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TIJDEMAN'S ITERATIVE ALGORITHM

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Abstract. R. Tijdeman introduced the following algorithm: Let $\theta > 2$, and define (a_n) inductively by $a_0 = 1$, $a_{n+1} = [\theta a_n]+1$, $(n \geq 0)$. Then the sequence (a_n/θ^n) converges to a limit x which has special properties relevant to "Mahler's $\frac{3}{2}$-problem". Here we study x as a function of θ in the interval $(1,\infty)$. We give some basic results concerning its behaviour at rational points, and we prove that it is discontinuous in a set of rationals which is dense in $(2,\infty)$. We also give a method for the evaluation of $x(\theta)$ when θ is in various families of algebraic numbers.

1. Introduction. Let $\theta > 2$ and define (a_n) inductively by

$$a_0 = 1, \qquad a_{n+1} = [\theta a_n] + 1, \quad n \geq 0, \tag{1.1}$$

and then set

$$x_n = x_n(\theta) = a_n/\theta_n, \qquad n \geq 0. \tag{1.2}$$

Tijdeman [1] proved that the sequence (x_n) converges to a limit x which has the remarkable property that

$$0 < \{x\theta^n\} \leq 1/(\theta - 1), \qquad n \geq 0; \tag{1.3}$$

here $\{z\}$ denotes the fractional part of z.

Actually the limit

$$x = x(\theta) = \lim_{n\to\infty} x_n(\theta) \tag{1.4}$$

still exists when $1 < \theta \leq 2$; but, of course, (1.3) now becomes a triviality. The purpose of this paper is to study x as a function of θ in the interval $(1,\infty)$. As we shall see, the function is always continuous from the right, and is continuous at each irrational number. The situation at a rational point is complicated, and we have not been able to prove that x is discontinuous at each rational number. Let D denote the set of discontinuities of x, so that D is a subset of the rationals in $(1,\infty)$. We shall prove

THEOREM 1. *The set D of discontinuities of x is dense in the interval $(2,\infty)$.*

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Although it is easy to see that

$$x(k) = k/(k-1), \qquad k = 2, 3, 4, \ldots .$$

the determination of $x(\theta)$ for any rational non-integer θ such as $3/2$ seems to be related to the "3/2-problem" of Mahler which was the inspiration of Tijdeman's paper [1]. We shall say more on this in Remark 2 of Section 5. On the other hand we can determine $x(\theta)$ for various families of algebraic numbers θ, the most interesting being

$$\theta = \lambda_k = \left(k + \sqrt{k^2 - 4}\right)/2, \qquad k = 3, 4, 5, \ldots . \tag{1.5}$$

Since λ_k is irrational, we know that x is continuous there. Furthermore, as we shall see in our proof of Theorem 1, the function x is "continuous when decreasing" in the sense that

$$\lim_{\theta \to \alpha_-} x(\theta) \le x(\alpha) = \lim_{\theta \to \alpha_+} x(\theta), \qquad \alpha > 1. \tag{1.6}$$

It is therefore rather surprising to have

THEOREM 2. *Each λ_k in (1.5) is a strong minimum of the function x, and*

$$x(\lambda_k) = \lambda_k^2/(\lambda_k^2 - 1), \qquad k = 3, 4, 5, \ldots . \tag{1.7}$$

The method used in the evaluation of $x(\lambda_k)$ can be applied to various families of algebraic numbers. We shall say more about the method after we prove:

THEOREM 3. *Let $h \ge 1$, and let λ be the root of $z^{h+1} - z^h - \cdots - z - 1 = 0$ satisfying $1 < \lambda < 2$. Then*

$$x(\lambda) = \frac{1}{h} \cdot \frac{(h+1)\lambda^h + h\lambda^{h-1} + \cdots + \lambda + 1}{(h+1)\lambda^h - h\lambda^{h-1} - \cdots - \lambda - 1}.$$

Figures 1, 2 and 3 are graphs of $x_n(\theta)$ for $1 < \theta < 4$ and $1 \le n \le 50$. In accordance with Theorem 2 it will be observed that there is rather strange behaviour in $x_n(\theta)$ in the neighbourhood of $1 + g = g^2$ where $g = (1 + \sqrt{5})/2$ is the *golden ratio.* It was our initial calculations for $a_n(\theta)$ in the neighbourhood of g^2 led us to the formula

$$a_n(\theta) = f_{2n+1}, \qquad \text{for} \quad \frac{f_{2n+1} - 1}{f_{2n-1}} \le \theta < \frac{f_{2n+1}}{f_{2n-1}},$$

where f_n is the n-th *Fibonacci number.* It follows easily from $f_n \sim g^{n+1}/\sqrt{5}$, as $n \to \infty$ that $\sqrt{5}x(g^2) = g^2$. The proof that g^2 is a minimum depends heavily on the identity $f_{2n+1}^2 - f_{2n+3}f_{2n-1} = 1$. Theorem 2 is a generalization of this discovery and Theorem 3 is an extension of the method used.

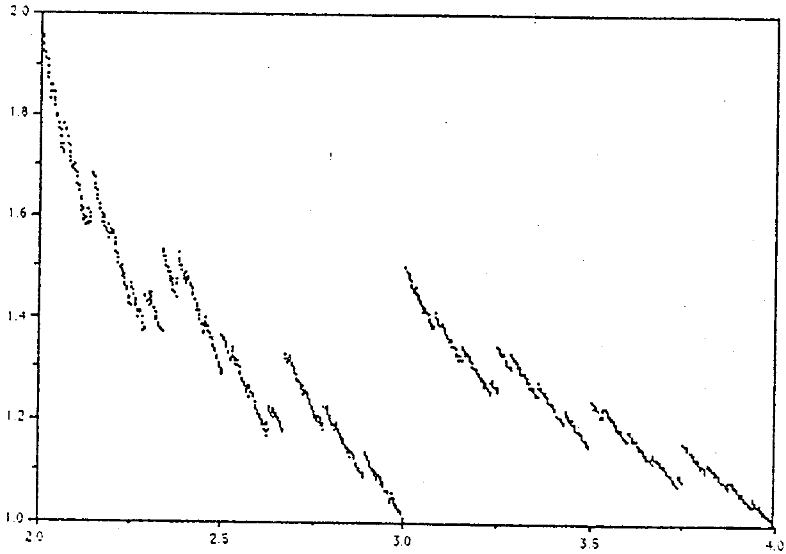


Figure 1. Graph of $x_{17}(\theta)$ with $2 < \theta < 4$.

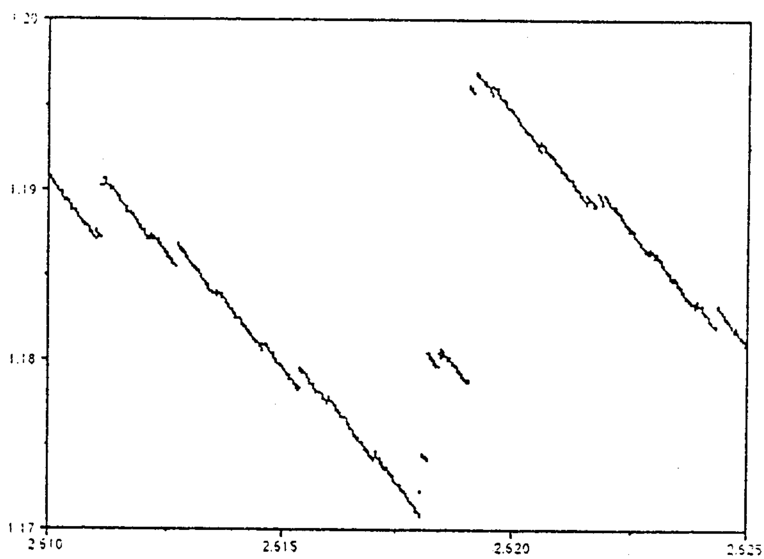


Figure 2. Graph of $x_{24}(\theta)$ with $2.61 < \theta < 2.625$.

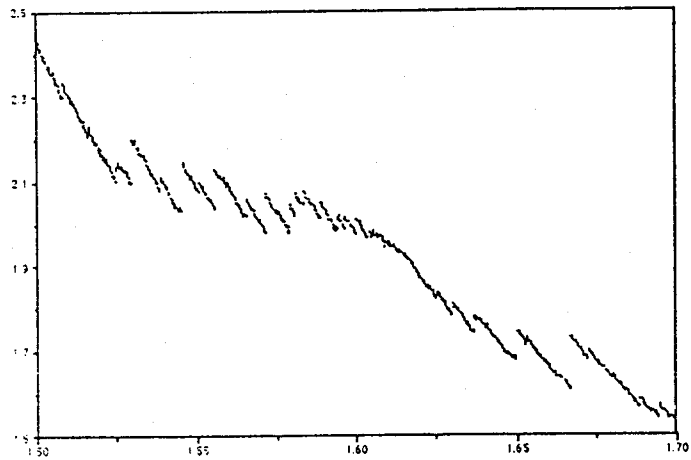


Figure 3. Graph of $x_{44}(\theta)$ with $1.5 < \theta < 1.7$.

2. Proof of Theorem 1. That (x_n) converges to x uniformly in $[1+\delta, \infty)$ for any $\delta > 0$ is implicit in Tijdeman's paper [1]. As we shall see, the uniformity aspect is crucial to our consideration. Since Tijdeman only dealt with the case $\theta > 2$, for completeness sake we repeat his simple and elegant argument. From $\theta a_n < [\theta a_n] + 1 = a_{n+1}$ and the definition of x_n in (1.2), it follows that $x_n < x_{n+1}$. Next, for $0 \le m \le n$, we have

$$\theta^m (x_n - x_m) = \theta^m \sum_{m<j\le n} (x_j - x_{j-1}) = \theta^m \sum_{m<j\le n} \theta^{-j}(a_j - \theta a_{j-1}),$$

and since $a_j - \theta a_{j-1} = [\theta a_{j-1}] + 1 - \theta a_{j-1} \le 1$ it follows that

$$\theta^m (x_n - x_m) \le \sum_{m<j\le n} \theta^{m-j} < \sum_{j>0} \theta^{-j} = \frac{1}{\theta - 1}.$$

Therefore, for $0 \le m \le n$ and $\theta > 1$,

$$x_m(\theta) \le x_n(\theta) \le x_m(\theta) + 1/\theta^m(\theta - 1).$$

That $x(\theta) = \lim_{n\to\infty} x_n(\theta)$ exists now follows at once. Indeed, we have

$$x_m(\theta) < x(\theta) \le x_m(\theta) + 1/\theta^m(\theta - 1) \tag{2.1}$$

for all $m \ge 0$ and $\theta > 1$. In particular, if $\theta \ge 1 + \delta$, then $0 < x(\theta) - x_m(\theta) \le 1/(1+\delta)^m \delta$, so that (x_m) converges to x uniformly in $[1+\delta, \infty)$.

We can now prove (1.6). The integer part function satisfies

$$\lim_{\theta\to\alpha_-}[\theta z] \le [\alpha z] = \lim_{\theta\to\alpha_+}[\theta z]; \tag{2.2}$$

indeed we can specify the left-hand side here by

$$\lim_{\theta\to\alpha_-}[\theta z] = \begin{cases} \alpha z - 1, & \text{if } \alpha z \in \mathbf{Z}, \\ [\alpha z], & \text{otherwise.} \end{cases} \tag{2.3}$$

It is clear that, for every fixed n,

$$\lim_{\theta\to\alpha_-} a_n(\theta) \le a_n(\alpha) = \lim_{\theta\to\alpha_+} a_n(\theta),$$

so that

$$\lim_{\theta\to\alpha_-} x_n(\theta) \le x_n(\alpha) = \lim_{\theta\to\alpha_+} x_n(\theta).$$

Now let $n \to \infty$, and the required result (1.6) follows from the uniform convergence of (x_n) to x.

This shows that the function x is always continuous from the right, and that it is also continuous at each irrational α. The discussion on whether x is continuous at a rational $\alpha = p/q$ is deeply involved with the numbers

$$j_n = j_n(\alpha) = a_n(\alpha) - \lim_{\theta\to\alpha_-} a_n(\theta), \qquad n \ge 0. \tag{2.4}$$

We shall require the following

LEMMA 1. *Let $\alpha = p/q > 1$, and let j_n be defined by* (2.4). *Then $j_{n+1} \ge j_n$. Furthermore, let r_n be defined by*

$$r_n \equiv pa_n(\alpha) \pmod q, \qquad 0 \le r_n < q;$$

then $j_{n+1} > j_n$ if either $r_n < (p-q)j_n$ or $j_n \equiv a_n(\alpha) \pmod q$.

Proof. Write $a_n = a_n(\alpha)$ and let θ be smaller than α but sufficiently close to α; for example close enough to ensure that the function a_{n+1} is continuous in the interval (θ, α). Then by the definition of j_n in (2.4) we have $a_n(\theta) = a_n - j_n$ and so $a_{n+1}(\theta) = [\theta(a_n - j_n)] + 1$. It now follows from (2.3) that

$$\lim_{\theta\to\alpha_-} a_{n+1}(\theta) = [\alpha(a_n - j_n)] - e_n + 1,$$

where $e_n = 1$ or 0 according to whether $j_n \equiv a_n(\alpha) \pmod q$ or not. But $a_{n+1}(\alpha) = [\alpha a_n] + 1$ so that $j_{n+1} = [\alpha a_n] - [\alpha(a_n - j_n)] + e_n$. Next, by the definition of r_n, we may write $pa_n = dq + r_n$, that is $\alpha a_n = d + r_n/q$ and so

$$\alpha(a_n - j_n) = d + (r_n - pj_n)/q = d - j_n + \big(r_n - (p-q)j_n\big)/q$$

which then gives

$$j_{n+1} = j_n - \big[\big(r_n - (p-q)j_n\big)/q\big] + e_n.$$

We can now use this iterative formula for j_n to prove the lemma. Consider the case $j_n = 0$ first. Then clearly $j_{n+1} \geq 0 = j_n$. The condition $r_n < (p-q)j_n$ cannot be satisfied, and, if $e_n = 1$ then $j_{n+1} = 0 - [r_n/q] + 1 = 1 > j_n$. Consider next the case $j_n \geq 1$. Since $r_n < q$ we now have

$$j_{n+1} \geq j_n - \left[(q-(p-q))/q\right] + e_n = j_n - [2-\alpha] + e_n.$$

But $2 - \alpha < 1$ so that we always have $j_{n+1} \geq j_n$, and with strict inequality when either $\alpha > 2$ or $e_n = 1$. Finally if $r_n < (p-q)j_n$, then we simply argue that

$$j_{n+1} \geq j_n - \left(r_n - (p-q)j_n\right)/q > j_n. \quad \square$$

LEMMA 2. *Let $\alpha = p/q > 1$. Suppose that there exists n such that*

$$j_n(\alpha) \geq 1/(\alpha - 1) = q/(p-q). \tag{2.5}$$

Then $\alpha \in D$. In particular, if $\alpha \geq 2$ and if there exists a positive $j_n(\alpha)$ then $\alpha \in D$.

Proof. From (2.1) and (1.2) we have

$$x(\theta) \leq x_n(\theta) + \frac{1}{\theta^n(\theta-1)} = \frac{1}{\theta^n}\left(a_n(\theta) + \frac{1}{\theta-1}\right).$$

It now follows from (2.4) and (2.5) that

$$\lim_{\theta\to\alpha_-} x(\theta) \leq \frac{1}{\alpha^n}\left(a_n(\alpha) - j_n(\alpha) + \frac{1}{\alpha-1}\right) \leq \frac{a_n(\alpha)}{\alpha^n} = x_n(\alpha);$$

but $x_n(\alpha) < x(\alpha)$ by (2.1), so that x is discontinuous at α. $\square$

We can now deduce Theorem 1. Let $2 < \alpha < \beta < \infty$, and suppose, if possible, that, for every $n \geq 0$, the function $a_n(\theta)$ is constant in $\alpha < \theta < \beta$. We then have

$$\frac{x_n(\alpha)}{x_n(\beta)} = \frac{a_n(\alpha)}{\alpha^n} \cdot \frac{\beta^n}{a_n(\beta)} = \left(\frac{\beta}{\alpha}\right)^n \to \infty \qquad \text{as } n \to \infty,$$

contradicting the existence of $x(\alpha)$ and $x(\beta)$. Therefore, for some n, the function $a_n(\theta)$ has a discontinuity in $[\alpha, \beta]$. By Lemma 2, the function x has a discontinuity at $\theta \in [\alpha, \beta]$ so that the theorem is proved. $\square$

3. Proof of Theorem 2. Let $k \geq 3$, and define (b_n) inductively by

$$b_0 = 1, \qquad b_1 = k, \qquad b_{n+2} = kb_{n+1} - b_n. \tag{3.1}$$

It follows easily by induction that b_n satisfies

$$b_n^2 - b_{n+1}b_{n-1} = 1; \tag{3.2}$$

indeed, on solving the difference equation (3.1) we find that

$$b_n = \frac{\lambda^{n+2} - \lambda^{-n}}{\lambda^2 - 1}, \qquad n \geq 0 \tag{3.3}$$

where $\lambda = \lambda_k$ is given by (1.5). Write

$$B_n = \left[\frac{b_n - 1}{b_{n-1}}, \frac{b_n}{b_{n-1}}\right), \qquad n \geq 1; \tag{3.4}$$

it then follows from (3.2) that (B_n) is a nested sequence of intervals whose intersection contains exactly one point. Moreover, by (3.3), this point is given by $\lim_{n\to\infty} b_n/b_{n-1} = \lambda$. The determination of $x(\lambda)$ is based on the following lemma.

LEMMA 3. *For $n \geq 1$, we have*

$$a_n(\theta) = b_n, \qquad \textit{for} \quad \theta \in B_n. \tag{3.5}$$

Furthermore, if θ satisfies

$$\theta > \lambda, \qquad \theta \in B_n, \qquad \theta \notin B_{n+1} \tag{3.6}$$

then

$$a_{n+1}(\theta) = b_{n+1} + 1. \tag{3.7}$$

Proof. We prove (3.5) by induction on n, the case $n = 1$ being trivial. Suppose then that (3.5) holds, and let $\theta \in B_{n+1}$. Then, since $\theta \in B_{n+1} \subset B_n$, we have $a_n(\theta) = b_n$ so that, by (1.1), $a_{n+1} = [\theta b_n] + 1$. Now, by the definition of B_{n+1} in (3.4), the condition $\theta \in B_{n+1}$ gives $b_{n+1} - 1 \leq \theta b_n < b_{n+1}$ which gives $a_{n+1} = (b_{n+1} - 1) + 1 = b_{n+1}$ and so completes the inductive step for (3.5).

Suppose now that θ satisfies (3.6). Then

$$b_{n+1}/b_n \leq \theta < b_n/b_{n-1}$$

so that, on applying (3.2),

$$b_{n+1} \leq \theta b_n < \frac{b_n^2}{b_{n-1}} = \frac{b_{n-1}b_{n+1} + 1}{b_{n-1}} \leq b_{n+1} + 1. \tag{3.8}$$

Therefore, by (3.5),

$$a_{n+1}(\theta) = [\theta a_n(\theta)] + 1 = [\theta b_n] + 1 = b_{n+1} + 1.$$

The lemma is proved. □

We can now establish (1.7). Since $\lambda \in B_n$ for all $n \geq 1$ we have $a_n(\lambda) = b_n$, and so $x_n = b_n/\lambda^n$. The required result now follows from (3.3).

Finally we show that there exists $\delta = \delta(\lambda) > 0$ such that $x(\theta) > x(\lambda)$ whenever $0 < |\theta - \lambda| < \delta$. According to (2.1) it suffices to find m, depending on θ, such that

$$x_m(\theta) > x(\lambda). \tag{3.9}$$

We shall specify δ later. Consider first the interval $\lambda - \delta < \theta < \lambda$. We choose m so that $\theta \in B_m$, $\theta \notin B_{m+1}$. Then, by (3.5) and (3.4), we have $a_m(\theta) = b_m$ and $\theta < (b_{m+1} - 1)/b_m$ so that

$$x_m(\theta) = \frac{a_m(\theta)}{\theta^m} > b_m \left(\frac{b_m}{b_{m+1} - 1}\right)^m; \tag{3.10}$$

we shall see later that this exceeds $x(\lambda)$.

Similarly, when $\lambda < \theta < \lambda + \delta$ we choose m so that $\theta \in B_{m-1}$ and $\theta \notin B_m$. Then, by (3.7) and (3.4), we have $a_m(\theta) = b_m + 1$, and $\theta < b_{m-1}/b_{m-2}$ so that

$$x_m(\theta) = \frac{b_m + 1}{\theta^m} > (b_m + 1)\left(\frac{b_{m-2}}{b_{m-1}}\right)^m. \tag{3.11}$$

Let us write U_m and V_m for the right hand sides of (3.10) and (3.11) respectively. In order to establish (3.9) it remains to show that U_m and V_m exceed $x(\lambda)$ which is given in (1.7). But it follows from (3.3) and the definitions of U_m and V_m that $\lim_{m\to\infty} U_m = \lim_{m\to\infty} V_m = x(\lambda)$. Therefore it suffices to show that (U_m) and (V_m) are decreasing sequences. We deal with the more delicate case of V_m and consider

$$\frac{V_{m+1}}{V_m} = \frac{b_{m+1}+1}{b_m+1} \cdot \frac{b_{m-1}}{b_m} \cdot \left(\frac{b_{m-1}^2}{b_m b_{m-2}}\right)^m = \frac{b_m^2 - 1 + b_{m-1}}{b_m(b_m+1)} \cdot \left(\frac{b_{m-1}^2}{b_{m-1}^2 - 1}\right)^m$$

using the identity (3.2). The first factor here is less than

$$\frac{b_m + 1/2}{b_m + 1} = 1 - \frac{1}{2b_m} + O\left(\frac{1}{b_m^2}\right)$$

whereas the second factor is

$$\left(1 + \frac{1}{b_{m-1}^2 - 1}\right)^m = 1 + O\left(\frac{m}{b_m^2}\right).$$

Therefore

$$\frac{V_{m+1}}{V_m} < 1 - \frac{1}{2b_m} + O\left(\frac{m}{b_m^2}\right) < 1$$

when m is large. Returning to the specification of δ, it is clear that m has to be large when δ is chosen to be small. The theorem is proved. □

4. Proof of Theorem 3.

We shall require the following:

LEMMA 4. *Let $h \geq 1$ and define (b_n) by*

$$\begin{aligned} b_0 = 1, \qquad b_n &= 0 \qquad \text{if } n < 0, \\ b_{n+1} = b_n + b_{n-1} + \cdots + b_{n-h} &+ 1 \qquad \text{if } n \geq 0. \end{aligned} \tag{4.1}$$

Then

$$0 \leq b_n^2 - b_{n+1}b_{n-1} \leq b_n - b_{n-1} \qquad \text{if } n \geq 1. \tag{4.2}$$

Proof. Let the two required inequalities be denoted by $P(n)$ and $Q(n)$ which we rewrite as

$$P(N): \quad \frac{b_{n+1}}{b_n} \leq \frac{b_n}{b_{n-1}}, \qquad Q(N): \quad \frac{b_n}{b_{n-1}} \leq \frac{b_{n+1} - 1}{b_n - 1}.$$

The cases $P(n)$, $Q(n)$ with $n < h$ can be verified directly. We now apply an inductive argument in the following manner:

$$\text{(i)} \qquad P(n-2), \dots, P(n-h-1) \Rightarrow P(n),$$
$$\text{(ii)} \qquad P(n-1), \dots, P(n-h) \Rightarrow Q(n).$$

We remark that the *mediant* of a collection of fractions lies between the minimum and the maximum of the fractions. For (i) we consider

$$\frac{1}{1} \le \frac{b_{n-1}}{b_{n-2}} \le \frac{b_{n-2}}{b_{n-3}} \le \cdots \le \frac{b_{n-h}}{b_{n-h-1}} \le \frac{b_{n-h-1}}{b_{n-h-2}}.$$

We then have

$$\frac{1 + b_{n-1} + \cdots + b_{n-h}}{1 + b_{n-2} + \cdots + b_{n-h-1}} \le \frac{1 + b_{n-1} + \cdots + b_{n-h-1}}{1 + b_{n-2} + \cdots + b_{n-h-2}} = \frac{b_n}{b_{n-1}}$$

so that

$$\frac{1 + b_{n-1} + \cdots + b_{n-h} + b_n}{1 + b_{n-2} + \cdots + b_{n-h-1} + b_{n-1}} \le \frac{b_n}{b_{n-1}};$$

which is just $P(n)$. Similarly, for (ii), we consider

$$\frac{b_n}{b_{n-1}} \le \frac{b_{n-1}}{b_{n-2}} \le \cdots \le \frac{b_{n-h+1}}{b_{n-h}} \le \frac{b_{n-h}}{b_{n-h-1}}$$

which gives

$$\frac{b_n}{b_{n-1}} \le \frac{b_n + b_{n-1} + \cdots + b_{n-h}}{b_{n-1} + b_{n-2} + \cdots + b_{n-h-1}}$$

which is just $Q(n)$. The lemma is proved. □

If we write

$$B_n = \left[\frac{b_n - 1}{b_{n-1}}, \frac{b_n}{b_{n-1}}\right)$$

then $B_{n+1} \subset B_n$, and we shall see later that the intersection of B_n is precisely the point λ in the theorem. If we now apply (4.2) in Lemma 4 (instead of the identity (3.2)) into (3.8) in the proof of Lemma 3, then we again have $a_n(\theta) = b_n$ for $\theta \in B_n$, $n \ge 1$. In particular, we have

$$x(\lambda) = \lim_{n\to\infty} a_n(\lambda)/\lambda^n = \lim_{n\to\infty} b_n/\lambda^n. \tag{4.3}$$

Let us write

$$f(z) = z^{h+1} - z^h - \cdots - z - 1,$$

so that $f(z) = 0$ is precisely the auxiliary equation associated with the difference equation (4.1), and that λ in the theorem is the real zero of $f(z)$ in the interval $(1, 2)$. We shall require the result that the other zeros λ_j of $f(z)$ are distinct and satisfy $|\lambda_j| < 1$, $1 \le j \le h$. This can be establised by applying Rouché's theorem to show that the two functions $(z-1)f(z) = z^{h+2} - 2z^{h+1} + 1$ and $2z^{h+1} - 1$ have the same number of zeros, namely $h + 1$ in the region $|z| < 1 + \epsilon$ for every ϵ satisfying

$0 < \epsilon < 1/2$. That $|\lambda_j| < 1$, $1 \le j \le h$ now follows easily; indeed consideration of the derivative of $(z-1)f(z)$ shows that these zeros are all simple.

Next, we write $\lambda_0 = \lambda$ and let

$$\Lambda = \begin{pmatrix} 1 & \lambda_0 & \cdots & \lambda_0^h \\ 1 & \lambda_1 & \cdots & \lambda_1^h \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_h & \cdots & \lambda_h^h \end{pmatrix}.$$

Then, on solving the difference equation (4.1) in Lemma 4, we have

$$b_n = A_0\lambda_0^n + \cdots + A_h\lambda_h^n - 1/h$$

where

$$(A_0, \ldots, A_h)\,\Lambda = (b_0 + 1/h, \ldots, b_h + 1/h)\,.$$

Since $|\lambda_j| < 1 < \lambda_0 = \lambda$ for $1 \le j \le h$, we now have $\lim_{n\to\infty} b_n/b_{n-1} = \lambda_0 = \lambda$. This not only shows that $\bigcap B_n = \{\lambda\}$ but also allows us to use (4.3) to write

$$x(\lambda) = A_0 = |\Lambda_0| \,/\, |\Lambda| \tag{4.4}$$

where Λ_0 is the matrix obtained from Λ with the first row $(1, \lambda_0, \ldots, \lambda^h)$ replaced by $(b_0 + 1/h, \ldots, b_h + 1/h)$. The Vandermond determinant $|\Lambda|$ and $|\Lambda_0|$ have the expansions

$$|\Lambda| = \prod_{0\le j<k\le h} (\lambda_j - \lambda_k) = \prod_{1\le j\le h} (\lambda_0 - \lambda_j) \times \prod_{1\le j<k\le h} (\lambda_j - \lambda_k)$$

$$|\Lambda_0| = \sum_{j+k=h} (-1)^j \sigma_j \left(b_k + \frac{1}{h}\right) \times \prod_{1\le j<k\le h} (\lambda_j - \lambda_k)$$

where σ_j is the elementary symmetric polynomial in $\lambda_1, \ldots, \lambda_h$ with degree j; that is

$$(z - \lambda_1)\cdots(z - \lambda_h) = \sigma_0 z^h - \sigma_1 z^{h-1} + \cdots + (-1)^h \sigma_h.$$

But the left hand side here can also be expressed as

$$\frac{f(z)}{z-\lambda} = \frac{z^{h+1} - z^h - \cdots - z - 1}{z - \lambda} = \tau_0 z^h + \tau_1 z^{h-1} + \cdots + \tau_h$$

where $\tau_0 = 1$, $\tau_j = \lambda\tau_{j-1} - 1$. Therefore $(-1)^j\sigma_j = \tau_j$ and so (4.4) now gives

$$h \prod_{1\le j\le h} (\lambda - \lambda_j)\, x(\lambda) = \sum_{j+k=h} (hb_k + 1)\,\tau_j. \tag{4.5}$$

But we also have

$$\prod_{1\le j\le h} (\lambda - \lambda_j) = f'(\lambda) = (h+1)\lambda^h - h\lambda^{h-1} - \cdots - 2\lambda - 1;$$

moreover, since $\tau_j = \lambda^j - \lambda^{j-1} - \cdots - \lambda - 1$, we find that the coefficient of λ^j in the polynomial $\sum_{j+k=h} (hb_k + 1) \tau_j$ is given by (with $k = h - j$)

$$-(hb_0+1)-\cdots-(hb_{k-1}+1)+(hb_k+1) = h(b_k - \cdots - b_0)-k+1 = h-k+1 = j+1;$$

in other words the polynomial concerned is simply

$$\sum_{j=0}^{h}(j+1)\lambda^h = 2(h+1)\lambda^h - f'(\lambda).$$

Therefore (4.5) now becomes $hf'(\lambda)x(\lambda) = 2(h+1)\lambda^h - f'(\lambda)$, so that

$$x(\lambda) = \frac{2(h+1)\lambda^h - f'(\lambda)}{hf'(\lambda)}$$

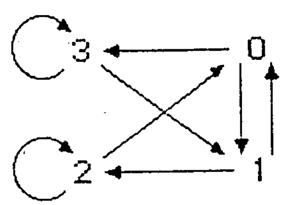
and the required result follows from this. □

5. Remarks. 1. Lemma 2 in the proof of Theorem 1 can be used to prove that $\alpha \in D$ for some special classes of rational α. For example, it can be proved by induction that if $\alpha = k + 1/q$ then $j_n = [a_n/q]$ for $n > 0$ so that x is discontinuous at such rationals. However, for a general rational α we are unable to establish that j_n in (2.4) must be eventually positive; in other words we cannot prove that q must divide $a_n(p/q)$ for some n. Nevertheless we conjecture that the hypothesis for Lemma 1 will always be satisfied by at least $q/(p-q)$ values of n; thereafter the hypothesis is satisfied trivially because $r_n < q$. This conjecture implies that every rational $\alpha > 1$ is a discontinuity of x.

2. The following description of the behaviour of $a_n \pmod 4$ when $\theta = 3/2$ may indicate the difficulty in the determination of $x(3/2)$. The sequence (a_n) is now given by

$$a_0 = 1, \qquad a_{n+1} = a_n + [a_n/2] + 1, \quad n \geq 0.$$

For $a_n \equiv 0, 1, 2, 3 \pmod 4$ we have $a_{n+1} \equiv 1, 0, 0, 1 \pmod 2$ respectively. The value of $a_n \pmod 4$ therefore has the following 'transition diagram':



It is conceivable that $a_n \pmod 4$ may cycle through any given path for arbitrarily long runs, but it must exit from the periodic path eventually. For example, it may happen that $a_n \equiv 2 \pmod 4$ for a long run of integers n, but eventually we must have $a_{n+K-1} \equiv 0 \pmod 4$, $a_{n+K} \equiv 1 \pmod 2$.

In order to demonstrate this, let A_k denote a number A satisfying $2^k | A$ but $2^{k+1} \not| A$. Suppose now that $a_n \equiv 2 \pmod 4$. Then there exists $K \geq 2$ such that $a_n = A_K - 2$, and so

$$a_{n+1} = (A_K - 2) + (A_K - 2)/2 + 1 = 3A_K/2 - 2 = A_{K-1} - 2.$$

Proceeding inductively, we find that $a_{n+k} = A_{K-k} - 2$, $0 \leq k \leq K$. Therefore $a_{n+k} \equiv 2 \pmod 4$ for $0 \leq k \leq K-2$, but

$$a_{n+K-1} = A_1 - 2 \equiv 0 \pmod 4, \qquad a_{n+K} = A_0 - 2 \equiv 1 \pmod 2.$$

A similar but more complicated argument ought to show that there cannot be any eventual periodicity for $a_n \pmod 4$. However, a proof or disproof of this would still be of no help in the determination of $x(3/2)$.

3. The method used in the proof of Theorem 2 also allows us to show that

$$x(\mu_k) = \frac{\mu_k^3}{(\mu_k^2 + 1)(\mu_k - 1)}, \quad \text{for } \mu_k = \frac{1}{2}\left(k + \sqrt{k^2+4}\right), \qquad k = 1, 2, \ldots .$$

It should be possible to identify some other families of algebraic numbers θ using linear recurrence relations more general than (4.1) thereby generalizing Theorem 3. More specifically, let $h, k \geq 1$, $b_0 = 1$, $b_1 = k+1$ and define (b_n) inductively by $b_{n+1} = (k+1)b_n - b_{n-h}$, with $b_n < 0$ for $n < 0$. The auxiliary equation associated with this difference equation has a root $\lambda_{k,h}$ in the interval $[k, k+1)$, and $\lambda_{k,1}$ is precisely λ_k of Theorem 2. Similarly, let $\mu_{k,h}$ and $\nu_{k,h}$ be the roots in $[k, k+1)$ of the auxiliary equations associated with the difference equations

$$b_{n+1} = kb_n + b_{n-h} + 1, \qquad \text{with } h \leq k,$$

and

$$\begin{aligned} b_{n+1} &= k(b_n + b_{n-1} + \cdots + b_{n-h}) + 1 \\ &= (k+1)b_n - kb_{n-h-1} \end{aligned}$$

respectively. One should then be able to evaluate x at $\mu_{k,h}$ and $\nu_{k,h}$, and in fact $\nu_{k,1}$ is precisely λ in Theorem 3.

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