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ON ORTHOGONAL SERIES SOLUTION FOR BOUNDARY LAYER PROBLEMS

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Abstract. The paper is concerned with linear boundary layer problems. It gives the solution as a sum of the reduced solution and the layer function which is approximated by the truncated orthogonal series. The domain of the layer function is obtained by determining, the so called, numerical layer length which depends on the small parameter ε in the differential equation and on the chosen degree of spectral approximation. The error estimate and the numerical example are given at the end.

1. Introduction. This paper will be concerned with the two point boundary value problem

$$L_\varepsilon y \equiv -\varepsilon^2 y''(x) + \alpha(x)y'(x) + \beta(x)y(x) = \gamma(x), \qquad x \in [0,1] \tag{1.1}$$

$$Gy \equiv (y(0), y(1)) = (A, B). \tag{1.2}$$

where ε is a small parameter and $\alpha(x), \beta(x), \gamma(x) \in C[0,1]$. It is well known that the solution of this kind of problem has a boundary layer at one of the endpoints. Without loss of generality we shall examine the case where $x = 1$ is the layer point. The solution of problem (1.1), (1.2) describes the stationary state of the evolution equation

$$\begin{aligned} &y_t - \varepsilon^2 y_{xx} + \alpha(x)y_x + \beta(x)y = \gamma(x), \qquad x \in [0,1],\ t > 0 \\ &y(0,t) = A, \quad y(1,t) = B, \qquad t > 0 \\ &y(x,0) = y_0(x), \qquad x \in [0,1], \end{aligned} \tag{1.3}$$

which arises among the others in the convective-diffusion type flow problems. In [6] we can find the sufficient conditions under which the solution of (1.1), (1.2) represents a stable state of (1.3). That is given by the following theorem:

THEOREM 1. *Let the conditions of one of the following cases hold for all* $x \in [0,1]$, *where* $\alpha_0, \beta_0, \gamma_0 \in \mathbf{R}$:

$$1^\circ \quad \alpha(x) \geq \alpha_0 > 0, \quad \beta(x) \geq \beta_0, \quad \alpha_0^2 + 4\varepsilon^2\beta_0 > 0.$$

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$$2^\circ \quad \alpha(x) \ge \alpha_0 > 0, \quad \beta(x) - \alpha'(x) \ge \gamma_0, \quad \alpha_0^2 + 4\varepsilon^2\gamma_0 > 0.$$

Then the problem (1.1), (1.2) *is inverse monotone, which ensures that it has a unique solution* $y(x) \in C^2[0,1]$ *which is stable.*

For the proof see [6, Th. 1]. In the further investigation in this paper we shall assume that condition 1° or 2° is fullfiled.

The aim of this paper is to construct the approximate solution for the problem (1.1), (1.2) using spectral methods and to estimate the error.

2. Transformation of the problem. Let us first examine the reduced problem for (1.1), (1.2):

$$L_r y_r \equiv \alpha(x)y_r'(x) + \beta(x)y_r(x) = \gamma(x), \qquad x \in [0,1]$$
$$y_r(0) = A.$$

It has a unique solution $y_r(x)$ which displays the boundary layer at $x = 1$. So, we are going to ask for the solution of (1.1), (1.2) in the form

$$y(x) = y_r(x) + u(x), \quad \text{where} \quad u(x) = \begin{cases} 0 & x \in [0, 1-\delta] \\ v(x) & x \in [1-\delta, 1] \end{cases} \tag{2.1}$$

is the layer function which satisfies

$$L_\varepsilon v \equiv -\varepsilon^2 v''(x) + \alpha(x)v'(x) + \beta(x)v(x) = \varepsilon^2 y_r''(x), \qquad x \in [1-\delta, 1] \tag{2.2}$$
$$G_0 v \equiv (v(1-\delta), v(1)) = (0, B_1), \qquad B_1 = B - y_r(1). \tag{2.3}$$

Here, $\delta > 0$ is, so called, numerical layer length, which is going to be determined latter.

Let us, now, construct the spectral approximation for the layer function $v(x)$ in the form of truncated series according to some orthogonal basis $\{Q_k,\ k = 0, \dots, n\}$, of the space $\mathcal{P}_n$ of all real polynomials of degree up to n. First we recall some properties of such a system.

The set of polynomials $\{Q_k(x)\}$ represents the classical orthogonal polynomials upon the interval $[-1,1]$ with respect to the weight function

$$p(x) = (1-x)^m(x+1)^n, \qquad m = B(1)/2, \ n = -B(-1)/2,$$

where $B(x)$ is the coefficient in the differential equation which determines $Q_k(x)$:

$$A(x)Q_k''(x) + B(x)Q_k'(x) + \lambda_k Q_k(x) = 0,$$
$$A(x) = 1 - x^2, \quad \lambda_k = -k\Big(\frac{k-1}{2}A''(0) + B'(0)\Big), \quad B(x) = ax + b, \qquad a, b \in \mathbf{R}.$$

It is well known that all classical orthogonal polynomials satisfy Bonnet's recurrent relation

$$Q_{k+1}(x) - (\alpha_k x + \beta_k)Q_k(x) + \gamma_k Q_{k-1}(x) = 0, \tag{2.4}$$

where $\alpha_k, \beta_k, \gamma_k$ are constants depending on the chosen basis, and the derivative equation

$$A(x)Q_k'(x) = (u_k x + v_k)Q_k(x) - w_k Q_{k-1}(x), \tag{2.5}$$

where

$$u_k = \frac{k}{2}A''(0), \quad v_k = kA'(0) - \frac{1}{2}r_k A''(0), \quad w_k = \frac{\gamma_k}{\alpha_k}\Big(B'(0) + \Big(k - \frac{1}{2}\Big)A''(0)\Big),$$

($r_k = b_k/a_k$, where a_k and b_k are the two oldest coefficients in $Q_k(x) = a_k x^k + b_k x^{k-1} + \cdots$). For these relations see [2, Ch. 2.2].

In order to construct the spectral approximation for the function $v(x)$ we have to transform the interval $[1-\delta, 1]$ into $[-1, 1]$ using the substitution

$$x = (\delta/2)(t-1) + 1 \tag{2.10}$$

which transforms (2.2), (2.3) into

$$L_\delta W \equiv -\mu^2 W''(t) + \eta(t)W'(t) + \xi(t)W(t) = \lambda(t), \qquad t \in [-1, 1] \tag{2.7}$$

$$G_1 W \equiv (W(-1), W(1)) = (0, B_1), \tag{2.8}$$

where

$$W(t) = v\Big(\frac{\delta}{2}(t-1) + 1\Big), \quad \mu = \frac{2\varepsilon}{\delta}, \quad \eta(t) = \frac{2}{\delta}\alpha\Big(\frac{\delta}{2}(t-1) + 1\Big),$$

$$\xi(t) = \beta\Big(\frac{\delta}{2}(t-1) + 1\Big), \quad \lambda(t) = \varepsilon^2 y_r''\Big(\frac{\delta}{2}(t-1) + 1\Big).$$

Thus, we are going to ask for the approximate solution of (2.7), (2.8) in the form

$$W(t) \approx W_n(t) = \sum_{k=0}^{n} q_k Q_k(t). \tag{2.9}$$

3. Numerical layer length. Numerical examples show that the accuracy of the approximation (2.9) vitaly depends on the choice of number δ so, we are going to construct the procedure which determines it in quite a natural way, adapting it to the chosen values of parameter ε and degree n of the spectral approximation. In fact, we are going to find how far from the layer pont $x = 1$ we have to go to provide the existence of certain n-th degree parabola which resembles the exact solution. This leeds to the following definitions:

Definition 1. A function $f(x) \in C^2[1-\delta, 1]$ is called resemblence function for the problem (2.2), (2.3) if 1° $G_0 f = (0, B_1)$; 2° $x = 1 - \delta$ is the stationary point for $f(x)$; 3° for $B_1 > 0$ $f(x)$ is concave and for $B_1 < 0$ $f(x)$ is convex.

Definition 2. The sufficiently small positive number $\delta = \delta(n, \varepsilon)$, for which a resemblence function satisfies the equation (2.2) at the layer point $x = 1$ is called the *numerical layer length*.

Now, we are able to prove the following lemma:

LEMMA 1. *The n-th degree polynomial*

$$p_n(x) = B_1\big((x-1)/\delta + 1\big)^n, \qquad n \geq 2 \tag{3.1}$$

is a resemblance function for the problem (2.2), (2.3).

Proof. It is obvious that $p_n(x) \in C^2[1-\delta, 1]$ and that $p_n(1) = B_1$ and $p_n(1-\delta) = 0$, so 1° holds. As

$$p_n'(x) = \frac{nB_1}{\delta}\left(\frac{x-1}{\delta}+1\right)^{n-1}$$

we have $p_n'(x) = 0$ only for $x = 1-\delta$, so 2° holds too. From

$$p_n''(x) = \frac{n(n-1)B_1}{\delta^2}\left(\frac{x-1}{\delta}+1\right)^{n-2}$$

we have that for $x \in [1-\delta, 1]$ $\operatorname{sgn} p_n''(x) = \operatorname{sgn} B_1$, which proves 3°. Thus, by Definition 1 (3.1) is resemblance function for the problem (2.2), (2.3).

THEOREM 2. *The numerical layer length is given by the expression*:

$$\delta = \left(rn - \sqrt{r^2n^2 - 4q\varepsilon^2 n(n-1)}\right)/(2q) \tag{3.2}$$

where q and r denote the constants

$$q = \varepsilon^2 y_r''(1)B_1^{-1} - \beta(1), \qquad r = \alpha(1). \tag{3.3}$$

Proof. By Definition 2, applying the resemblance polynomial (3.1), we come to the equality

$$-\frac{\varepsilon^2 n(n-1)B_1}{\delta^2}\left(\frac{x-1}{\delta}+1\right)^{n-2} + \alpha(x)\frac{nB_1}{\delta}\left(\frac{x-1}{\delta}+1\right)^{n-1} + \\ + \beta(x)B_1\left(\frac{x-1}{\delta}+1\right)^n = \varepsilon^2 y_r''(x),$$

which, at the layer point $x = 1$ gives

$$(\varepsilon^2 y_r''(x)B_1^{-1} - \beta(1))\delta^2 - \alpha(1)n\delta + \varepsilon^2 n(n-1) = 0. \tag{3.4}$$

The solution of (3.4), using notation (3.3), is the expression (3.2), which existence, for sufficiently small ε, is ensured under the conditions of Theorem 1, i.e.

$$D = r^2n^2 - 4q\varepsilon^2 n(n-1) > n(n-1)(r^2 - 4q\varepsilon^2) \geq \big(\alpha_0^2 + 4\varepsilon^2\beta_0 - 4\varepsilon^4 y_r''(1)B_1^{-1}\big)n(n-1).$$

In the case 1° as $\alpha_0^2 + 4\varepsilon^2\beta_0 > 0$, for sufficiently small ε we have that $D \geq 0$. Similary, in the case 2°, as $\alpha_0^2 + 4\varepsilon^2\gamma_0 > 0$ from

$$D > \big(\alpha_0^2 + 4\varepsilon^2\gamma_0 - 4\varepsilon^4 y_r''(1)B_1^{-1} + 4\varepsilon^2\alpha'(1)\big)n(n-1)$$

we, again, have $D \geq 0$ when ε is small enough.

Remark 1. The other solution of the equation (3.4) is either negative (if $q < 0$), or too big (if $q > 0$), so in both cases it exceedes the original interval $[0, 1]$.

4. Construction of the spectral approximation. After the numerical layer length δ is determined, using formula (3.2), we can proceed to determine the approximate solution (2.9) of (2.7), (2.8). One of the main problems is approximation of the functions $\eta(t)$ and $\xi(t)$. If these functions are approximated by the power series or some orthogonal series of relatively large degree that might leed to a rather complicated calculations which demande a large computational time and a lot of memory space. Under the additional assumptions that $\alpha(x), \beta(x) \in C^3[1-\delta, 1]$ it is sufficient to approximate them by the low degree polynomials, let say of second degree, which gives

$$\eta(t) \approx c_1 t^2 + c_2 t + c_3, \qquad \xi(t) \approx c_4 t^2 + c_5 t + c_6.$$

The order of such an approximation is $O(\delta^3)$, which is very small, so it doesn't effects the accuracy of the spectral solution.

As for the function $\lambda(t)$, it has to be approximated by the aproporiate orthogonal series as

$$\lambda(t) \approx \lambda_n(t) = \sum_{k=0}^{n} l_k Q_k(t).$$

So, we come to the problem

$$-\mu^2 W_n''(t) + (c_1 t^2 + c_2 t + c_3) W_n'(t) + (c_4 t^2 + c_5 t + c_6) W_n(t) = \lambda_n(t),$$
$$W_n(-1) = 0, \qquad W_n(1) = B_1.$$

In the process of constructing the system for determining the coefficients q_k, $k = 0, \ldots, n$ of the spectral solution (2.9) we have to overcome two difficulties. The first one is to express the first and second derivative of (2.9), i.e. to construct the relation which expresses their coeficients through q_k. In general this is achieved by the repeted use of formula (2.5). The second difficulty is how to multiply (2.9) and its derivatives by t and t^2. This is achieved by the use of Bonnet's relation (2.4) and multiplying it by t. After a tedious calculations this leeds to an explict system for q_k which we shall give for two orthogonal basis, Chebyshev and Legendre polynomials:

I. *Chebyshev basis*. The first $n-1$ equations in the system are obtained by Horner in [4], and their construction is given in Table 1 for $k = 2, 3, \ldots, n$, using the notation $q_{-k} = q_k$, $l_{-k} = l_k$, where, for the left hand side of the equation we multiply each element of the table by the column constant and row variable and summarise the obtained terms, and for the right-hand side we make the linear combination of the elements of the last column and the appropriate values of l_i, $i = k-4, \ldots, k+4$. Two additional equations are obtained from the boundary

	$-\mu^2$	c_1	c_2	c_3	c_4	c_5	c_6
q_{k-4}					$k+1$		
q_{k-3}		$2(k-3)(k+1)$				$2(k+1)$	
q_{k-2}			$4(k-2)(k+1)$		2		$4(k+1)$
q_{k-1}		$2(k+3)(k-1)$		$8(k^2-1)$		$-2(k-1)$	
q_k	$16k(k^2-1)$		$8k$		$-2k$		$-8k$
q_{k+1}		$-2(k-3)(k+1)$		$-8(k^2-1)$		$-2(k+1)$	
q_{k+2}			$-4(k+2)(k-1)$		-2		$4(k-1)$
q_{k+3}		$-2(k+3)(k-1)$				$2(k-1)$	
q_{k+4}					$k-1$		

Table 1.

conditions and they give $\sum_{k=0}^{n}{}'q_k(-1)^k = 0$, $\sum_{k=0}^{n}{}'q_k = B_1$, where we have used that for Chebyshev polynomials $T_k(t)$ we have $T_k(-1) = (-1)^k$ and $T_k(1) = 1$. (The notation $'q_k$ means that summation envolves $0{,}5 \cdot q_0$ rather than q_0).

When the obtained system is solved for $q_0, \dots, q_n$ the following algorithm is used to evaluate (2.9):

Let $b_{k-1}^{-1} = 0{,}5 \cdot q_k$ for $k = 0, \dots, n$.

Let $b_{n+2} = b_{n+1} = 0$.

Evaluate $b_k = 2b_{k+1}^{-1} + 2tb_{k+1} - b_{k+2}$ for $k = n, \dots, 0$.

Let $W_n(t) = 0{,}5 \cdot (b_0 - b_2)$.

II. *Legendre basis.* The system in this case is obtained in slightly different way by the author in [5]. The first $n-1$ equations are constructed form the Table 2. in the same manner as in the first case, and the two equations obtained from the boundary conditions are

$$\sum_{k=0}^{n}(-1)^k q_k = 0, \qquad \sum_{k=0}^{n} q_k = B_1,$$

because for Legendre basis $P_k(t)$ we also have $P_k(-1) = (-1)^k$ and $P_k(1) = 1$.

In this case for the evaluation of (2.9) we need the following algorithm:

Let $b_{n+2} = b_{n+1} = 0$.

Evaluate $b_{n-k} = q_{n-k} - \dfrac{n-k+1}{n-k+2} b_{n-k+2} + t \cdot \dfrac{2n-2k+1}{n-k+1} b_{n-k+1}$ for $k = 0, \dots, n$.

Let $W_n(t) = q_0$.

5. The error estimate. It can be easily seen from (2.1) that the error function has the form

$$d(x) = \begin{cases} |y(x) - y_r(x)| & x \in [0, 1-\delta] \\ |v(x) - W_n(2(x-1)/\delta + 1)| & x \in [1-\delta, 1]. \end{cases}$$

	$-\mu^2$	c_1	c_2	c_3	c_4	c_5	c_6
q_{k-2}					$\frac{k(k-1)}{(2k-1)(2k-3)}$		
q_{k-1}		$\frac{k(k-1)}{2k-1}$				$\frac{k}{2k-1}$	
q_k			k		$\frac{2k^2+2k-1}{(2k+3)(2k-1)}$		1
q_{k+1}		$\frac{3k^2+5k+1}{2k+3}$		$2k+1$		$\frac{k+1}{2k+3}$	
q_{k+2}	$(2k+1)(2k+3)$		$-(2k+1)$		$\frac{(k+1)(k+2)}{(2k+5)(2k+3)}$		
q_i, $i \geq k+3$ step 2		$2k+1$		$2k+1$			
q_i, $i \geq k+4$ step 2	$(2k+1)\frac{i-k}{2}(i+k+1)$		$-2(k+1)$				

Table 2.

In [1] the following theorem is given

THEOREM 3. *For $x \in [0, 1-\delta]$ the following estimate holds*

$$d(x) \leq C\big(\varepsilon^2 + \exp\{\varepsilon^{-2}\alpha(1)(x-1)\}\big),$$

where C is constant independent of x and ε.

For the proof see [1, Ch. II, Th. 3.2].

The problem of the error estimate of the layer function we shall solve by generalizing Oliver's estimate in [3] for the one dimensional case. First we have to remark that the error, upon the subinterval $[1-\delta, 1]$, by the use of (2.6), may be written down as

$$z(t) = \big|v\big((\delta/2)(t-1)+1\big) - W_n(t)\big| = |W(t) - W_n(t)|. \tag{5.1}$$

As $W(t) \in C^2[-1,1]$ it can be exactly represented by the infinite orthogonal series according to the chosen basis as $W(t) = \sum_{k=0}^{\infty} a_k Q_k(t)$. The coefficients a_k, $k = 0, \dots,$ are, in fact, determined by the infinite system of the same form as for q_k, $k = 0, \dots, n$. These two systems can be written down in the vector form as

$$\sum_{i=0}^{\infty} A_i a_i = R \quad \text{and} \quad \sum_{i=0}^{n} A_i q_i = R,$$

where A_i and R are the appropriate column matrixes. By substracting these two equalities we come to

$$\sum_{i=0}^{n} A_i(q_i - a_i) = \sum_{i=n+1}^{\infty} A_i a_i. \tag{5.2}$$

Let us now define the values $\alpha_i^{(j)}$, $j \geq n+1$, $i = 0, \dots, n$ as the solutions of $\sum_{i=0}^{n} A_i \alpha_i^{(j)} = A_j$.

The equality (5.2) now gives

$$\sum_{i=0}^{n} A_i (q_i - a_i) = \sum_{j=n+1}^{\infty} \Big(\sum_{i=0}^{n} A_i \alpha_i^{(j)} \Big) a_j = \sum_{i=0}^{n} \Big(\sum_{j=n+1}^{\infty} \alpha_i^{(j)} a_j \Big) A_i,$$

which implies

$$q_i - a_i = \sum_{j=n+1}^{\infty} \alpha_i^{(j)} a_j \qquad i = 0, \dots, n. \tag{5.3}$$

As the examined error is

$$z(t) = \Big| \sum_{k=n+1}^{\infty} a_k Q_k(t) - \sum_{k=0}^{n} (q_k - a_k) Q_k(t) \Big|$$

(5.3) gives

$$z(t) = \Big| \sum_{j=n+1}^{\infty} \Big(\sum_{k=0}^{n} \alpha_k^{(j)} Q_k(t) - Q_j(t) \Big) a_j \Big|. \tag{5.4}$$

This proves the following theorem:

THEOREM 4. *The error* (5.1) *can be approximately estimated by*

$$z_m(t) = \sum_{j=n+1}^{n+m} e_j q_j^*, \quad m \in \mathbf{N}, \quad \textit{where}$$

$$e_j = \sum_{k=0}^{n} \alpha_k^{(j)} Q_k(t) - Q_j(t), \quad j = n+1, \dots, n+m,$$

and q_j^* *determines the magnitude of* q_j.

In the proof we have, also, to use the fact that the summ (5.4) is dominantly determined by the first few terms and that the magnitude of the coefficients in the infinite series is of the same order as for the finite one (of degree $n+m$) when n is large enough.

In practice it is sufficient to take $m = 1$ or $m = 2$ to obtain a quite satisfying estimate.

6. Numerical example. We shall construct the spectral approximation for the boundary value problem due to [1]

$$-\varepsilon^2 y'' + \frac{2 + 4\varepsilon^2 - 2\varepsilon^2 x}{(2-x)^2} y' = \frac{\varepsilon^2 \pi^2}{(2-x)^4} \cos \frac{\pi(1-x)}{2-x} + \frac{2\pi}{(2-x)^4} \sin \frac{\pi(1-x)}{2-x},$$
$$y(0) = y(1) = 0.$$

$\varepsilon^2 = 10^{-5}$	$n = 10$	$[1-\delta, 1] = [0.99991, 1]$	
x	$y(x)$	$d(x)$	$z_1(t)$
0.99995	0.993	$1 \cdot 10^{-4}$	$1 \cdot 10^{-3}$
0.99997	0.950	$1 \cdot 10^{-4}$	$9 \cdot 10^{-4}$
0.99999	0.632	$4 \cdot 10^{-5}$	$4 \cdot 10^{-4}$
0.999993	0.503	$4 \cdot 10^{-6}$	$7 \cdot 10^{-4}$
0.999996	0.330	$2 \cdot 10^{-5}$	$6 \cdot 10^{-5}$
0.999998	0.181	$1 \cdot 10^{-5}$	$7 \cdot 10^{-4}$
0.999999	0.095	$5 \cdot 10^{-6}$	$7 \cdot 10^{-4}$

Table 3.

The reduced problem has the solution $y_r(x) = \cos \dfrac{\pi(1-x)}{2-x}$ and for the approximation (2.9) of the layer function we are going to use Chebyshev basis. The absolute error $d(x)$ and the error estimate $z_1(t)$ are given in the Table 3 for $n = 10$ and $\varepsilon^2 = 10^{-5}$ in several points displayed through the boundary layer.

The similar results can be obtained for other orthogonal basis.

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