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ON COVARIANCE OF SPECTRAL ESTIMATES OF STATIONARY RANDOM SEQUENCE

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Abstract. The asymptotic behaviour of the covariance of estimates of the spectral density of stationary random sequence is investigated.

1. Introduction. Let $X(t), t \in \{\dots, -1, 0, 1, \dots\} = D$ be a strictly stationary real random sequence with the mean $EX(t) = 0$ and the spectral density $f(\lambda)$, $\lambda \in [-\pi, \pi] = \Pi$. We shall consider the following spectral density estimate of the Grenander-Rosenblatt type

$$\widehat{f}_N(\lambda) = \int_R \varphi_N(x - \lambda) I_N(x)\, dx,$$

which is based on the periodogram

$$I_N(x) = \frac{1}{2\pi N} \left| \sum_{t=1}^{N} X(t) e^{-itx} \right|^2,$$

where $\varphi_N(x) = B_N^{-1} \varphi(x B_N^{-1})$, $x \in \Pi$; $\varphi(x)$ is a weight function that is symmetric about 0, has a bounded first derivative and such that $\varphi(0) = 1$, $\int_\Pi \varphi(x)\, dx = 1$, $\varphi(x) = 0$ for $|x| \geq \pi$ and the sequence (B_N) is such that $B_N \to 0$, $NB_N \to \infty$ when $N \to \infty$. We assume that the functions f and φ_N are defined on the whole real line and 2π-periodic.

The cumulant spectral densities of the sequence $\{X(t), t \in D\}$ are defined as follows

(1)

$$f_n(\lambda_1, \lambda_2, \dots, \lambda_{n-1}) = \frac{1}{(2\pi)^{n-1}} \sum_{t_1, t_2, \dots, t_{n-1}} S_n(t_1, \dots, t_{n-1}, 0) \exp\left(-\sum_{j=1}^{n-1} \lambda_j t_j \right)$$

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where $S_n(t_1, \dots, t_{n-1}, 0)$ denote the n-th cumulant of the sequence $X(t)$, $t \in D$. (The function f_n is defined if the series on the right-hand side of (1) converges.) Then, one has the following inverse formula

$$S_n(t_1, \dots, t_{n-1}, 0) = \int_{\Pi^{n-1}} \exp\left(i \sum_{j=1}^{n-1} \lambda_j t_j\right) f_n(\lambda_1, \dots, \lambda_{n-1}) d\lambda_1 \dots d\lambda_{n-1},$$

or in symmetric form

$$(2) \quad S_n(t_1, \dots, t_{n-1}, t_n) = \int_{\Pi^n} \exp\left(i \sum_{j=1}^{n} \lambda_j t_j\right) \times \\ \times f_n(\lambda_1, \dots, \lambda_n) \eta(\lambda_1 + \dots + \lambda_n) d\lambda_1 \dots d\lambda_n,$$

where $\eta(\lambda) = \sum_{j=-\infty}^{+\infty} \delta(\lambda + 2\pi j)$, $\delta(\cdot)$ is Dirac function and $\sum_{j=1}^{n} \lambda_j = 0 \pmod{2\pi}$.

2. Covariance of Spectral Estimate. It is well known that for $\lambda, \mu \in \Pi$ the relation

$$\operatorname{cov}(\widehat{f}_N(\lambda), \widehat{f}_N(\mu)) = O(1/NB_N), \qquad N \to \infty,$$

is valid under some assumptions on the higher order spectra of the random sequence $X(t)$. We shall prove the following

THEOREM 1. *Let* $\sup_\lambda |f'(\lambda)| = C_1 < +\infty$ *and* $\sup_{\lambda_1, \lambda_2, \lambda_3, \lambda_4} |f_4(\lambda_1, \lambda_2, \lambda_3, \lambda_4)| = C_2 < +\infty$. *Then we have*

$$(3) \quad \begin{aligned} cov(\widehat{f}_N(\lambda), \widehat{f}_N(\mu)) &= \frac{2\pi}{N} f(\lambda) f(\mu) \int_\Pi \varphi_N(x - \lambda)[\varphi_N(x - \mu) + \varphi_N(x + \mu)] dx \\ &+ |\lambda - \mu| O\left(\frac{\ln^2 N}{N} \int_\Pi \varphi_N^2(x) dx\right) + o\left(\frac{1}{N} \int_\Pi \varphi_N^2(x) dx\right) \\ &+ O\left(\frac{\ln^4 N}{N^2} \int_\Pi x^2 \varphi_N^2(x) dx \int_\Pi \varphi_N^2(x) dx\right), \end{aligned}$$

when $N \to \infty$, *uniformly for* $\lambda, \mu \in \Pi$.

(The asymptotic behaviour of the variance of spectral estimates of the Grenander-Rosenblatt type is investigated in [1]. The formula (3) is given in [3] without proof.)

THEOREM 2. *Let us denote*

$$\tilde{\xi}_N(\lambda) = \sqrt{NB_N}[\widehat{f}_N(\lambda B_N) - E\widehat{f}_N(\lambda B_N)], \quad |\lambda| \le \pi B_N^{-1},$$
$$\tilde{Z}_N(\lambda) = \sqrt{NB_N}[\widehat{f}_N(\lambda B_N) - f(\lambda B_N)], \quad |\lambda| \le \pi B_N^{-1},$$
$$r(\lambda, \mu) = 2\pi f^2(0) \left\{ \int_R \varphi(x - \lambda)\varphi(x - \mu)\, dx + \int_R \varphi(x - \lambda)\varphi(x + \mu)\, dx \right\}.$$

Under assumptions of Theorem 1 we have

$$\lim_{N\to\infty} \operatorname{cov}\left(\tilde{\xi}_N(\lambda), \tilde{\xi}_N(\mu)\right) = r(\lambda,\mu) \tag{4}$$

for every λ and μ. If $NB_N^3 \to 0$ when $N\to\infty$, then we also have

$$\lim_{N\to\infty} \operatorname{cov}\left(\tilde{Z}_N(\lambda), \tilde{Z}_N(\mu)\right) = r(\lambda,\mu). \tag{5}$$

3. Proofs. Denote $F_N(x) = \sin\frac{Nx}{2}\sin^{-1}\frac{x}{2}$. We shall use the following results that are given in [1]:

LEMMA 1. a) *Let $\sup_\lambda |f'(\lambda)| = C_1 < +\infty$. Then for every λ, λ_1, λ_2 and $\varepsilon > 0$ we have*

$$\left|\frac{1}{2\pi N}\int_\Pi f(\lambda+x)F_N(-x+\lambda_1)F_N(x+\lambda_2)\,dx - f(\lambda)\frac{F_N(\lambda_1+\lambda_2)}{N}\right| \le$$
$$\le C_1(|\lambda_1|+|\lambda_2|+2\varepsilon)\max_{|\theta|<\varepsilon}\left|\frac{F_N(\lambda_1+\lambda_2+\theta)}{N}\right|\ln(2N\varepsilon)+\frac{C_1\pi^2}{8N\varepsilon^2}.$$

b) *For every α and β we have*

$$\frac{1}{N^2}\int_{\Pi^2}\varphi_N(x+\alpha)\varphi_N(y+\beta)F_N^2(x-y)\,dx\,dy =$$
$$\frac{2\pi}{N}\int_\Pi \varphi_N(x+\alpha)\varphi_N(x+\beta)\,dx + o\left(\frac{1}{N}\int_\Pi \varphi_N^2(x)\,dx\right).$$

Proof of Theorem 1. We have

$$\operatorname{cov}(\widehat{f}_N(\lambda),\widehat{f}_N(\mu)) = \left(\frac{1}{2\pi N}\right)^2\int_{\Pi^2}\varphi_N(x)\varphi_N(y)\times$$
$$\times \sum_{t_1,t_2,t_3,t_4=1}^{N}[E(X(t_1)X(t_2)X(t_3)X(t_4)) - E(X(t_1)X(t_2))E(X(t_3)X(t_4))]\times$$
$$\times e^{-it_1(x+\lambda)+it_2(x+\lambda)+it_3(y+\mu)-it_4(y+\mu)}dx\,dy$$
$$= \left(\frac{1}{2\pi N}\right)^2\int_{\Pi^2}\varphi_N(x)\varphi_N(y)\sum_{t_1,t_2,t_3,t_4=1}^{N}[S_4(t_1,t_2,t_3,t_4)+S_2(t_1,t_3)S_2(t_2,t_4)+$$
$$+S_2(t_1,t_4)S_2(t_2,t_3)]e^{-it_1(x+\lambda)+it_2(x+\lambda)+it_3(y+\mu)-it_4(y+\mu)}dx\,dy.$$

Using the formula (2) we get $\operatorname{cov}(\widehat{f}_N(\lambda),\widehat{f}_N(\mu)) = J_1+J_2+J_3$, where

$$J_1 = \left(\frac{1}{2\pi N}\right)^2\int_{\Pi^2}\varphi_N(x)\varphi_N(y)\int_{\Pi^4}f_4(\lambda_1,\lambda_2,\lambda_3,\lambda_4)\eta(\lambda_1+\lambda_2+\lambda_3+\lambda_4)\times \tag{6}$$
$$\times F_N(\lambda_1-x-\lambda)F_N(\lambda_2+x+\lambda)F_N(\lambda_3+y+\mu)F_N(\lambda_4-y-\mu)d\lambda_1 d\lambda_2 d\lambda_3 d\lambda_4 dx\,dy$$
$$= o\left(\frac{1}{N}\int_\Pi \varphi_N^2(x)dx\right)$$

(e.g. Žurbenko [1]) and

$$J_2 = \left(\frac{1}{2\pi N}\right)^2 \int_{\Pi^2} \varphi_N(x)\varphi_N(y) \int_{\Pi^2} f(u)f(v)F_N(u-x-\lambda)\times$$
$$\times F_N(v+x+\lambda)F_N(-u+y+\mu)F_N(-v-y-\mu)\,du\,dv\,dx\,dy,$$

$$J_3 = \left(\frac{1}{2\pi N}\right)^2 \int_{\Pi^2} \varphi_N(x)\varphi_N(y) \int_{\Pi^2} f(u)f(v)F_N(u-x-\lambda)\times$$
$$\times F_N(v+x+\lambda)F_N(-v+y+\mu)F_N(-u-y-\mu)\,du\,dv\,dx\,dy.$$

Using the periodicity of the functions f, φ_N and F_N and Lemma 1.a) we get the following form for the integral J_2:

$$J_2 = \int_{\Pi^2} \varphi_N(x)\varphi_N(y)\frac{1}{2\pi N}\int_\Pi F_N(u-x)F_N(-u+y-\lambda+\mu)f(u+\lambda)\,du\times$$
$$\times \frac{1}{2\pi N}\int_\Pi F_N(v-y)F_N(-v-\mu+x+\lambda)f(v-\mu)\,dv\,dx\,dy$$
$$= \sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Pi^2}\varphi_N(x)\varphi_N(y)a_i b_j\,dx\,dy \equiv \sum_{i=1}^{3}\sum_{j=1}^{3} J_{ij},$$

where

$$a_1 = \frac{f(\lambda)}{N}F_N(x-y+\lambda-\mu), \quad b_1 = \frac{f(\mu)}{N}F_N(x-y+\lambda-\mu),$$
$$a_2 = C_1\theta_1(|x|+|y-\lambda+\mu|+2\varepsilon)|F_N(x-y+\lambda-\mu-\theta)|\frac{\ln(2N\varepsilon)}{N},$$
$$b_2 = C_1\theta_3(|x|+|y-\lambda+\mu|+2\varepsilon)|F_N(x-y+\lambda-\mu-\theta)|\frac{\ln(2N\varepsilon)}{N},$$
$$a_3 = \frac{C_1\theta_2\pi^2}{8N\varepsilon^2}, \quad b_3 = \frac{C_1\theta_4\pi^2}{8N\varepsilon^2},$$

and $|\theta_i| \le \varepsilon$ for $i \in \{1,2,3,4\}$, where $\varepsilon > 0$. Further, using Lemma 1.b) we have

$$J_{11} = f(\lambda)f(\mu)\int_{\Pi^2}\varphi_N(x)\varphi_N(y)\frac{1}{N^2}F_N^2(x-y+\lambda-\mu)\,dx\,dy$$
$$= f(\lambda)f(\mu)\int_{\Pi^2}\varphi_N(x-\lambda)\varphi_N(y-\mu)\frac{1}{N^2}F_N^2(x-y)\,dx\,dy$$
$$= \frac{2\pi}{N}f(\lambda)f(\mu)\int_\Pi \varphi_N(x-\lambda)\varphi_N(x-\mu)\,dx + o\left(\frac{1}{N}\int_\Pi \varphi_N^2(x)\,dx\right).$$

Let $\varepsilon = \ln^{-\delta} N$, where $\delta > 2$ and

$$G_1 = \{(x,y)| -\pi+\lambda \le x \le \pi+\lambda, -\pi+\mu \le y \le \pi+\mu\},$$
$$G = \{(x,y)| -\pi+\lambda \le x \le \pi+\lambda, -\pi+\mu \le x-z \le \pi+\mu\}.$$

For the addend J_{12} we have (we consider the addend J_{21} similarly):

$$
\begin{aligned}
|J_{12}| &\le C_1 f(\lambda) \int_{\Pi^2} |\varphi_N(x)\varphi_N(y)|(|y| + |x+\lambda-\mu| + 2\varepsilon)\times \\
&\qquad\qquad \times |F_N(x-y+\lambda-\mu)F_N(x-y+\lambda-\mu-\theta)| \frac{\ln N}{N^2}\, dx\, dy \\
&= C_1 f(\lambda)\frac{\ln N}{N^2} \int_{G_1} |\varphi_N(x-\lambda)\varphi_N(y-\mu)|(|y-\mu| + |x-\mu| + 2\varepsilon)\times \\
&\qquad\qquad \times |F_N(x-y)F_N(x-y-\theta)|\, dx\, dy \\
&= C_1 f(\lambda)\frac{\ln N}{N^2} \int_G (|x-z-\mu| + |x-\mu| + 2\varepsilon) \\
&\qquad \times |\varphi_N(x-\lambda)\varphi_N(x-z-\mu)||F_N(z)F_N(z-\theta)|\, dx\, dz \equiv J_{12}^{(1)} + J_{12}^{(2)} + J_{12}^{(3)},
\end{aligned}
$$

$$
\begin{aligned}
J_{12}^{(1)} &= C_1 f(\lambda)\frac{\ln N}{N^2} \int_G |\varphi_N(x-\lambda)\varphi_N(x-z-\mu)(x-z-\mu)|\times \\
&\qquad\qquad \times |F_N(z)F_N(z-\theta)|\, dx\, dz \\
&\le C_1 f(\lambda)\frac{\ln N}{N^2} \int_{-2\pi}^{2\pi} \Big\{ |F_N(z)F_N(z-\theta)|\times \\
&\qquad\qquad \times \int_\Pi |(x-z-\mu)\varphi_N(x-\lambda)\varphi_N(x-z-\mu)|\, dx \Big\}\, dz \\
&\le \frac{C_1 f(\lambda)\ln N}{2\pi N} \int_\Pi F_N^2(z)dz \left(\frac{2\pi}{N}\int_\Pi x^2\varphi_N^2(x)\, dx\, \frac{2\pi}{N}\int_\Pi \varphi_N^2(x)\, dx\right)^{1/2} \\
&= O\left(\frac{\ln^2 N}{N}\int_\Pi x^2\varphi_N^2(x)\, dx \frac{1}{N}\int_\Pi \varphi_N^2(x)\, dx\right)^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
J_{12}^{(2)} &= C_1 f(\lambda)\frac{\ln N}{N^2} \int_\Pi |\varphi_N(x-\lambda)\varphi_N(x-z-\mu)(x-\mu)F_N(z)F_N(z-\theta)|\, dx\, dz \\
&\le C_1 f(\lambda)\frac{\ln N}{N^2} \int_G |\varphi_N(x-\lambda)\varphi_N(x-z-\mu)|\times \\
&\qquad\qquad \times (|x-\lambda| + |\lambda-\mu|)|F_N(z)F_N(z-\theta)|\, dx\, dz \\
&= O\left(\frac{\ln^2 N}{N^2}\int_\Pi x^2\varphi_N^2(x)dx \int_\Pi \varphi_N^2(x)dx\right)^{1/2} + |\lambda-\mu| O\left(\frac{\ln N}{N}\int_\Pi \varphi_N^2(x)dx\right),
\end{aligned}
$$

$$
\begin{aligned}
J_{12}^{(3)} &= 2C_1 f(\lambda)\frac{\ln^{1-\delta} N}{N^2} \int_\Pi |\varphi_N(x-\lambda)\varphi_N(x-z-\mu)F_N(z)F_N(z-\theta)|\, dx\, dz \\
&= o\left(\frac{1}{N}\int_\Pi \varphi_N^2(x)\, dx\right).
\end{aligned}
$$

The same way we get

$$
J_{ij} = o\left(\frac{1}{N}\int_\Pi \varphi_N^2(x)\, dx\right), \quad \text{for} \quad ij = 13, 23, 31, 32, 33,
$$

$$
\begin{aligned}
J_{22} &\le C_1^2 \frac{\ln^2 N}{N^2} \int_{\Pi^2} |\varphi_N(x)\varphi_N(y)|(|x|+|y-\lambda+\mu|+2\varepsilon)(|y|+|x+\lambda-\mu|+2\varepsilon)\times \\
&\qquad \times |F_N(x+\lambda-y-\mu)F_N(x+\lambda-y-\mu-\theta)|\, dx\, dy \\
&= C_1^2 \frac{\ln^2 N}{N^2} \int_{G_1} |\varphi_N(x-\lambda)\varphi_N(y-\mu)|(|x-\lambda|+|y-\lambda|+2\varepsilon)\times \\
&\qquad \times (|x-\mu|+|y-\mu|+2\varepsilon)|F_N(x-y)F_N(x-y-\theta)|\, dx\, dy \\
&\le C_1^2 \frac{\ln^4 N}{N^2} \int_{G_1} |\varphi_N(x-\lambda)\varphi_N(y-\mu)|(|x-\lambda|+|y-\mu|+|\lambda-\mu|+2\varepsilon)^2\times \\
&\qquad \times |F_N(x-y)F_N(x-y-\theta)|\, dx\, dy \\
&= O\Big(\frac{\ln^2 N}{N}\int_\Pi x^2\varphi_N^2(x)\, dx \frac{1}{N}\int_\Pi \varphi_N^2(x)\, dx\Big)^{1/2} \\
&\quad + |\lambda-\mu| O\Big(\frac{\ln^2 N}{N}\int_\Pi \varphi_N^2(x)\, dx\Big) + o\Big(\frac{1}{N}\int_\Pi \varphi_N^2(x)\, dx\Big),
\end{aligned}
$$

and finally we have

$$
\begin{aligned}
J_2 &= \frac{2\pi}{N} f(\lambda) f(\mu) \int_\Pi \varphi_N(x-\lambda)\varphi_N(x-\mu)\, dx + |\lambda-\mu| O\left(\frac{\ln^2 N}{N}\int_\Pi \varphi_N^2(x)\, dx\right) \\
&\quad + O\left(\frac{\ln^4 N}{N^2}\int_\Pi x^2\varphi_N^2(x)\, dx \int_\Pi \varphi_N^2(x)\, dx\right)^{1/2} + o\left(\frac{1}{N}\int_\Pi \varphi_N^2(x)\, dx\right).
\end{aligned} \tag{7}
$$

Analogously one can find

$$
\begin{aligned}
J_3 &= \frac{2\pi}{N} f(\lambda) f(\mu) \int_\Pi \varphi_N(x-\lambda)\varphi_N(x+\mu)\, dx + |\lambda-\mu| O\left(\frac{\ln^2 N}{N}\int_\Pi \varphi_N^2(x)\, dx\right) \\
&\quad + O\left(\frac{\ln^4 N}{N^2}\int_\Pi x^2\varphi_N^2(x)\, dx \int_\Pi \varphi_N^2(x)\, dx\right)^{1/2} + o\left(\frac{1}{N}\int_\Pi \varphi_N^2(x)\, dx\right).
\end{aligned} \tag{8}
$$

Then, (3) follows by (6), (7) and (8).

Proof of Theorem 2. The equality (4) is an easy corollary of Theorem 1. Since

$$
\sup_{-\pi\le\lambda\le\pi} |f(\lambda) - Ef_N(\lambda)| \le C_1 \int_\Pi |x\varphi_N(x)|\, dx + o\left(\int_\Pi |x\varphi_N(x)|\, dx\right),
$$

(this inequality is given in [2]) and

$$
\sup_{-\pi\le\lambda\le\pi} \sqrt{NB_N}|E\widehat{f}_N(\lambda) - f(\lambda)| = O\left(\sqrt{NB_N}\int_\Pi |x\varphi_N(x)|\, dx\right)
$$

$$= O\left(\sqrt{NB_N} \int_{-\pi/B_N}^{\pi/B_N} B_N |t\varphi(t)|\, dt\right) = O(N^{1/2} B_N^{3/2}) = o(1), \quad N \to \infty,$$

$$\sup_{-\pi \le \lambda \le \pi} |\tilde{\xi}_N(\lambda) - \tilde{Z}_N(\lambda)| = \sup_{-\pi \le \lambda \le \pi} \sqrt{NB_N} |E\widehat{f}_N(\lambda) - f(\lambda)| = o(1), \quad N \to \infty,$$

it follows that the equality (5) is also valid.

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