PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 49 (63), 1991, 51–60

ENANTIOMORPHISM OF THREE-DIMENSIONAL SPACE AND LINE MULTIPLE ANTISYMMETRY GROUPS

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Abstract. Only space and line groups, among all three-dimensional crystallographic l-multiple antisymmetry group categories, can differ exclusively with regards to enantiomorphism. For all values of l, total numbers of different non-enantiomorphic groups of these categories are computed. It is pointed out that these groups give complete interpretation of all symmetry groups belonging to multidimensional categories $G_{(l+3)(l+2)(l+1)l\ldots 3}$ and $G_{(l+3)(l+2)(l+1)l\ldots 31}$ retaining invariant $l+2$, $l+1$, l, ... -dimensional planes of the $(l+3)$-dimensional space, which contain one another in succession.

1. Basic assumptions of l-multiple antisymmetry.

The l-multiple antisymmetry is an extension of antisymmetry realized by ascribing to the points not only one, but l qualitatively different signs $+$ or $-$; isometric transformation of l-signed figure is called a symmetry transformation if it preserves the signs; if it changes only the j-sign, only the j- and k-signs, ... or all the l signs, it is called, respectively, the antisymmetry transformation of the j-pattern, (j,k)-pattern, ... or $(1,2,\ldots,l)$-pattern. Altogether there are 2^l-1 antisymmetry patterns [2]. Every antisymmetry transformation is the product of a symmetry transformation and an antiidentity transformation of a given pattern (exclusive sign change).

In an l-multiple antisymmetry group the presence of two patterns entails the presence of the third pattern dependent on them. In the general case, p patterns are called dependent if the antiidentities of these patterns are dependent, i.e. if one of them is the product of two or more antiidentities from the remaining $p-1$ patterns $(3 \le p \le 2^l - 1)$; otherwise, these p patterns are independent.

The group is called the k independent-pattern senior group and m independent-pattern junior group (S^kM^m-type) if it contains exactly $k+m$ independent patterns, where exactly k are represented by antiidentities. For $k=0$ the group is called junior of m independent patterns (M^m-type) and for $m=0$ the senior group of k independent patterns (S^k-type); the classic groups ($k=m=0$

AMS Subject Classification (1985): Primary 20H15 case) are called generating groups (P-type). There are altogether $(l+1)(l+2)$, types of l-multiple antisymmetry groups. For $l > 1$ the group types are subdivide into the forms according to their antisymmetry pattern and antiidentity transfo mation. Thus, for $l = 2$, to the 6 types P, S, S^2, M, SM, M^2 there correspond th 12 forms: P, S_1, S_2, S_{12} (generating, senior of 1-pattern, 2-pattern, (1, 2)-pattern S_1S_2 (senior of two independent patterns); M_1, M_2, M_{12} (junior of the correspond ing pattern); S_1M_2, S_2M_1, $S_{12}M_1$ (senior of one pattern and junior of the other pattern); M_1M_2 (junior of two independent patterns) [2].

To every group G (of any type) corresponds, so called, generating group S consisting of symmetry transformations geometrically coinciding with the transformations from G. The set of l-multiple antisymmetry groups with a common generating group S determines a family. The derivation of S, S^2, ..., S^l -type groups is trivial (the multiple use of generating group by addition of antiidentity transformations of one, two or more independent patterns). The derivation of M, SM, ..., $S^{l-1}M$ -type groups is also not difficult when the simple antisymmetry groups are known. Nontrivial is only the derivation of M^m-type groups for $m > 1$. They are obtained by Kishinev geometricians from generating groups by the generalized Shubnikov generator — substitution method [2]. For the derivation of the M^m-type groups, the method founded on the use of antisymmetric characteristics of symmetry groups, introduced in [9, 10, 11] is also very efficient.

The notation $G_{r...}{}^l$ is for the category of l multiple antisymmetry groups derived from the category $G_{r...}$. Two l-multiple antisymmetry groups G and G' are equal if $G' = aGa^{-1}$ for some orientation preserving affine transformation a. Therefore, there is the isomorphism of G onto G' which maps every symmetry or antisymmetry transformation from G into a transformation of the same character from G': rotation into rotation, reflection into reflection, screw into screw with the same orientation, antirotation of a definite pattern into antirotation of the same pattern, etc.

For every category $G_{r...}{}^l$, the number of different groups of any form and fixed M^m-type is equal $N_m =$ the number of different M^m-type groups (of $M_1M_2 \dots M_m$ form) of the category $G_{r...}{}^m$. The same is the number of different groups of any form and S^kM^m-type for $1 \le k \le l-m$. For the S^k-type, this number is equal to $N_0 =$ the number of generating groups, i.e. the number of the groups of the category $G_{r...}{}^m$. Therefore, in the transition from $l-1$ signs to l signs, nontrivial is only the computation of the M^l-type groups of the given category. The number P_l of different groups of the category $G_{r...}{}^l$ is expressed by $N_0, N_1, \dots, N_l$ in the formula from §3 of the monograph [2]:

$$P_l = \sum_{m=0}^{l} \left\{ \sum_{k=0}^{l-m} C(l,k,m) \right\} N_m. \tag{1}$$

For $l = 1, \dots, 6$ we get:

$P_1 = 2N_0 + N_1,$

$P_2 = 5N_0 + 6N_1 + N_2,$

$$P_3 = 16N_0 + 35N_1 + 14N_2 + N_3,$$
$$P_4 = 67N_0 + 240N_1 + 175N_2 + 30N_3 + N_4,$$
$$P_5 = 374N_0 + 2077N_1 + 2480N_2 + 775N_3 + 62N_4 + N_5,$$
$$P_6 = 2825N_0 + 23562N_1 + 43617N_2 + 22320N_3 + 3255N_4 + 126N_5 + N_6,$$

so the problem is completely solved if all $N_m > 0$ are found.

2. l-multiple antisymmetry space groups

In [2] for $l = 1, 2, 6$, as well as in [11] for $l = 3, 4, 5$ M^m-type groups of the category ${G_3}^l$ are completely investigated. For these groups, $N_0 = 230$, $N_1 = 1191$, $N_2 = 9511$, $N_3 = 109139$, $N_4 = 1640955$, $N_5 = 28331520$, $N_6 = 419973120$ and $N_l = 0$ for $l \geq 7$. In accordance with the formula (1) it is possible to find the total numbers of all different groups of the category ${G_3}^l$. These numbers are: $P_0 = 230$, $P_1 = 1651$, $P_2 = 17807$, $P_3 = 287658$, $P_4 = 6880800$, $P_5 = 240900462$, $P_6 = 12210589024$.

Two l-multiple antisymmetry groups G and G' are enantiomorphic if $G' = aGa^{-1}$ holds only for some orientation reversing affine transformation(s) a. Among Fedorov groups G_3 there are 11 pairs of enantiomorphic groups, among Shubnikov groups ${G_3}^1$ there are 57 such pairs: 11 among the generating, 11 among the senior and 35 among the junior groups [4, 6, **12**, **13**]. Enantiomorphic pairs exist also for $l \geq 2$ among Zamorzaev groups ${G_3}^l$.

By the results [2, 4] it is understood, and proved as well, that from every symmetry group belonging to an enantiomorphic group pair, the same number of simple and multiple antisymmetry groups is derived, and to every group of one family corresponds the enantiomorphic group of the other family. The 11 pairs of enantiomorphic Fedorov groups from [2, **13**], followed by the corresponding numbers $N_m(S)$, are: a) $\mathbf{P4_1}$, $\mathbf{P4_3}$, $N_1 = 2$, $N_2 = 3$; 2) $\mathbf{P4_122}$, $\mathbf{P4_322}$, $N_1 = 5$, $N_2 = 24$, $N_3 = 84$; 3) $\mathbf{P4_12_12}$, $\mathbf{P4_32_12}$, $N_1 = 3$, $N_2 = 6$; 4) $\mathbf{P3_1}$, $\mathbf{P3_2}$, $N_1 = 1$; 5) $\mathbf{P3_121}$, $\mathbf{P3_221}$, $N_1 = 2$, $N_2 = 3$; 6) $\mathbf{P3_112}$, $\mathbf{P3_212}$, $N_1 = 2$, $N_2 = 3$; 7) $\mathbf{P6_1}$, $\mathbf{P6_5}$, $N_1 = 1$; 8) $\mathbf{P6_2}$, $\mathbf{P6_4}$, $N_1 = 3$, $N_2 = 6$; 9) $\mathbf{P6_122}$, $\mathbf{P6_522}$, $N_1 = 3$, $N_2 = 6$; 10) $\mathbf{P6_222}$, $\mathbf{P6_422}$, $N_1 = 5$, $N_2 = 24$; $N_3 = 84$; 11) $\mathbf{P4_132}$, $\mathbf{P4_332}$, $N_1 = 1$. Except from these 11 pairs, the M^m-type enantiomorphic group pairs are derived from the Fedorov groups $\mathbf{I4_1}$, $\mathbf{P4_2}$, $\mathbf{I4_122}$, $\mathbf{P4_222}$, $\mathbf{P4_22_12_1}$ and $\mathbf{I4_132}$. In [**13**], the registration of enantiomorphic pairs among the M^1 and M^2-type groups of the category ${G_3}^l$ for $l = 1, 2$ is realised by comparing International symbols of the same group obtained from a Fedorov group by Shubnikov-Zamorzaev and Belov method.

The problem of registration of enantiomorphic pairs among Zamorzaev M^m-type groups is very efficiently solved using the antisymmetric characteristics of Fedorov groups, introduced in [**11**].

Definition 1. Let all products of generators of a symmetry group S, within which every generator participates once at most, be given, and then subsets of transformations equivalent with regard to symmetry, be separated. The resulting system is called the antisymmetric characteristic of S ($AC(S)$).

THEOREM 1. *Two groups G and G' of the M^m-type derived from the same symmetry group are equal iff they possess equal antisymmetric characteristics.*

In every enumerated enantiomorphic pair 1–11, the Fedorov groups belonging to it differ only by the orientation of screws with screw axes $\mathbf{n}_j(\frac{jc}{n}\mathbf{n})$ and $\mathbf{n}_{n-j}(-\frac{jc}{n}\mathbf{n})$ $(1 \leq j \leq [n:2])$. The product of a screw motion $\mp\frac{jc}{n}\mathbf{n}$ with the translation $\pm c$ is $\pm\frac{jc}{n}\mathbf{n}$. In this way, the multiplication of any screw motion from one of the 11 enumerated pairs with the translation $\pm c$ results in the transition from the discussed group to its enantiomorph.

This is illustrated by example of the M^1-type groups of families with the generating Fedorov groups enumerated by 10 (**84a** and **85a**). In accordance to the proposed method, every M^1-type group is followed by its antisymmetric characteristic AC and transformed antisymmetric characteristic AC^*, i.e. by the corresponding antisymmetric characteristic of the second family. Since we are dealing only with the antiidentities or their products, the signs $\pm$ and $\mp$ are omitted.

$$\mathbf{84a} \sim \mathbf{P6_222} \sim \{(a,b),c\}\left(\frac{c}{3}6:2\right)$$

$m=1$ $\quad AC: \left\{\frac{c}{3}6\right\}\{2,2c\}, \quad AC^*: \left\{c\frac{c}{3}6\right\}\{2,2c\}$

1) $\{(a,b),c\}\left(\frac{c}{3}6:\underline{2}\right)$: $\{E\}\{e_1,e_1\}$, $\quad \{E\}\{e_1,e_1\}$ 1′)

2) $\{(a,b),c\}\left(\frac{c}{3}\underline{6}:2\right)$: $\{e_1\}\{E,E\}$, $\quad \{e_1\}\{E,E\}$ 2′)

3) $\{(a,b),c\}\left(\frac{c}{3}\underline{6}:\underline{2}\right)$: $\{e_1\}\{e_1,e_1\}$, $\quad \{e_1\}\{e_1,e_1\}$ 3′)

4) $\{(a,b),\underline{c}\}\left(\frac{c}{3}6:2\right)$: $\{E\}\{E,e_1\}$, $\quad \{e_1\}\{E,e_1\}$ 4′)

5) $\{(a,b),\underline{c}\}\left(\frac{c}{3}\underline{6}:2\right)$: $\{e_1\}\{E,e_1\}$, $\quad \{E\}\{E,e_1\}$ 5′)

$$\mathbf{85a} \sim \mathbf{P6_422} \sim \{(a,b),c\}\left(-\frac{c}{3}6:2\right)$$

$m=1$ $\quad AC: \left\{-\frac{c}{3}6\right\}\{2,2c\}$

1′) $\{(a,b),c\}\left(-\frac{c}{3}6:\underline{2}\right)$: $\{E\}\{e_1,e_1\}$

2′) $\{(a,b),c\}\left(-\frac{c}{3}\underline{6}:2\right)$: $\{e_1\}\{E,E\}$

3′) $\{(a,b),c\}\left(-\frac{c}{3}\underline{6}:\underline{2}\right)$: $\{e_1\}\{e_1,e_1\}$

4′) $\{(a,b),\underline{c}\}\left(-\frac{c}{3}\underline{6}:2\right)$: $\{e_1\}\{E,e_1\}$

5′) $\{(a,b),\underline{c}\}\left(-\frac{c}{3}6:2\right)$: $\{E\}\{E,e_1\}$

The same procedure is used for finding the enantiomorph of the given M^m-type group $(m \geq 2)$. This is realized by finding the group with the antisymmetric characteristic identical to in such a manner transformed antisymmetric characteristic of the initial group. We find such a group among the M^m-type groups derived form the second group of the discussed pair. These two groups make the M^m-type enantiomorphic group pair derived from the two enantiomorphic Fedorov groups. In the discussed families to each of 24 M^2-type groups and 84 M^3-type groups from the first, corresponds the enantiomorphic group from the second family.

Classifying in that manner all M^m-type groups derived from the 11 Fedorov enantiomorphic group pairs, it is possible to acknowledge that all M^m-type groups of the families derived from enantiomorphic generating groups are mutually enantiomorphic. Therefore, if we add the numbers $N_i(S)$ $(1 \leq i \leq m)$, previously quoted for every Fedorov enantiomorphic group pair 1–11, we can conclude that among M^m-type groups derived from them there are 28 M^1-type, 75 M^2-type and 168 M^3-type enantiomorphic group pairs [13].

The total number of all different enantiomorphic pairs among all Shubnikov and Zamorzaev groups is not exhausted by the given numbers. The new enantiomorphic pairs exist also among the M^m-type groups of every family with a generating group **I4$_1$**, **P4$_2$**, **I4$_1$22**, **P4$_2$22**, **P4$_2$2$_1$2$_1$** and **I4$_1$32** [13].

We cite these groups from [2], denoted by three different systems of symbols, comprising their antisymmetric characteristic AC and the existential conditions from [11].

1) $\mathbf{32a} \sim \mathrm{I4_1} \sim \left\{a, b, \frac{a+b+c}{4}\right\}\left(\frac{c}{4}4\right) \quad : \left\{\frac{c}{4}4\right\}\left\{\frac{a+b+c}{2}\right\}$
$a \neq \bar{a},\ b \neq \bar{b}$

2) $\mathbf{33a} \sim \mathrm{P4_1} \sim \{a, b, c\}\left(\frac{c}{2}4\right) \quad : \{c\}\left\{\frac{c}{2}4, a\frac{c}{2}4\right\}$
$a = b$

3) $\mathbf{46a} \sim \mathrm{I4_122} \sim \left\{a, b, \frac{a+b+c}{2}\right\}\left(\frac{c}{4}4 : 2\right) \quad : \left\{\frac{a+b+c}{2}\right\}\left\{\frac{c}{4}4\right\}$
$a = b$

4) $\mathbf{47a} \sim \mathrm{P4_222} \sim \{a, b, c\}\left(\frac{c}{2}4 : 2\right) \quad : \left\{\frac{c}{2}4, a\frac{c}{2}4\right\}\{2, 2c\}$
$a \neq \bar{a},\ b \neq \bar{b}$

5) $\mathbf{50a} \sim \mathrm{P4_22_12_1} \sim \{a, b, c\}\left(\frac{c}{2}4 : \frac{a}{2}2_{b/4}\right) \quad : \left\{\frac{c}{2}4\right\}\left\{\frac{a}{2}2_{b/4}, c\frac{a}{2}2_{b/4}\right\}$
$a \neq \bar{a},\ b \neq \bar{b}$

6) $\mathbf{96a} \sim \mathrm{I4_132} \sim \left\{a, b, \frac{a+b+c}{2}\right\}\left(3_{(b-a/8)} : \frac{c}{4}4\right) \quad : \left\{\frac{a+b+c}{2}\right\}\left\{\frac{c}{4}4\right\}$
$a \neq \bar{a},\ b \neq \bar{b},\ 3 \neq \bar{3}.$

In the case of Fedorov symmetry groups **P4$_2$**, **P4$_2$22**, **P4$_2$2$_1$2$_1$** or **I4$_1$**, **I4$_1$22**, **I4$_1$32**, with a screw axis $\frac{c}{4}4$ or $\frac{c}{4}4$, the transition from the initial AC to the transformed (AC^*) is realized multiplying $\frac{c}{4}4$ or $\frac{c}{4}4$ by c or by $(a+b+c):2$. For the recognition of M^m-type enantiomorphic group pairs we have only to compare the initial and transformed antisymmetric characteristics and find the equal ones. By treating in this manner all simple and multiple antisymmetry M^m-type groups of each of six families enumerated by 1–6, we will obtain all missing enantiomorphic pairs of the M^m-type Shubnikov and Zamorzaev groups.

The suggested method is illustrated by example of the junior Shubnikov and Zamorzaev groups derived from the Fedorov group enumerated by **1 (32a)**.

$$\mathbf{32a} \sim \mathbf{I4_1} \sim \left\{a, b, \frac{a+b+c}{2}\right\}\left(\frac{c}{4}4\right)$$

$m = 1$ $\qquad AC : \{\frac{a+b+c}{2}\}\{\frac{c}{4}4\}, \quad AC^* : \{\frac{a+b+c}{2}\}\{\frac{a+b+c}{2}\,\frac{c}{4}4\}$

1) $\left\{a, b, \frac{a+b+c}{\underline{2}}\right\}\left(\frac{c}{4}4\right) \; : \; \{e_1\}\{E\}, \qquad \{e_1\}\{e_1\} \qquad 3')$

2) $\left\{a, b, \frac{a+b+c}{2}\right\}\left(\frac{c}{4}\underline{4}\right) \; : \; \{E\}\{e_1\}, \qquad \{E\}\{e_1\} \qquad 2')$

3) $\left\{a, b, \frac{a+b+c}{\underline{2}}\right\}\left(\frac{c}{4}\underline{4}\right) \; : \; \{e_1\}\{e_1\}, \qquad \{e_1\}\{E\} \qquad 1')$

From the initial and transformed antisymmetric characteristics of the written Shubnikov groups we conclude that the antisymmetry groups 1 and 3 are enantiomorphic.

$m = 2$

1) $\left\{a, b, \frac{a+b+c}{\underline{2}}\right\}\left(\frac{c}{4}{}^{*}4\right) \; : \; \{e_1\}\{e_2\}, \qquad \{e_1\}\{e_1 e_2\} \quad 6')$

2) $\left\{a, b, {}^{*}\!\left(\frac{a+b+c}{\underline{2}}\right)\right\}\left(\frac{c}{4}{}^{*}4\right) \; : \; \{e_1 e_2\}\{e_2\}, \qquad \{e_1 e_2\}\{e_1\} \quad 5')$

3) $\left\{a, b, {}^{*}\!\left(\frac{a+b+c}{2}\right)\right\}\left(\frac{c}{4}\underline{4}\right) \; : \; \{e_2\}\{e_1\}, \qquad \{e_2\}\{e_1 e_2\} \quad 4')$

4) $\left\{a, b, {}^{*}\!\left(\frac{a+b+c}{2}\right)\right\}\left(\frac{c}{4}{}^{*}\underline{4}\right) \; : \; \{e_2\}\{e_1 e_2\}, \qquad \{e_2\}\{e_1\} \quad 3')$

5) $\left\{a, b, {}^{*}\!\left(\frac{a+b+c}{\underline{2}}\right)\right\}\left(\frac{c}{4}\underline{4}\right) \; : \; \{e_1 e_2\}\{e_1\}, \qquad \{e_1 e_2\}\{e_2\} \quad 2')$

6) $\left\{a, b, \frac{a+b+c}{\underline{2}}\right\}\left(\frac{c}{4}{}^{*}\underline{4}\right) \; : \; \{e_1\}\{e_1 e_2\}, \qquad \{e_1\}\{e_2\} \quad 1')$

Comparing the initial and transformed AC for $m = 2$, we notice that 1 and 6, 2 and 5, 3 and 4, are enantiomorphic group pairs. Consequently, among the M^m-type groups derived from $\mathbf{I4_1}$ there are two non-enantiomorphic groups for $m = 1$ and three for $m = 2$.

Moreover, directly from a generating group S it is possible to derive all different non-enantiomorphic M^m-type groups and compute their number $\bar{N}_m(S)$, avoiding the aforementioned finding procedure. Having this purpose, it is necessary to transform the initial $AC(S)$ into a new antisymmetric characteristic $\overline{AC}(S)$, estabilishing the equivalence between every M^m-type group derived from S, and its enantiomorph. If S is one of the Fedorov groups denoted by 1–6, and $AC(S)^*$

is derived by replacing in $AC(S)$ the c-screw motion by its product with the c-component translation, then $\overline{AC}(S) = \{AC(S), AC(S)^*\}$.

The construction of $\overline{AC}$ is illustrated by example of the group $\mathbf{I4_1} \sim \{a, b, \frac{a+b+c}{2}\}(\frac{c}{4}4)$ with the antisymmetric characteristic $\{\frac{a+b+c}{2}\}\{\frac{c}{4}4\}$. By multiplying the screw motion $\frac{c}{4}4$ with the translation $\frac{a+b+c}{2}$, the initial AC is transformed into the $AC^*\{\frac{a+b+c}{2}\}\{\frac{a+b+c}{2}\ \frac{c}{4}4\}$. The addition of the obtained AC^* to the initial AC results in the $\overline{AC}$:

$$\{\{\tfrac{a+b+c}{2}\}\{\tfrac{c}{4}4\}, \{\tfrac{a+b+c}{2}\}\{\tfrac{a+b+c}{2}\ \tfrac{c}{4}4\}\} = \{\tfrac{a+b+c}{2}\}\{\tfrac{c}{4}4, \tfrac{c}{4}4\ \tfrac{a+b+c}{2}\}$$

with the reduced form $\{\frac{c}{4}4, \frac{c}{4}4\ \frac{a+b+c}{2}\}$, permiting direct derivation of non-enantiomorphic M^m-type groups.

THEOREM 2. *Two symmetry groups S and S' with isomorphic antisymmetric characteristics generate the same number of the M^m-type groups for every fixed m $(1 \leq m \leq l)$, wich correspond to each other with regards to structure.*

Proof. Let i be an isomorphism such that $i(AC(S)) = AC(S')$. Its extension i' is defined as follows: $i' = i$ on S, and $i'(e_i) = e_i$ for every i $(i \in \{1, 2, \dots, l\})$. For every fixed m, i' transforms equal antisymmetric characteristics into the equal ones, and different into the different ones, and preserves their structure. So, according to Theorem 1, groups S and S' generate the same number of the M^m-type groups for every m $(1 \leq m \leq l)$.□

Since the reduced $\overline{AC}$ of the group $\mathbf{I4_1}$ is isomorphic to the antisymmetric characteristic of the Fedorov group **27s**, we can directly conclude that $\bar{N}_1(\mathbf{I4_1}) = \bar{N}_1(\mathbf{32a}) = N_1(\mathbf{27s}) = 2$, $\bar{N}_2(\mathbf{I_1}) = \bar{N}_2(\mathbf{32a}) = N_2(\mathbf{27s}) = 3$ [**11**]. The complete catalogue of non-enantiomorphic M^m-type groups derived from $\mathbf{I4_1}$ (**32a**) can be obtained from the partial catalogue corresponding to the group **27s** [**11**, Appendix].

By the same method, the antisymmetric characteristics $\overline{AC}$ of Fedorov groups, previously enumerated by 1–6, are obtained. We cite them including the symbol of the corresponding group S' with an isomorphic AC [**11**]. This makes possible the complete cataloguing of non-enantiomorphic M^m-type groups derived from the group S (1–6) and computing the numbers $\bar{N}_m(S) = N_m(S')$.

1) $\mathbf{32a} \sim \mathbf{I4_1}$, $\overline{AC} : \left\{\frac{c}{4}4, \frac{a+b+c}{2}\ \frac{c}{4}4\right\}$, XXI) **27s**,
$\bar{N}_1(\mathbf{32a}) = 2$, $\bar{N}_2(\mathbf{32a}) = 3$; $N_1(\mathbf{32a}) = 3$, $N_2(\mathbf{32a}) = 6$.

2) $\mathbf{33a} \sim \mathbf{P4_2}$, $\overline{AC} : \left\{\frac{c}{2}4, a\frac{c}{2}4\right\}\left\{c\frac{c}{2}4, ac\frac{c}{2}4\right\}$, IV) **4s**,
$\bar{N}_1(\mathbf{33a}) = 4$, $\bar{N}_2(\mathbf{33a}) = 15$, $\bar{N}_3(\mathbf{33a}) = 42$;
$N_1(\mathbf{33a}) = 5$, $N_2(\mathbf{33a}) = 24$, $N_3(\mathbf{33a}) = 84$.

3) $\mathbf{46a} \sim \mathbf{I4_122}$, $\overline{AC} : \{2\}\left\{\frac{c}{4}4, \frac{a+b+c}{2}\ \frac{c}{4}4\right\}$, VI) **6s**,
$\bar{N}_1(\mathbf{46a}) = 5$, $\bar{N}_2(\mathbf{46a}) = 24$, $\bar{N}_3(\mathbf{46a}) = 84$;
$N_1(\mathbf{46a}) = 7$, $N_2(\mathbf{46a}) = 42$, $N_3(\mathbf{46a}) = 168$.

4) **47a** $\sim$ **P4**$_2$**22**, $\overline{AC}: \{2, 2c\}\{2a, 2ac\}\Big\{\Big\{\frac{c}{2}4, a\frac{c}{2}4\Big\}, \Big\{c\frac{c}{2}4, ac\frac{c}{2}4\Big\}\Big\}$, **XXVI) 3h**, $\bar{N}_1(\mathbf{47a}) = 7$, $\bar{N}_2(\mathbf{47a}) = 54$, $\bar{N}_3(\mathbf{47a}) = 420$, $\bar{N}_4(\mathbf{47a}) = 2520$; $N_1(\mathbf{47a}) = 8$, $N_2(\mathbf{47a}) = 75$, $N_3(\mathbf{47a}) = 714$, $N_4(\mathbf{47a}) = 5040$.

5) **50a** $\sim$ **P4**$_2$**2**$_1$**2**$_1$, $\overline{AC}: \Big\{\frac{a}{2}2_{b/4}, c\frac{a}{2}2_{b/4}\Big\}\Big\{\frac{c}{2}4, c\frac{c}{2}4\Big\}$, **VI) 6s**, $\bar{N}_1(\mathbf{50a}) = 4$, $\bar{N}_2(\mathbf{50a}) = 15$, $\bar{N}_3(\mathbf{50a}) = 42$; $N_1(\mathbf{50a}) = 5$, $N_2(\mathbf{50a}) = 24$, $N_3(\mathbf{50a}) = 84$.

6) **96a** $\sim$ **I4**$_1$**32**, $\overline{AC}: \Big\{\frac{c}{4}4, \frac{a+b+c}{2}\frac{c}{4}4\Big\}$, **XXI) 27s**, $\bar{N}_1(\mathbf{96a}) = 2$, $\bar{N}_2(\mathbf{96a}) = 3$; $N_1(\mathbf{96a}) = 3$, $N_2(\mathbf{96a}) = 6$.

The number of M^m-type enantiomorphic group pairs derived from S is given by the formula $N_m(S) - \bar{N}_m(S)$. Consequently, among the M^m-type groups derived from Fedorov groups 1–6 there are 7 M^1-type, 63 M^2-type, 262 M^3-type and 2520 M^4-type enantiomorphic group pairs. Summarizing the obtained results we conclude that among M^m-type groups of the category $G_3{}^l$ there are 35 M^1-type, 138 M^2-type (not 132, as in [13]), 430 M^3-type, and 2520 M^4-type enantiomorphic group pairs. The numbers $\bar{N}_m$ of all different non-enantiomorphic M^m-type groups of this category are: $\bar{N}_1 = N_1 - 35 = 1191 - 35 = 1156$, $\bar{N}_2 = N_2 - 138 = 9511 - 138 = 9373$ (not 9379, as in [4, 13]), $\bar{N}_3 = N_3 - 430 = 109139 - 430 = 108709$, $\bar{N}_4 = N_4 - 2520 = 1640955 - 2520 = 1638435$, $\bar{N}_5 = N_5 = 28331520$, $\bar{N}_6 = N_6 = 419973120$, and $\bar{N}_m = N_m = 0$ for $m \geq 7$. According to the formula (1), knowing the numbers $\bar{N}_m$ we have the numbers $\bar{P}_l$ of all different non-enantiomorphic groups of the category $G_3{}^l$: $\bar{P}_0 = 219$, $\bar{P}_1 = 1594$, $\bar{P}_2 = 17404$ (not 17410, as in [**4**, **13**]), $\bar{P}_3 = 243435$, $\bar{P}_4 = 6832093$, $\bar{P}_5 = 239891923$, $\bar{P}_6 = 12185913933$.

3. Three-dimensional line (rod) groups of l-multiple antisymmetry

The crystallographic l-multiple antisymmetry rod groups $G_{31}{}^l$ are thoroughly discussed for $l = 1, 2, \ldots, 5$ in [2]. In this category, numbers N_m are: $N_0 = 75$, $N_1 = 244$, $N_2 = 945$, $N_3 = 4074$, $N_4 = 15120$, $N_l = 5$ for $l \geq 5$; the numbers P_l ($l = 0, 1, 2, \ldots, 6$) are: $P_0 = 75$, $P_1 = 394$, $P_2 = 2764$, $P_3 = 27044$, $P_4 = 366300$, $P_5 = 6973228$, $P_6 = 187326348$.

Among the classic rod groups G_{31} there are 8 enantiomorphic pairs, among Shubnikov $G_{31}{}^1$ there are 34 such pairs: 8 among the generating, 8 among the senior and 18 among the junior groups [2, 13]. Among Zamorzaev groups $G_{31}{}^l$ for $l \geq 2$ there are 39 M^2-type and 63 M^3-type enantiomorphic groups pairs. Their registration is simple enough in the case of M^m-type groups derived from 8 enantiomorphic pairs of rod symmetry groups, since from each symmetry group of a pair the same number of M^m-type groups is derived for every fixed m, and to every group of one family corresponds the enantiomorphic group form the other family [2, 13]. We cite these 8 enantiomorphic pairs and the corresponding numbers $N_m(S)$: 1) **p3**$_1$, **p3**$_2$, $N_1 = 1$; 2) **p4**$_1$, **p4**$_3$, $N_1 = 1$; 3) **p6**$_1$, **p6**$_5$, $N_1 = 1$; 4) **p6**$_2$, **p6**$_4$, $N_1 = 3$, $N_2 = 6$; 5) **p3**$_1$**2**, **p3**$_2$**2**, $N_1 = 2$, $N_2 = 3$; 6) **p4**$_1$**22**, **p4**$_3$**22**, $N_1 = 2$, $N_2 = 3$; 7) **p6**$_1$**22**, **p6**$_5$**22**, $N_1 = 2$, $N_2 = 3$; 8) **p6**$_2$**22**, **p6**$_4$**22**, $N_1 = 4$, $N_2 = 15$,

$N_3 = 42$. So, from these 8 enantiomorphic group pairs are derived 16 M^1-type, 30 M^2-type and 42 M^3-type enantiomorphic group pairs.

Enantiomorphic pairs exist also among the M^m-type groups of every family with the generating rod group $\mathbf{p4_2}$ and $\mathbf{p4_222}$ [**13**]. These enantiomorphic pairs are registrated in [**2**, **13**] by comparing the symbols of the derived M^m-type groups (International and proposed by Zamorzaev [**2**]).

For the registration of the M^m-type enantiomorphic group pairs derived from the rod groups $\mathbf{p4_2}$ and $\mathbf{p4_222}$ very efficient is t he application of the antisymmetric characteristic method (AC-method). These two groups are given by two systems of symbols, comprising AC and $\overline{AC}$, followed by the symbol [**11**] of the corresponding groups S' with the isomorphic AC. This makes it possible to use directly the partial catalogue [**11**] and the numbers $\bar{N}_m(S) = N_m(S')$ and $N_m(S)$.

1) $\mathbf{p4_2} \sim \{c\}\left(\frac{c}{2}4\right)$, $AC : \{c\}\left\{\frac{c}{2}\right\}4$, $\overline{AC} : \left\{\frac{c}{2}4, c\frac{c}{2}4\right\}$, VI) 6s,
$\bar{N}_1(\mathbf{p4_2}) = 2$, $\bar{N}_2(\mathbf{p4_2}) = 3$; $N_1(\mathbf{p4_2}) = 3$, $N_2(\mathbf{p4_2}) = 6$.

2) $\mathbf{p4_222} \sim \{c\}\left(\frac{c}{2}4 : 2\right)$, $AC : \left\{\frac{c}{2}4\right\}\left\{\{2, 2c\}, \left\{2\frac{c}{2}4, 2c\frac{c}{2}4\right\}\right\}$,
$\overline{AC} : \left\{\frac{c}{2}4, c\frac{c}{2}4\right\}\left\{2, 2c, 2\frac{c}{2}4, 2c\frac{c}{2}4\right\}$, XXVIII) **8h**,
$\bar{N}_1(\mathbf{p4_222}) = 3$, $\bar{N}_2(\mathbf{p4_222}) = 9$, $\bar{N}_3(\mathbf{p4_222}) = 21$;
$N_1(\mathbf{p4_222}) = 4$, $N_2(\mathbf{p4_222}) = 15$, $N_3(\mathbf{p4_222}) = 42$.

Therefore, in two families with the generating rod group $\mathbf{p4_2}$ and $\mathbf{p4_222}$ there are 2 M^1-type, 9 M^2-type and 21 M^3-type enantiomorphic group pairs.

Among all M^m-type groups of rods there are 18 M^1-type, 39 M^2-type and 63 M^3-type enantiomorphic group pairs [**2**, **13**].

According to this, the total numbers $\bar{N}_m$ are: $\bar{N}_0 = 67$, $\bar{N}_1 = 226$, $\bar{N}_2 = 906$, $\bar{N}_3 = 4011$, $\bar{N}_4 = 15120$, and $\bar{N}_m = 0$ for $m \geq 5$. According to the formula (1), the numbers $\bar{P}_l$ $(l = 0, 1, 2, \dots, 6)$ are: $\bar{P}_0 = 67$, $\bar{P}_1 = 360$, $\bar{P}_2 = 2597$, $\bar{P}_3 = 25677$, $\bar{P}_4 = 352729$, $\bar{P}_5 = 6787305$, $\bar{P}_6 = 183772409$.

4. Geometrical applications of non-enantiomorphic space and rod simple and multiple antisymmetry groups

Antisymmetry and its l-multiple extension described in this paper has enriched symmetry in physical sense, but without changing its geometrical essence. The l qualitatively different signs $+$, $-$ ascribed to the points of a figure possess also a definitive geometrical meaning with respect to the space in which the figure is discussed. In additional dimensions these signs can be interpreted geometrically, making possible the application of the groups obtained to the computation and modelling of the certain subperiodic multidimensional symmetry group categories [**2**, **8**].

In [**2**, **8**] it is pointed out that the hyper-layer groups G_{43} are completely described by Shubnikov groups $G_3{}^1$. However, a three-dimensional plane in the four-dimensional space can be transformed onto itself by a rotation (orientation preserving transformation) around in it inserted two-dimensional plane, resulting in the elimination of the difference between the left and right screw motion. To every enantiomorphic pair of Shubnikov groups corresponds exactly one hyper-layer group. hence, the number of different hyper-layer groups is not 1651, as it looks at the first glance, but 1594 [4, 13].

In the same way, by all different non-enantiomorphic Zamorzaev groups ${G_3}^l$ are completely described multidimensional symmetry groups of the category $G_{(l+3)(l+2)(l+1)l\dots 3}$. Therefore, in the $(l+3)$-dimensional Euclidian space for $l = 1, 2, 3, \dots, 6$, there are 1594 symmetry groups of the category G_{43}, 17404 G_{543} (not 17410, as in [4, 13], 243435 G_{6543}, 6832093 G_{76543}, 239891923 G_{876543} and 12185913933 $G_{9876543}$ (in accordance to the numbers $\bar{P}_l$ $(l = 2, 3, \dots, 6)$ of Zamorzaev non-enantiomorphic groups of the category ${G_3}^l$).

By all different non-enantiomorphic crystallographic groups of the category ${G_{31}}^l$ are completely described all different crystallographic symmetry groups of the category $G_{(l+3)(l+2)\dots 31}$. Consequently, in the $(l+3)$-dimensional Euclidean space, for $l = 1, 2, \dots, 6$, there are 360 crystallographic symmetry groups of the category G_{431}, 2597 G_{5431}, 25677 G_{65431}, 352729 G_{765431}, 6787305 $G_{8765431}$ and 183772409 $G_{98765431}$ (in accordance to the numbers $\bar{P}_l$ $(l = 1, 2, \dots, 6)$ corresponding to the category ${G_{31}}^l$).

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(Received 25 08 1988)