots, c \neq -1, -2, \ldots$,

$$\operatorname{Re}\left(z^{-(1+c)}\int_0^z f(t)t^{c-1}\left(\log(z/t)\right)^{a-1}dt\right) \geq \Gamma(a)\sum_{n=1}^{\infty}\frac{(-r)^{n-1}A_n(\alpha)}{(n+c)^a},$$

where Γ *is the Gamma function.*

(iv) *For* c *real and* $c \neq 0, -1, -2, \ldots$,

$$\operatorname{Re}\left(z^{-(1+c)}\int_0^z f(t)(z-t)^{c-1}dt\right) \geq \sum_{n=1}^{\infty}(-r)^{n-1}B(c, n+1)A_n(\alpha),$$

where B *is the Beta function.*

In all cases, equality occurs for the function $f_0 \in C(\alpha)$, *where*

$$f_0(z) = \sum_{n=1}^{\infty}(-1)^{n-1}A_n(\alpha)z^n = \begin{cases} \dfrac{1-(1+z)^{2\alpha-1}}{(1-2\alpha)}, & \text{for } \alpha \neq 1/2 \\ \log(1+z), & \text{for } \alpha = 1/2. \end{cases}$$

THEOREM 2. *Let* $f \in C(\alpha)$ *and* $z = re^{i\theta} \in D$. *For* $n = 1, 2, \ldots$, *define*

$$I_n(z) = \frac{1}{z}\int_0^z I_{n-1}(t)\,dt,$$

where $I_0(z) = f(z)/z$. *Then for* $n \geq 0$,

$$\operatorname{Re} I_n(z) \geq \gamma_n(r),$$

where

$$\frac{1}{2} \leq \gamma_n(r) = \sum_{k=1}^{\infty}\frac{(-r)^{k-1}A_k(\alpha)}{k^n} < 1.$$

The result is sharp for f_0 *as given in Theorem* 1.

We note that when $n = 0$, we obtain the following result of Brickman et al. [2] which we shall use in the proofs of Theorem 1 and 2.

LEMMA. *Let* $f \in C(\alpha)$ *and* $z = re^{i\theta}$. *Then for* $0 \leq \alpha < 1$,

$$\operatorname{Re}\left(\frac{f(z)}{z}\right) \geq \begin{cases} \dfrac{1-(1+r)^{2\alpha-1}}{(1-2\alpha)r}, & \text{for } \alpha \neq 1/2 \\ \dfrac{\log(1+r)}{r}, & \text{for } \alpha = 1/2. \end{cases}$$

The results are sharp for the function f_0 given above.

Proof of Theorem 1. In each case, we will give the proof when $\alpha \neq 1/2$. When $\alpha = 1/2$, the proofs are similar. Write $t = \rho e^{i\theta}$, then applying the Lemma in each of the following, we have

(i).
$$\begin{aligned}\operatorname{Re}\left(\frac{1}{z^{1+a}}\int_0^z f(t)t^{a-1}dt\right) &= r^{-(1+a)}\int_0^r \rho^a \operatorname{Re}\left(f(\rho e^{i\theta})/\rho e^{i\theta}\right)d\rho \\ &\geq \frac{r^{-(1+a)}}{(1-2\alpha)}\int_0^r \rho^{a-1}\left(1-(1+\rho)^{2\alpha-1}\right)d\rho \\ &= r^{-(1+a)}\sum_{n=1}^{\infty}(-1)^{n-1}A_n(\alpha)\int_0^r \rho^{n+a-1}d\rho,\end{aligned}$$

The result now follows at once.

(ii)
$$\begin{aligned}&\operatorname{Re}\left(\frac{1}{z^2}\int_0^z f(t)\left[(t/z)^{c_1-1}-(t/z)^{c_2-1}\right]dt\right) \\ &= \frac{1}{r^2}\int_0^r \rho\left[(\rho/r)^{c_1-1}-(\rho/r)^{c_2-1}\right]\operatorname{Re}\left(f(\rho e^{i\theta})/\rho e^{i\theta}\right)d\rho \\ &\geq \frac{1}{r^2}\int_0^r \left[(\rho/r)^{c_1-1}-(\rho/r)^{c_2-1}\right]\sum_{n=1}^{\infty}(-1)^{n-1}A_n(\alpha)\rho^n d\rho \\ &= \sum_{n=1}^{\infty}(-r)^{n-1}A_n(\alpha)\int_0^1 x^n\left(x^{c_1-1}-x^{c_2-1}\right)dx \\ &= (c_2-c_1)\sum_{n=1}^{\infty}\frac{(-r)^{n-1}A_n(\alpha)}{(n+c_1)(n+c_2)},\end{aligned}$$

for $c_2 > c_1$ and $c_1, c_2 \neq -1, -2, \dots$.

(iii)
$$\begin{aligned}&\operatorname{Re}\left(\frac{1}{z^{1+c}}\int_0^z f(t)t^{c-1}\left(\log(z/t)\right)^{a-1}dt\right) \\ &= \frac{1}{r^{1+c}}\int_0^r \rho^c\left(\log(r/\rho)\right)^{a-1}\operatorname{Re}\left(f(\rho e^{i\theta})/\rho e^{i\theta}\right)d\rho \\ &\geq \frac{1}{r^2}\int_0^r (\rho/r)^{c-1}\left(\log(r/\rho)\right)^{a-1}\sum_{n=1}^{\infty}(-1)^{n-1}A_n(\alpha)\rho^n d\rho \\ &= \sum_{n=1}^{\infty}(-r)^{n-1}A_n(\alpha)\int_0^1 x^{n+c-1}\left(\log(1/x)\right)^{a-1}dx \\ &= \Gamma(a)\sum_{n=1}^{\infty}\frac{(-r)^{n-1}A_n(\alpha)}{(n+c)^a},\end{aligned}$$

for $a \neq 0, -1, -2, \dots$, $c \neq -1, -2, \dots$.

(iv)
$$\begin{aligned}
&\mathrm{Re}\left(\frac{1}{z^{1+c}}\int_0^z f(t)(z-t)^{c-1}dt\right)\\
&= \frac{1}{r^{1+c}}\int_0^r \rho(r-\rho)^{c-1}\,\mathrm{Re}\left(f(\rho e^{i\theta})/\rho e^{i\theta}\right)d\rho\\
&\geq \frac{1}{r^2}\int_0^r \left(1-\frac{\rho}{r}\right)^{c-1}\sum_{n=1}^{\infty}(-1)^{n-1}A_n(\alpha)\rho^n d\rho\\
&= \sum_{n=1}^{\infty}(-r)^{n-1}A_n(\alpha)\int_0^1 (1-x)^{c-1}x^n dx\\
&= \sum_{n=1}^{\infty}(-r)^{n-1}B(c,n+1)A_n(\alpha), \qquad \text{for } c \neq 0,-1,-2,\ldots
\end{aligned}$$

This completes the proof of Theorem 1.

Proof of Theorem 2. It follows easily from the Lemma that for $0 \leq \alpha < 1$,

$$\mathrm{Re}\, I_0(z) \geq \sum_{k=1}^{\infty}(-r)^{k-1}A_k(\alpha) = \gamma_0(r).$$

Next, writing $t = \rho e^{i\theta}$ we have,

$$\begin{aligned}
\mathrm{Re}\, I_n(z) &= \mathrm{Re}\,\frac{1}{z}\int_0^z I_{n-1}(t)dt\\
&\geq \frac{1}{r}\int_0^r \sum_{k=1}^{\infty}\frac{(-\rho)^{k-1}A_k(\alpha)}{k^{n-1}}d\rho\\
&= \sum_{k=1}^{\infty}\frac{(-r)^{k-1}A_k(\alpha)}{k^n} = \gamma_n(r),
\end{aligned}$$

where we have used induction. For $n \geq 0$ and $0 \leq \alpha < 1$, $\gamma_n(r)$ is absolutely convergent for $0 \leq r < 1$ and hence rearranging the terms appropriately shows that $1/2 < \gamma_n(r) < 1$.

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(Received 16 10 1989)