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DYHEDRAL HYPERHOMOLOGY OF A CHAIN ALGEBRA WITH INVOLUTION

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Abstract. It is shown that Goodwillie's results from [G] (see also Rinehart [R]), on homotopy invariance of certain groups associated with $\mathrm{HC}_\bullet(A)$, with respect to the action of a (graded) derivation $D : A \to A$ of a (graded) k-algebra A can be extended to the case of dihedral (hyper)homology $HD_\bullet(A)$ of a (graded) algebra A with involution. These results provide a tool for computations of the Hermitian algebraic K-theory of A in terms of dihedral homology.

0. Introduction.

Goodwillie showed in [G] that the homotopy invariance of the de Rham cohomology can be meaningfully extended to the case of the cyclic homology $\mathrm{HC}_*(A)$ of a (graded) k-algebra A. In this generalization, the homotopy invariance of the de Rham cohomology is replaced by the invariance of certain groups, which are directly tied to $\mathrm{HC}_*(A)$, with respect to the action of a (graded) derivation $D : A \to A$ of a (graded) k-algebra A. Let us note that an early result of this type (and the one that plays a key role in [G] and our paper, see Proposition 1.7) has been proved by Rinehart long before the cyclic homology was defined (see [R]). One of the groups which is left invariant under the action of a derivation is the so called periodic cyclic homology $\mathrm{HC}_*^{\mathrm{per}}(A)$ of an algebra A. As a direct consequence Goodwillie obtains a useful result which states that under some not too restrictive conditions (one-connectedeness) a homomorphism $f : A \to B$ of (chain) algebras induces an isomorphism $f_* : \mathrm{HC}_*^{\mathrm{per}}(A) \to \mathrm{HC}_*^{\mathrm{per}}(B)$. This result plays one of the key roles in his computation of the relative rational algebraic K-theory $K_*(f) \otimes Q$ in terms of the cyclic homology for one-connected map of simplicial rings.

The objective of this note is to extend Goodwillie's results to the case of dihedral (hyper) homology $\mathrm{HD}_*(A)$ of a (graded) algebra A with involution. Our intention is to use these results in computations of the Hermitian algebraic K-theory of A in terms of dihedral homology.

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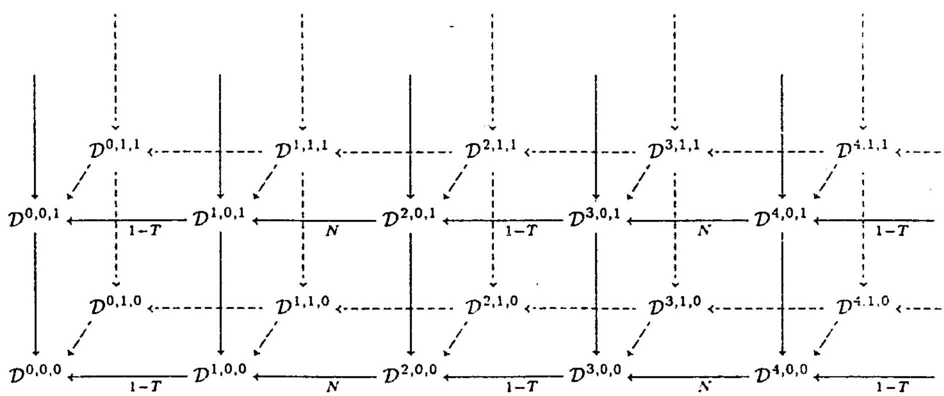
1. Periodic Dihedral Homology and Derivations

The reader is refered to [**KLS**], [**Lod**], and [**Ts**] for the foundations of dihedral homology. For convenience we give a brief overview. Ξ denotes the small category (see [**KLS**]) which produces dihedral objects as contravariant functors from Ξ into an appropriate category. The category Ξ is briefly described as the usual small category Δ, used in the definition of simplicial objects, enriched by the dihedral group of automorphisms of $[n] \in \mathrm{Ob}(\Delta)$. This group is generated by τ_n (cyclic permutation) and ρ_n (reflection) and the interaction of these new morphisms with the usual face and degeneracy operations is given by the following list:

$$\begin{cases} \tau_n\partial_n^i = \partial_n^{i-1}\tau_{n-1}, & 1 \le i \le n, \\ \tau_n\sigma_n^j = \sigma_n^{j-1}\tau_{n+1}, & 1 \le j \le n, \\ \tau_n^{n+1} = 1_{[n]}, & \end{cases} \qquad \begin{cases} \rho_n\partial_n^i = \partial_n^{n-i}\rho_{n-1}, & 0 \le i \le n, \\ \rho_n\sigma_n^j = \sigma_n^{n-j}\rho_{n+1}, & 0 \le j \le n, \\ \rho_n^2 = 1_{[n]}, & \tau_n\rho_n = \rho_n\tau_n^{-1}. \end{cases}$$

If $D : \Xi^{\mathrm{op}} \to \mathcal{T}$ is a dihedral object in category $\mathcal{T}$, we put $t_n := D(\tau_n)$ and $r_n := D(\rho_n)$. In analogy with the bicomplex which is used to define the cyclic homology, one defines, (see [**KLS**]), for every dihedral k-module $U : \Xi^{\mathrm{op}} \to \mathrm{Mod}_k$, or more generally for every dihedral object in an abelian category, a tricomplex $\mathcal{D} = (\mathcal{D}^{p,q,r}, \delta_1, \delta_2, \delta_3 \mid p, q, r, \ge 0)$, where $\mathcal{D}^{p,q,r} := D([q])$.



(1.1)

The differentials of this tricomplex, $\delta_1, \delta_2, \delta_3$, are defined as follows. The horizontal differentials, δ_1 and δ_2, are the differentials used in the usual definition of the cyclic homology. The veritical differential $\delta_3 : \mathcal{D}^{p,q,r+1} \to \mathcal{D}^{p,q,r}$ is defined so that the bicomplex which arises in the x, z plane of the tricomplex above defines a free biresolution for the dihedral group (compare [**Lodd**, Lemma 1.3.1]). The explicit definition of these differentials is given by the following formulas [KLS].

$$\delta_3 = \begin{cases} (-1)^q(1+(-1)^{r+1}R_q), & p \equiv 0 \pmod 4 \\ (-1)^{q+1}(1+(-1)^r R_q T_q), & p \equiv 1 \pmod 4 \\ (-1)^q(1+(-1)^{r+1}R_q), & p \equiv 2 \pmod 4 \\ (-1)^{q+1}(1+(-1)^r R_q T_q), & p \equiv 3 \pmod 4, \end{cases}$$

where $T_q := (-1)^q t_q$ and $R_q := (-1)^{q(q+1)/2} r_q$.

By definition, the dihedral homology $\mathrm{HD}_*(D)$ of the dihedral object D is the homology of the total complex of $\mathcal{D}$.

Definition 1.2. An algebra with involution $(A, *)$ is a k-algebra supplied with an anti-automorphism of period two, i.e. $* : A \to A$, $a \to a^*$, is an algebra automorphism satisfying the conditions (a) $(a^*)^* = a$ and (b) $(ab)^* = b^*a^*$. A derivation D of the algebra A is said to be compatible with $*$ if $D(a^*) = (D(a))^*$. In that case D will be called a derivation of the algebra with involution $(A, *)$.

Each algebra with involution produces naturally a dihedral module (see [KLS]) which we denote by $D(A)$, so $D(A)$ is a contravariant functor from the category Ξ into modules defined by $D(A)([n]) := A^{\otimes(n+1)}$ hence the corresponding tricomplex $\mathcal{D}(A)$ is described by $\mathcal{D}(A)^{p,q,r} = A^{\otimes(q+1)}$. Dihedral homology of this object is denoted by $\mathrm{HD}_*(A)$.

Our next objective is to define periodic dihedral homology. For this purpose, we extend the tricomplex $\mathcal{D}$ periodically in the negative direction of the x axes. Actually, $\mathcal{D}$ can be extended in the direction of any vector $-e \in Z^3$ if e represents a translation of the tricomplex into itself which preserves the differentials. The semigroup of these translations is generated by the three minimal translations $c_1 = (4,0,0)$, $c_2 = (2,0,1)$ and $c_3 = (0,0,2)$. Let us denote by s_i the map of the tricomplex (1) described as the composition of the translation in the direction of $-c_i$ and restriction to the first quadrant. These maps s_1, s_2 and s_3 induce chain maps on the total complex $\mathrm{Tot}_*(\mathcal{D}) := \bigotimes_{p+q+r=*} \mathcal{D}^{p,q,r}$ of (1) of degrees $i-5$, $i = 1,2,3$. We will denote both these chain maps and the induced homomorphisms of dihedral groups $\mathrm{HD}_*(D)$ by the same letters s_i, $i = 1,2,3$.

Definition 1.3. Let us define ${}_i\mathcal{D}_*^{\mathrm{per}} := \varprojlim\{\mathrm{Tot}_*(\mathcal{D}), s_i\}$ where $s_i : \mathrm{Tot}_*(\mathcal{D}) \to \mathrm{Tot}_{*-d_i}(\mathcal{D})$ is the epimorphism defined above, $i = 1,2,3$ and $d_i = 5 - i$. The homology of ${}_i\mathcal{D}_*^{\mathrm{per}}$ is denoted by ${}_i\,\mathrm{HD}_*^{\mathrm{per}}$.

The following isomorphism is immediate

$$(1.4) \qquad s_i := {}_i\,\mathrm{HD}_*^{\mathrm{per}}(D) \longrightarrow {}_i\,\mathrm{HD}_{*-d_i}^{\mathrm{per}}(D).$$

One has the following short exact sequence:

$$(1.5) \qquad 0 \longrightarrow \varprojlim_n{}^1 \mathrm{HD}_{*+d_in+1} \longrightarrow {}_i\,\mathrm{HD}_*^{\mathrm{per}} \longrightarrow \varprojlim_n \mathrm{HD}_{*+d_in} \longrightarrow 0$$

which is analogous to the sequence given in [G, II.3.2]. Also note that a short exact sequence of dihedral objects induces a long exact sequence in ${}_i\,\mathrm{HD}_*^{\mathrm{per}}$.

Let us equip the total complex $\mathrm{Tot}_*(\mathcal{D})$ with the filtration $F_m = F_m(\mathrm{Tot}_*(\mathcal{D})) := \bigotimes\{\mathcal{D}^{p,q,r} \mid p,q, \geq 0, 0 \leq r \leq m\}$. One observes that F_m/F_{m-1} is the double complex which produces the cyclic homology $\mathrm{HC}_*(A)$ so the E^1-term of the corresponding first quadrant spectral sequence converging to $\mathrm{HD}_{m+n}(A)$ is given by $E^1_{m,n} = \mathrm{HC}_n(A)$. From here we obtain a first quadrant spectral sequence:

$$\{E^1_{p,q} = \mathrm{HC}_q(D), \delta\} \Longrightarrow \mathrm{HD}_*(D) \tag{1.6}$$

where δ is induced by δ_3.

Let D be a derivation of an algebra with involution $(A, *)$. One defines, following Goodwillie, a map $L_D : D(A) \to D(A)$ of the associated dihedral objects by the formula

$$L_D(a_0, \dots, a_n) = \sum_{i=0}^{n}(a_0, \dots, Da_i, \dots, a_n), \qquad (a_0, \dots, a_n) \in A^{\otimes(n+1)}.$$

L_D induces a homomorphism of dihedral homology groups which will be denoted by the same letter. The key observation is the following result which is directly motivated by Corollary II.4.6 in [G]. Actually, this corollary was already proved by G.S. Rinehart, which was kindly pointed to us by the referee, in [R].

PROPOSITION 1.7. *$L_D \circ s_i := \mathrm{HD}_*(A) \to \mathrm{HD}_{*-d_i}(A)$, $i \in \{1,2\}$, is a zero homomorphism.*

Proof. We will reduce this result to Corollary II.4.6 mentioned above which claims that $L_D \circ s = 0 : \mathrm{HC}_*(A) \to \mathrm{HC}_{*-2}(A)$. Let us note that the map $s_i : \mathrm{Tot}_*(A) \to \mathrm{Tot}_*(A)$ is compatible with the filtration F_m defined in (1.6) above and that the induced map on F_m/F_{m-1} is s^{3-i}, $i = 1, 2$. So $L_D \circ s_i$ induces a map of spectral sequences (1.6) which is $L_D \circ s^{3-i} = 0$ on E^1-terms by the corollary above. Hence, by passing to the limit one obtains $L_D \circ s_i = 0$.□

The following proposition shows that in the case $1/2 \in k$ neither s_2 nor s_3 give an interesting periodic homology.

PROPOSITION 1.8. *If $1/2 \in k$ then*

(a) $s_i = 0 : \mathrm{HD}_*(A) \to \mathrm{HD}_{*-d_i}(A)$, *for $i = 2, 3$ and*

(b) ${}_i\mathrm{HD}^{\mathrm{per}}_*(A) = 0$, *for $i = 2, 3$.*

Proof. (a) Let us give the proof for the case $i = 2$; the proof for the other case is similar. Recall (see [KLS]) that $\mathrm{HD}_*(A)$ comes with its companion homology ${}^-\mathrm{HD}_*(A)$ which is coming from the tricomplex obtained by deleting the first horizontal stratum of the original tricomplex. By Theorem 3.3 in [KLS], if $1/2 \in k$, then one has a natural isomorphism $\mathrm{HC}_*(A) \cong \mathrm{HD}_*(A) \otimes {}^-\mathrm{HD}_*(A)$. This isomorphism is obtained from the long exact sequence associated with the short sequence of chain complexes $0 \longrightarrow \mathrm{Tot}_*\,\mathcal{C}(A) \longrightarrow \mathrm{Tot}_*\,\mathcal{D}(A) \longrightarrow \mathrm{Tot}^-_*\,\mathcal{D}(A)[-1] \longrightarrow 0$. Here, $\mathcal{C}(A)$ is the bicomplex producing cyclic homology while ${}^-\mathcal{D}(A)$ is the tricomplex which produces the companion dihedral homology. One observes that s_2 is well defined as a map of this short sequence into itself which induces a zero homomorphism $s_2 = 0 : \mathrm{HC}_*(A) \to \mathrm{HC}_{*-3}(A)$. Hence, by the splitting above, the same holds for the dihedral homology.

(b) This equality follows from (a) and the short exact sequence (1.5).

Proposition 1.7 above is all we need to establish, by mimicking closely the argument given in [G], the following theorem.

THEOREM 1.9. *Let k be a field of characteristic zero, A a k-algebra with involution $*$ and $I \subseteq A$ a nilpotent ideal invariant with respect to $*$, i.e. $*(I) \subseteq I$. Then the natural map $A \to A/I$ induces an isomorphism* $\mathrm{HD}^{\mathrm{per}}(A) \longrightarrow \mathrm{HD}^{\mathrm{per}}(A/I)$, *for $i \in \{1,2\}$.*

Proof. We refer the reader to the proof of T.II.5.1 in [G]. Let us mention only that our Proposition 1.7 plays the role of Corollary II.4.6 which provides a key step in the proof of T.II.5.1 in [G].□

2. Dihedral Hyperhomology

Chain Algebras with Involution. Since we are mainly interested in the case of chain algebras, our next objective is to extend the technique of the preceding paragraph to this case. The following is an extension of the definition 1.2 to the case of differential graded algebras or chain algebras as they are called in [G].

Definition 2.1. A *chain algebra with involution* (CAI) $(A, D, *)$ over k is a nonnegatively graded associative k-algebra A with identity together with k-linear maps $d : A \to A$ of degree -1 and $* : A \to A$ of degree 0 satisfying

$$d(ab) = (da)b + (-1)^a a(db) \qquad (a^*)^* = a$$
$$d^2 = 0 \qquad (ab)^* = (-1)^{ab} b^* a^*$$
$$d(a^*) = (da)^*.$$

A map of CAI is a map of algebras which commutes with differential and involution.

Example 2.2. If X is a topological monoid with involution then $S(X, k)$, the singular chain complex of X, has a structure of CAI.

The Dihedral Chain Complex (DCC) Associated to a CAI. *Definition* 2.3. Let $(A, D, *)$ be a CAI over k which is flat as a k-module. Then we define a dihedral object $D(A, d, *)$ in the category of CAI (and hence in the category of chain complexes also) by $D_n(A, d, *) = (A, d, *)^{\otimes(n+1)}$ where involution is defined by $(a_0, \dots, a_n)^* = (-1)^{\Sigma a_i a_j}(a_n^*, \dots, a_0^*)$ and the sum is taken over all pairs (i, j) with $0 \le i < j \le n$. We write a_i in the exponent instead of $|a_i|$, the degree of a_i. It is easy to check that $D_n(A, D, *)$ is a CAI.

Face and degeneracy maps as well as the cyclic group action are defined as in [G]. The involution $r_{n+1} : D_n \to D_n$ is given by:

$$r_{n+1}(a_0, \dots, a_n) = (-1)^{\Sigma a_i a_j}(a_0^*, a_n^*, \dots, a_1*)$$

where the sum is taken over all pairs (i,j) with $1 \leq i < j \leq n$. It is easy to check that all the necessary equalities hold.

Dihedral Hyperhomology. *Definition* 3.4. Let $(C,d) = (C_{*,*}, d)$ be a dihedral chain complex. Then the three-complex $\mathcal{D}^{*,*,*}$ in the category of chain complexes is in fact a 4-complex of k-modules whose homology we call *dihedral hyperhomology* of $(C_{*,*}, d)$ and denote by $\mathsf{HD}_*(C_{*,*}, d)$. This definition is parallel to the definition of cyclic hyperhomology and is equivalent to the definition of dihedral hyperhomology given by **[Lodd]**. This holds because total complexes and differentials of the 4-complex and the appropriate bicomplex coincide.

Homomorphisms s_1, s_2, s_3. The 4-complex $\mathcal{D} = \{\mathcal{D}^{*,*,*,*}, \delta_1, \delta_2, \delta_3, \delta_4\}$ associated to a dihedral chain complex (C,d) is the following (Here $(C_{n,*}, d)$ is a chain complex for each $n \geq 0$): $\mathcal{D}^{n,m,l,k} = C_{m,k}$, $n,m,l,k \geq 0$ and the differentials are

$$(2.5)\qquad \left.\begin{aligned} \delta_1 &: \mathcal{D}^{n,m,l,k} \longrightarrow \mathcal{D}^{n-1,m,l,k} \\ \delta_2 &: \mathcal{D}^{n,m,l,k} \longrightarrow \mathcal{D}^{n,m-1,l,k} \\ \delta_3 &: \mathcal{D}^{n,m,l,k} \longrightarrow \mathcal{D}^{n,m,l-1,k} \end{aligned}\right\} \quad \text{same as in (1.1)}$$
$$\delta_4 : \mathcal{D}^{n,m,l,k} \longrightarrow \mathcal{D}^{n,m,l,k-1}, \qquad \delta_4 = d.$$

As in §1 there are three periodic maps $s_i : \mathcal{D} \to \mathcal{D}$ of degrees $(-4,0,0,0)$, $(-2,0,-1,0)$ and $(0,0,-2,0)$ respectively. They induce three surjective chain maps $s_i : \mathrm{Tot}_* \mathcal{D} \to \mathrm{Tot}_{*-d_i} \mathcal{D}$, where $d_i = 5 - i$, $i = 1,2,3$ and the maps in HD_* as well.

Long Exact Sequence. Let ${}^{-}\mathcal{D}$ be a 4-complex obtained by cutting off the level $l = 0$ from $\mathcal{D}$. Define **[KLS]**

$$(2.6)\qquad {}^{-}\mathsf{HD}_*(C,d) = H_*(\mathrm{Tot}_*({}^{-}\mathcal{D})).$$

As in **[KLS]** we obtain the following results.

LEMMA 2.7. *There is a long exact sequence*

$$\cdots \longrightarrow {}^{-}\mathsf{HD}_*(C,d) \longrightarrow \mathsf{HC}_*(C,d) \longrightarrow \mathsf{HD}_*(C,d) \longrightarrow {}^{-}\mathsf{HD}_{*-1}(C,d) \longrightarrow \cdots$$

Proof. We have a short exact sequence of 4-complexes

$$0 \longrightarrow \mathcal{C} \xrightarrow{\ i\ } \mathcal{D} \xrightarrow{\ p\ } {}^{-}\mathcal{D} \longrightarrow 0$$

where $\mathcal{C}$ is a cyclic 3-complex viewed as a 4-complex which is zero if $1 \neq 0$, and p is a projection of degree $(0,0,-1,0)$. It induces a short exact sequence of total complexes which completes the proof.

THEOREM 2.8. *If* $1/2 \in k$, *then, there is a splitting,* $\mathsf{HC}_*(C,d) \cong \mathsf{HD}_*(C,d) \oplus {}^{-}\mathsf{HD}_*(C,d)$.

Proof. First note that $p_{\natural} : \mathsf{HD}_*(C,d) \to {}^{-}\mathsf{HD}_{*-1}(C,d)$ is the zero homomorphism if $1/2 \in k$. Hence the long exact sequence (2.7) splits into short exact sequences

$$0 \longrightarrow {}^{-}\mathsf{HD}_*(C,d) \xrightarrow{j} \mathsf{HC}_*(C,d) \xrightarrow{i} \mathsf{HD}_*(C,d) \longrightarrow 0.$$

We also have the other short excat sequence with HD and $^{-}$HD intercahanged and $j \circ i = 1$ implies the splitting.□

COROLLARY 2.9. *If* $1/2 \in k$, *then*:

(a) $s_i = 0 : \mathsf{HD}_*(C,d) \to \mathsf{HD}_{*-d_i}(C,d)$, *for* $i = 2, 3$.

(b) ${}_i\mathsf{HD}^{\mathrm{per}}_*(C,d) = 0$, *for* $i = 2, 3$.

Proof. Note that $s_i : \mathcal{D} \to \mathcal{D}$ is a zero map when restricted to cyclic subcomplex $\mathcal{C}$. Hence induced map on HC_* is zero. Now use Theorem 2.8. Part (b) follows directly from (a).□

Spectral sequences. If we filter $\mathcal{D}$ by

$$F_q^{n,m,l,k} = \begin{cases} \mathcal{D}^{n,m,l,k}, & l \le q \\ 0, & \text{otherwise} \end{cases}$$

and take homology of the total complexes we get a natural spectral sequence

$$(2.10) \qquad E^1_{p,q} = \left\{ \begin{array}{ll} \mathsf{HC}_p(C,d), & p,q \ge 0 \\ 0, & \text{otherwise} \end{array} \right\} \Longrightarrow \mathsf{HD}_*(C,d)$$

with differential $d^1 : E^1_{p,q} \to E^1_{p,q-1}$ induced by δ^3.

We may filter $\mathcal{D}$ in δ_2-direction by taking two new levels at a time. Note that 3-complex $\mathcal{D}^{n,*,*,*}$ is acyclic for n odd, while for n even the homology of its total complex is $H_*(\mathbf{Z}_2, C^h_*(C,d))$, (the hyperhomology of $\mathbf{Z}_2$ with coefficients $C^h_*(C,d)$, the Hochschild bicomplex of chain algebra (C,d)), for n even. We get a spectral sequence

$$(2.11) \qquad E^1_{p,q} = \left\{ \begin{array}{ll} H_{q-p}(\mathbf{Z}_2, C^h_*(C,d)), & \text{for } q \ge p \ge 0 \\ 0, & \text{otherwise} \end{array} \right\} \Longrightarrow \mathsf{HD}_*(C,d)$$

with differential $d : E^1_{p,q} \to E^1_{p-1,q}$ induced by δ_1.

Let us list, for the record, a few standard observations.

LEMMA 2.12. *A short exact sequence*

$$0 \longrightarrow (C',d) \longrightarrow (C,d) \longrightarrow (C'',d) \longrightarrow 0$$

of DCC induces a long exact sequence

$$\cdots \longrightarrow \mathsf{HD}_{n+1}(C'',d) \longrightarrow \mathsf{HD}_n(C',d) \longrightarrow \mathsf{HD}_n(C,d) \longrightarrow \mathsf{HD}_n(C'',d) \longrightarrow \cdots.$$

Proof. A short exact sequence of DCC induces a short exact sequence of 4-complexes and hence of their total complexes and finally we get a long exact sequence involving dihedral hyperhomology.

Note that a short exact sequence of CAI induces a short exact sequence of appropriate DCC and hence a long exact sequence in HD.

Definition 2.13. A map $(X,d) \to (Y,d)$ of dihedral chain complexes is and *equivalence* if for each $n \geq 0$ the chain map $(X_n, d) \to (Y_n, d)$ is a quasi-isomorphism i.e. induces isomorphisms in homology.

PROPOSITION 2.14. *An equivalence induces an isomorphisms of* HD_*.

Proof. Note that an equivalence between DCC is also an equivalence between cyclic chain complexes and hence it induces an isomorphisms of HC_*, [G, Prop. III.2.7]. Now use the spectral sequence (2.10).

Definition 2.15. A map $(A, d, *) \to (B, d, *)$ of CAI is an *equivalence* if it is an equivalence as a chain map.

PROPOSITION 2.16. *Any equivalence* $f : (A, d, *) \to (B, d, *)$ *of flat CAI induces isomorphisms of* HD_*.

The proof is similar to the one of Proposition 2.14. Here we use [G, Prop. III.2.9].

Relative HD_*. As in the cyclic case we may define relative dihedral hyperhomology of a map $f : (X,d) \to (Y,d)$ between DDC's by $\mathsf{HD}_*(f) = \mathsf{HD}_*(M^f)$, where M^f is a DCC with $M^f_n =$ algebraic mapping cone of the chain map $f_n : (X_n, d) \to (Y_n, d)$. The short exact sequence

$$0 \longrightarrow (X,d) \longrightarrow (Y',d) \longrightarrow (M^f,d) \longrightarrow 0$$

of DCC's induces a long exact sequence in HD_*

$$\cdots \longrightarrow \mathsf{HD}_n(X,d) \longrightarrow \mathsf{HD}_n(Y,d) \longrightarrow \mathsf{HD}_n(f) \longrightarrow \mathsf{HD}_{n-1}(X,d) \longrightarrow \cdots . \tag{2.17}$$

Periodic HD. As in Definition 1.3 we have three periodic dihedral hyperhomologies. To be more precise define

$${}_iD^{\mathrm{per}}_*(X,d) = \varprojlim_k \{\mathrm{Tot}_{*+kd_i}(\mathcal{D}, s_i)\} = \prod_{n+m+l+k=*} \mathcal{P}^{n,m,l,k} = \mathrm{Tot}_*({}_i\mathcal{P})$$

where the 4-complex ${}_i\mathcal{P}$ is obtained by extending the 4-complex $\mathcal{D}$ along the differential δ^1, for $i = 1$; along the differentials δ^1 and δ^3, for $i = 2$; and along δ^3, for $i = 3$.

Define the i-th periodic dihedral hyperhomology of DCC(X,d) by

$$\mathsf{HD}^{\mathrm{per}}_*(X,d) = H_*({}_iD^{\mathrm{per}}_*(X,d), \delta), \qquad i = 1,2,3. \tag{2.18}$$

Here δ is the induced differential from the appropriate 4-complex. Obviously they are periodic with periods 4, 3 and 2 respectively. If $(A, d, *)$ is a CAI we write ${}_i\mathsf{HD}_*^{\mathrm{per}}(A, d, *)$ for ${}_i\mathsf{HD}_*^{\mathrm{per}}(D(A, d, *))$.

LEMMA 2.19. *We have an exact sequence*

$$0 \longrightarrow \varprojlim_k{}^1 \mathsf{HD}_{*+4k+1}(X, d) \longrightarrow {}_1\mathsf{HD}_*^{\mathrm{per}}(X, d) \longrightarrow \varprojlim_k \mathsf{HD}_{*+4k}(X, d) \longrightarrow 0$$

where limits are taken with respect to s_1.

There are similar exact sequences for s_2 and s_3.

Proof. There is a short exact sequence of chain complexes

$$0 \longrightarrow \mathrm{Tot}_*({}_1\mathcal{P}) \longrightarrow \prod_{k \equiv * \pmod 4} \mathrm{Tot}_k(\mathcal{D}) \xrightarrow{1-s_1} \prod_{k \equiv * \pmod 4} \mathrm{Tot}_k(\mathcal{D}) \longrightarrow 0.$$

Note that the homology of the first complex is ${}_1\mathsf{HD}_*^{\mathrm{per}}(X, d)$ while the homology of the other two complexes is the product of $\mathsf{HD}_k(X, d)$, $k \equiv * \pmod 4$. Hence we have a short exact sequence

$$0 \longrightarrow \mathrm{coker}_{n+1}(1 - s_1) \longrightarrow {}_1\mathsf{HD}_n^{\mathrm{per}}(X, d) \longrightarrow \ker_n(1 - s_1) \longrightarrow 0$$

which proves the lemma. □

Note also that a short exact sequence of DCC induces long exact sequences involving ${}_1\mathsf{HD}_*^{\mathrm{per}}$.

Derivation and HD_*. *Definition* 2.20. A derivation of CAI is a map of CAI $D : (A, d, *) \to (A, d, *)$ of degree 0 such that $D(ab) = D(a)b + aD(b)$.

If D is a derivation of $(A, d, *)$ define $L_D : D(A, d, *) \to D(A, d, *)$ by $L_D(a_0, \dots, a_n) = \sum_{i=0}^{n}(a_0, \dots, Da_i, \dots, a_n)$. One checks that $L_D : D_n(A, d, *) \to D_n(A, d, *)$ is a map of CAI of degree 0 and that it is a map of DCC $D(A, d, *)$ to itself.

LEMMA 2.21. *With the setting as above we have*

$$L_D \circ s_1 = 0 : \mathsf{HD}_*(X, d) \to \mathsf{HD}_{*-4}(X, d), \qquad \textit{and}$$
$$L_D \circ s_2 = 0 : \mathsf{HD}_*(X, d) \to \mathsf{HD}_{*-3}(X, d).$$

Proof. L_D and s_1 induce maps from the 4-complex $\mathcal{D}(A, d, *)$ to itself of degrees $(0, 0, 0, 0)$ and $(-4, 0, 0, 0)$ respectively. They induce maps from the spectral sequence (2.10) to itself of degrees $(0, 0)$ and $(-4, 0)$ respectively. Now note that

$$s_1 = s^2 : E^1_{p,q} = \mathsf{HC}_p(X, d) \to E^1_{p-4,q} = \mathsf{HC}_{p-4}(X, d)$$

and use a result from [G, Corollary III.4.4] to get that $L_D \circ s_1$ is the zero map on the level of spectral sequences and therefore it is t he zero map of dihedral hyperhomology.

For s_2 the proof is similar. One has to note that

$$s_2 = s : E^1_{p,q} = \mathsf{HC}_p(X,d) \to E^1_{p-2,q-1} = \mathsf{HC}_{p-2}(X,d). \quad \square$$

Observe that Lemma 2.21 is not valid for s_3.

Now we have the following result. Since we made all the necessary preparations the proof of [G, Theorem III.5.1] works here as well.

THEOREM 2.22. *Let $(A, d, *)$ be a CAI over a field k of characteristic zero, and let $I \subset A$ be a CAI ideal (a graded ideal satisfying $dI \subset I$ and $I^* \subset I$) and assume $I_0 = 0$. Then the quotient map $(A, d, *) \to (A/I, d, *)$ induces the isomorphisms*

$$_i\,\mathsf{HD}^{\mathrm{per}}_*(A, d, *) \longrightarrow {}_i\,\mathsf{HD}^{\mathrm{per}}_*(A/I, d, *), \qquad \textit{for } i = 1, 2.$$

The Non-Flat Case. As in [G, IV.1] we may extend the definition of dihedral for non-flat CAI. If $(A, d, *)$ is a CAI then there is a flat CAI $(R_A, d, *)$ which is naturally equivalent to it. The same construction as in [G, IV.1.1] works here (the involution on R_A is obvious). Now define $\mathsf{HD}_*(A, d) := \mathsf{HD}_*(R_A, d)$.

THEOREM 2.23. *Any one-connected map $(A, d, *) \to (B, d, *)$ of chain algebras with involution over a field k of characteristic zero induces the isomorphisms $_i\,\mathsf{HD}^{\mathrm{per}}_*(A, d, *) \longrightarrow {}_i\,\mathsf{HD}^{\mathrm{per}}_*(D, d, *)$, for $i = 1, 2$.*

Proof. We need two lemmas to reduce to the case of the projection map $(A, d, *) \to (A/I, d, *)$.

LEMMA 2.24. *Any zero-connected map of CAI over a commutative ring k with $\mathbb{Q} \subseteq k$ can be factored as an equivalence composed by a surjection.*

Proof. Let $f : (A, d, *) \to (B, d, *)$ be a zero-connected chain map. Define $(C, d, *)$ to be the following chain algebra with involution:

$$C_n = A_n \oplus B_{n+1} \oplus B_n, \qquad \text{if } n > 0$$

$$C_0 = \{(a, x, y) \in A_0 \oplus B_1 \oplus B_0 \mid dy = z - f(a)\}$$

$$(a, y, z)(a', y', z') = \left(aa', \tfrac{i+j+1}{2}\left[\tfrac{f(a)+z}{j+1}y' + (-1)^j y \tfrac{f(a')+z'}{i+1}\right], zz'\right)$$
$$\text{for } (a, y, z) \in C_i,\ (a', y', z') \in C_j$$

$$d(a, y, z) = \left(da, (-1)^n n f(a) + \tfrac{n}{n+1} dy + (-1)^{n-1} nz, dz\right)$$
$$\text{for } (a, y, z) \in C_n$$

$$(a, y, z)^* = (a^*, y^*, z^*).$$

Straightforward calculations show that $(C, d, *)$ is a chain algebra with involution.

One easily checks that the following maps are maps between chain algebras with involution

$$\begin{aligned} h &: (A, d, *) \to (C, d, *), & h(a) &= (a, 0, f(a)) \\ g &: (C, d, *) \to (B, d, *), & g(a, y, z) &= z \\ l &: (C, d, *) \to (A, d, *), & l(a, y, z) &= a. \end{aligned}$$

It is also easy to see that: $f = g \circ h$, g is a surjection, h is an equivalence with homotopy inverse l. For the last assertion note that $l \circ h = 1$ and that

$$h_n : C_n \to C_{n+1}, \qquad h_n(a, y, z) = \left(0, 0, \tfrac{(-1)^n}{n+1} y\right)$$

is a chain homotopy between $h \circ l$ and 1.

We are not able to avoid the condition $\mathbb{Q} \subseteq k$ although it looks reasonable to expect that this assumption is not necessary.

LEMMA 2.25. *For any one-connected surjection $(A, d, *) \to (A/J, d, *)$ of chain algebras with involution there is a commutative diagram*

$$\begin{array}{ccc} (T, d, *) & \longrightarrow & (A, d, *) \\ \downarrow & & \downarrow \\ (T/I, d, *) & \longrightarrow & (A/J, d, *) \end{array}$$

of chain algebras with involution such that horizontal maps are equivalences, T is a tensor algebra, the chain ideal I is generated as an ideal by a subset of tensor basis, and $I_0 = 0$.

The proof is the same as of [G, Lemma IV.2.3]. The only difference is that one has to check that the obvious involution on the chain algebras involved gives the structure of CAI and that all maps are maps of CAI.

Now using Lemma 2.24 we reduce to the case of a surjection $(A, d, *) \to (A/I, d, *)$ and by Lemma 2.25 we assume that $(A, d, *)$ is free and that $I_0 = 0$. By Theorem 2.22 we are done.

COROLLARY 2.26. *Let a be a CAI. Then, ${}_i\mathrm{HD}_*^{\mathrm{per}}(A) = {}_i\mathrm{HD}_*^{\mathrm{per}}(H_0(A))$.*

Proof. We see H_0 as CAI with $H_0(A)$ on the 0^{th} level and 0 otherwise. It is easy to check that $H_0(A)$ is a CAI.

The projection $A \to H_0(A)$ is a map of CAI which is 1-connected, hence it induces an isomorphism ${}_i\mathrm{HD}_*^{\mathrm{per}}(A) = {}_i\mathrm{HD}_*^{\mathrm{per}}(H_0A)$, $* \geq 0$, $i = 1, 2$, by Theorem 2.23. Finally, let us note that ${}_i\mathrm{HD}_*^{\mathrm{per}}(H_0A) = {}_i\,\mathrm{HD}_*^{\mathrm{per}}(H_0A)$.

The same proof as of [G, Theorem IV.2.6] gives the following

THEOREM 2.26. *Let $f : (A, d, *) \to (B, d, *)$ be a one-connected map of chain algebras over a commutative ring k, with $\mathbb{Q} \subseteq k$. The map $\mathrm{HD}_{*+kd_i}(f) \xrightarrow{s_i^k} \mathrm{HD}_*(f)$ is zero for $* < k$ an[illegible] $1, 2$.*

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