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A SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract. Let f be analytic and normalised in the unit disc D and satisfy $\operatorname{Re}(zf'(z)/\phi(z)) > 0$ for $z \in D$ where ϕ is a normalised convex function. Such functions form a subset of the close-to-convex functions. Various extremal problems are considered.

Introduction. Denote by S the class of functions f which are analytic and univalent in $D = \{z : |z| < 1\}$ and normalised so that $f(0) = 0$ and $f'(0) = 1$. Thus for $f \in S$ we may write

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Let C, S^* and K be those subsets of S which are convex, starlike and close-to-convex respectively. Then $f \in K$ if, and only if, there exists $g \in S^*$ such that for $z \in D$

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0. \tag{2}$$

Since C is a subclass of S^*, we can define a subclas J of K such that $g \in C$ in (2). Thus we have

Definition. Let f be analytic in D and be given by (1). Then $f \in J$, if, and only if, there exists $\phi \in C$ such that for $z \in D$,

$$\operatorname{Re} \frac{zf'(z)}{\phi(z)} > 0. \tag{3}$$

Results. We first give some distortion theorems.

THEOREM 1. *Let $f \in J$, then for $z = re^{i\theta} \in D$,*

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$$\frac{1-r}{(1+r)^2} \le |f'(z)| \le \frac{1+r}{(1-r)^2},$$

$$-\log(1+r) + \frac{2r}{1+r} \le |f(z)| \le \log(1-r) + \frac{2r}{1-r}.$$

Each inequality is sharp for F defined by

$$F(z) = \bar{x}\log(1-xz) + \frac{2z}{1-xz}, \quad \textit{with } |x| = 1. \tag{4}$$

Proof. It follows from (3) that we can write $zf'(z) = \phi(z)p(z)$, for $p \in P$, the class of analytic functions satisfying $\operatorname{Re} p(z) > 0$ in D and $p(0) = 1$. The inequalitis for $|f'(z)|$ in Theorem 1 follow at once from the well-known distortion theorems for C and P. Integrating along a straight line segment from $z = 0$ to $z = re^{i\theta}$ gives the upper bound for $|f(z)|$. For the lower bound let z_1 be such that $|z_1| = r$ and satisfies $|f(z_1)| \le |f(z)|$ for all z with $|z| = r$. Writing $w = f(z)$ it follows that the line segment λ from $w = 0$ to $w = f(z)$ lies entirely in the image of f. Let Λ be the pre-image of λ. Then

$$|f(z)| \ge |f(z_1)| = \int_\lambda |dw| = \int_\Lambda \left|\frac{dw}{dz}\right| |dz|$$

$$\ge \int_0^r \frac{1-t}{(1+t)^2}\, dt = -\log(1+r) + \frac{2r}{1+r}.$$

Equality is attained on choosing $\phi(z) = \dfrac{z}{1-xz}$ and $p(z) = \dfrac{1+xz}{1-xz}$ for $|x| = 1$ in the representation $zf'(z) = \phi(z)p(z)$.

COROLLARY. *Let $f \in J$ and $f(z) \ne w$ for $z \in D$, then $|w| > 1 - \log 2$.*

The proof follows using a standard argument (see e.g. [2])

THEOREM 2. *Let $f \in J$, then for $z = re^{i\theta} \in D$,*

$$|\arg f'(z)| \le \arcsin \frac{2r}{1+r^2} + \arcsin r.$$

The result is sharp.

Proof. From (3) write

$$\frac{zf'(z)}{\phi(z)} = \frac{1+\omega(z)}{1-\omega(z)},$$

so that ω is analytic in D with $\omega(0) = 0$ and $|\omega(z)| \le 1$. Since the image of the disc $\{z : |z| \le r\}$ by the transformation $\dfrac{1+\omega(z)}{1-\omega(z)}$ is contained in closed disc centre $\dfrac{1+r^2}{1-r^2}$, radius $\dfrac{2r}{1-r^2}$, it follows that

$$\left|\frac{zf'(z)}{\phi(z)} - \frac{1+r^2}{1-r^2}\right| \le \frac{2r}{1-r^2}.$$

Thus

$$\left|\arg \frac{zf'(z)}{\phi(z)}\right| \leq \arcsin \frac{2r}{1+r^2},$$

and since $\phi \in C$, it follows that $|\arg(\phi(z)/z)| \leq \arcsin r$, see for example [6], and so the result is proved.

To show that the inequality is sharp, choose θ_1 and θ_2 so that

$$\frac{zf'(z)}{\phi(z)} = \frac{1+\theta_1 z}{1-\theta_1 z}$$

and

$$\phi(z) = \frac{z}{1+\theta_2 z},$$

where $\theta_1 = ir/z$ and $\theta_2 = (r/z)(-r + i\sqrt{1-r^2})$ at any point z such that $|z| = r$.

We next give some coefficient results.

THEOREM 3. *Let $f \in J$ and be given by* (1). *Then $|a_n| \leq 2 - 1/n$ for $n \geq 2$, with equality for F defined in* (4). *Also*

$$|a_3 - \mu a_2^2| \leq \begin{cases} 5/3 - 9\mu/4, & \text{if } \mu \leq 2/9 \\ 2/3 + 1/(9\mu), & \text{if } 2/9 \leq \mu \leq 2/3 \\ 5/6, & \text{if } 2/3 \leq \mu \leq 1. \end{cases}$$

For each μ, there is a function in J such that equality holds.

Proof. The first inequality follows at once on equating coefficients in the representation $zf'(z) = \phi(z)p(z)$ and using the well-known coeficient estimates for C and P.

Next write

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \sum_{n=2}^{\infty} \alpha_n z^n,$$

and

$$\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{5}$$

so that $\omega(0) = 0$ and $|\omega(z)| \leq 1$ for $z \in D$. Then equating coefficents, we have

$$2a_2 = b_2 + 2\alpha_1,$$

and

$$3a_3 = b_3 + 2\alpha_2 + 2\alpha_1^2 + 2\alpha_1 b_2.$$

Thus

$$a_3 - \mu a_2^2 = \frac{1}{3}\left(b_3 - \frac{3}{4}\mu b_2^2\right) + \frac{2}{3}\left(\alpha_2 + \left(1 - \frac{3}{2}\mu\right)\alpha_1^2\right) + \left(\frac{2}{3} - \mu\right)\alpha_1 b_2. \tag{6}$$

We first consider the case $2/9 \le \mu \le 2/3$. It follows from (6) that

$$\begin{aligned} |a_3 - \mu a_2^2| &\le \frac{1}{3}\left|b_3 - \frac{3}{4}\mu b_2^2\right| + \frac{2}{3}\left|\alpha_2 - \frac{1}{2}(3\mu - 2)\alpha_1^2\right| + \frac{2-3\mu}{3}|\alpha_1 b_2| \\ &\le \frac{4-3\mu}{12} + \frac{2}{3} - \mu|\alpha_1^2| + \frac{2-3\mu}{3}|\alpha_1| \\ &= \Phi(t), \qquad \text{say, with } t = |\alpha_1|, \end{aligned}$$

where we have used the fact that $|b_2| \le 1$ and the inequalities

$$|\alpha_2 - s\alpha_1^2| \le 1 + (|s| - 1)|\alpha_1|^2 \tag{7}$$

and

$$|b_3 - sb_2^2| \le \max\{1/3, |s-1|\}, \tag{8}$$

for any complex number s, see e.g. [1]. Since the function Φ attains its maximum at $t_0 = (2-3\mu)/(6\mu)$, it follows that the second inequality in Theorem 3 is established, if $\mu \le 2/3$. Choosing $b_1 = b_2 = 1$, $\alpha_1 = (2-3\mu)/(6\mu)$ and $\alpha_2 = 1 - \alpha_1^2$ in (6), shows that the result is sharp if $\mu \ge 2/9$, since $|\alpha_1| \le 1$.

Next suppose that $\mu \le 2/9$. Then

$$|a_3 - \mu a_2^2| \le \frac{9\mu}{2}\left|a_3 - \frac{2}{9}a_2^2\right| + \left(1 - \frac{9\mu}{2}\right)|a_3| \le \frac{5}{3} - \frac{9\mu}{4},$$

where we have used the result already proved in the case $\mu = 2/9$, and the inequality $|a_3| \le 5/3$ proved in the first part of the theorem. Equality is attained on choosing $b_2 = b_3 = 1$, $\alpha_1 = 1$, and $\alpha_2 = 0$ in (6).

Suppose next that $2/3 \le \mu \le 1$. Then when $\mu = 1$, (6) gives

$$a_3 - a_2^2 = \frac{1}{3}\left(b_3 - \frac{3}{4}b_2^2\right) + \frac{2}{3}\left(\alpha_2 - \frac{1}{2}\alpha_1^2\right) - \frac{\alpha_1 b^2}{3}$$

and so

$$\begin{aligned} |a_3 - a_2^2| &\le \frac{1}{3}\left|b_3 - \frac{3}{4}b_2^2\right| + \frac{2}{3}\left|\alpha_2 - \frac{1}{2}\alpha_1^2\right| + \frac{|\alpha_1 b_2|}{3} \\ &\le \frac{1}{9}(1 - |b_2|^2) + \frac{|b_2|^2}{12} + \frac{2}{3} - \frac{|\alpha_1|^2}{3} + \frac{|\alpha_1 b_2|}{3} \\ &= \frac{7}{9} + \frac{|b_2|^2}{18} - \frac{1}{3}\left(|\alpha_1| - \frac{|b_2|}{2}\right)^2 \le \frac{5}{6}, \end{aligned}$$

where we have used (7), (8), the inequality $|b_2| \le 1$ and the fact that $|\alpha_2| \le 1 - |\alpha_1|^2$, (see e.g. [5]). Now write,

$$a_3 - \mu a_2^2 = (3\mu - 2)(a_3 - a_2^2) + 3(1-\mu)(a_3 - (2/3)a_2^2),$$

and the result follows at once on using the the theorem already proved for $\mu = 1$ and $\mu = 2/3$. Equality is attained when $b_2 = b_3 = 1$, $\alpha_2 = 1 - \alpha_1^2$ with

$$\alpha_1 = \frac{2-3\mu}{6\mu} \pm i\frac{\sqrt{6\mu - 4}}{6\mu}.$$

We now obtain the radius of convexity for J.

THEOREM 4. *Let $f \in J$, then f maps $\{z : |z| < 1/3\}$ onto a convex set. The function F given by* (4) *shows that this result is best possible.*

Proof. Differentiating $zf'(z) = \phi(z)p(z)$ logarithmically, we have

$$1 + \frac{zf''(z)}{f'(z)} = \frac{z\phi'(z)}{\phi(z)} + \frac{zp'(z)}{p(z)},$$

from which it follows that

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq \mathrm{Re}\,\frac{z\phi'(z)}{\phi(z)} - \left|\frac{zp'(z)}{p(z)}\right|. \tag{9}$$

Since $\phi \in C$,

$$\mathrm{Re}\,\frac{z\phi'(z)}{\phi(z)} \geq \frac{1}{1+r},$$

see e.g. [4]. Also [3],

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2r}{1-r^2}.$$

The result now follows at once from (9).

We finally consider a problem concerning the partial sums of functions in J. We prove the following:

THEOREM 5. *Let $f \in J$ and be given by* (1). *For $n \geq 2$, define f_n*

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k.$$

Then for $n = 2$ and $n \geq 4$,

$$\mathrm{Re}\,\frac{zf_n'(z)}{\phi_n(z)} > 0 \tag{10}$$

for $z \in \{z : |z| < 1/3\}$, where $\phi_n(z) = z + \sum_{k=2}^{n} b_k z^k$ and where b_k is given by (5) *for $2 \leq k \leq n$. The result is sharp.*

We are unable to prove the result in the case $n = 3$.

Proof. For $n \geq 2$, write $f(z) = f_n(z) + r_n(z)$ and $\phi(z) = \phi_n(z) + s_n(z)$. Then since $|a_n| \leq 2 - 1/n$ and $|b_n| \leq 1$, it follows that

$$|zr_n'(z)| \leq \frac{r^{n+1}}{(1-r)^2}[2n(1-r) + (1+r) \tag{11}$$

and

$$|s_n(z)| \leq \frac{r^{n+1}}{1-r}. \tag{12}$$

Thus for $|z| = r = 1/3$

$$\begin{aligned}
\operatorname{Re}\frac{zf_n'(z)}{\phi_n(z)} &= \operatorname{Re}\frac{zf'(z)}{\phi(z)} + \operatorname{Re}\left(\frac{[zf'(z)/\phi(z)]s_n(z) - zr_n'(z)}{\phi(z) - s_n(z)}\right)\\
&\geq \operatorname{Re}\frac{zf'(z)}{\phi(z)} - \frac{|zf'(z)/\phi(z)|\,|s_n(z)| + |zr_n'(z)|}{\big|\,|\phi(z)| - s_n(z)\,\big|}\\
&\geq \frac{1-r}{1+r} - \left(\frac{1+r}{1-r}\frac{r^{n+1}}{1-r} + \frac{r^{n+1}}{(1-r)^2}[2n(1-r) + (1+r)]\right)\left(\frac{r}{1+r} - \frac{r^{n+1}}{1-r}\right)^{-1}\\
&= \frac{1}{2} - \frac{4(2+n)}{3^n - 2} > 0,
\end{aligned}$$

when $r = 1/3$, where we have use (11) and (12) and distortion theorems for the classes C and P. Thus (10) is proved when $n \geq 4$ and when $|z| = 1/3$. Since $\operatorname{Re}(zf_n'(z)/\phi_n(z))$ is harmonic, it follows from the minimum principle that (10) is valid in $\{z : |z| \leq 1/3\}$ and $n \geq 4$.

When $n = 2$ we note that

$$\operatorname{Re}\frac{zf_2'(z)}{\phi_2(z)} = \operatorname{Re}\frac{1 + 2a_2z}{1 + b_2z} = 1 + \operatorname{Re}\frac{(2a_2 - b_2)z}{1 + b_2z} \geq 1 - \frac{|2a_2 - b_2|\,|z|}{1 + |b_2z|} > 0$$

for $|z| \leq 1/3$, where we have used the fact that $|2a_2 - b_2| \leq 2$, which follows easily on equating coefficients in the representation $zf'(z) = \phi(z)p(z)$ and the inequality $|b_2| \leq 1$. We note that when $n = 2$ and F is given by (4),

$$\frac{zF_2'(z)}{\phi_2(z)} = \frac{1 + 3z}{1 + b_2z} = 0$$

when $z = -1/3$ and so the result is best possible.

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