PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 49 (63), 1991, 233–238

ON WEAK CONVERGENCE OF SPECTRAL DENSITY ESTIMATE

Pavle Mladenović

Abstract. We study the periodogram based estimate

$$\widehat{f}_N(\lambda) = \int_R B_N^{-1}\varphi(xB_N^{-1} - \lambda B_N^{-1})(2\pi N)^{-1}\left|\sum_{t=1}^{N} X(t)e^{-itx}\right|^2 dx,$$

where $-\pi \leq \lambda \leq \pi$, ($\varphi$ is a weight function and $B_N \to 0$, $NB_N \to +\infty$, when $N \to \infty$) of the spectral density $f(\lambda), -\pi \leq \lambda \leq \pi$, of a strictly stationary random sequence. We renormalize the scale in λ and define the random process

$$Z_N(\lambda) = (NB_N)^{1/2}[\widehat{f}_N(\lambda B_N) - E\widehat{f}_N(\lambda B_N)], \quad |\lambda| \leq \pi B_N^{-1},$$

in order to obtain the limiting (Gaussian) process whose sample part functions are continuous with probability one. A weak convergence of the sequence $\{Z_N(\lambda), a \leq \lambda \leq b\}_{N=1,2,\dots}$ is investigated.

1. Introduction. Let $X(t)$, $t \in \{\dots,-1,0,1,\dots\} = D$ be a strictly stationary real random sequence with the mean $EX(t) = 0$ and the spectral density $f(\lambda)$, $\lambda \in [-\pi,\pi] = \Pi$.

Assumption A. All cumulant spectral densities of the random sequence $\{X(t), t \in D\}$ are bounded.

This assumption is valid if all moments of the sequence $\{X(t), t \in D\}$ exist and for the Rosenblatt mixing coefficients

$$\alpha(\tau) = \sup_{A\in\Re_{-\infty}^{t},\, B\in\Re_{t+\tau}^{\infty}} |P(AB) - P(A)P(B)|,$$

the inequality $\alpha(\tau) \leq Ke^{-\delta\tau}$, $K > 0$, $\delta > 0$ holds, where $\Re_a^b$ is the σ-algebra generated by the random variables $X(t)$, $a \leq t \leq b$ (e.g. Žurbenko [4]).

$$\widehat{f}_N(\lambda) = \int_R \varphi_N(x - \lambda)I_N(x)dx \tag{1}$$

AMS Subject Classification (1985): Primary 62 M 15, 60 G 10, 60 G 15

Key words: stationary time series, spectral density, periodogram based estimate

which is based on the periodogram $I_N(x) = \dfrac{1}{2\pi N}\left|\sum_{t=1}^{N} X(t)e^{-itx}\right|^2$, where $\varphi_N(x) = B_N^{-1}\varphi(xB_N^{-1})$, $|x| \le \pi$; $\varphi(x), |x| \le \pi$, is a weight function that is symmetric about 0, has a bounded first derivative and such that $\varphi(0) = 1$, $\int_{-\pi}^{\pi}\varphi(x)dx = 1$ and the sequence (B_N) is such that $0 < B_N < 1$ and $B_N \to 0, NB_N \to +\infty$ when $N \to \infty$ (We assume that $\varphi(x) = 0$ for $|x| > \pi$ and that the functions f and φ_N are defined on the whole real line and 2π-periodic.) Note that

$$(2) \qquad (\forall x, y)|\varphi_N(x) - \varphi_N(y)| \le HB_N^{-2}|x - y|,$$

where $H = \sup|\varphi'(x)|$. Under Assumption A one has the following inequality for the cumulants of the random process $\widehat{f}_N(\lambda), -\pi \le \lambda \le \pi$:

$$(3) \qquad \left|S_N(\widehat{f}_N(\lambda_1), \widehat{f}_N(\lambda_2), \ldots, \widehat{f}_N(\lambda_n))\right| \le \frac{K_n}{(NB_N)^{n-1}}$$

where the constant K_n does not depend on the particular choice of points λ_1, λ_2, $\ldots$, λ_n, (e.g. Bentkus, [1]).

2. Results. Let $\tilde{\xi}_N(\lambda) = \sqrt{NB_N}[\widehat{f}_N(\lambda B_N) - E\widehat{f}_N(\lambda B_N)]$ for $|\lambda| \le \pi/B_N$ and $\tilde{\xi}_N(\lambda) = \tilde{\xi}_N(\pi/B_N)$ for $|\lambda| > \pi/B_N$. Let $Z(\lambda), -\infty < \lambda < +\infty$, be the Gaussian random process with the mean $EZ(t) = 0$ and the covariance function

$$EZ(\lambda)Z(\mu) == 2\pi f^2(0)\left\{\int_R \varphi(x-\lambda)\varphi(x-\mu)\,dx + \int_R \varphi(x-\lambda)\varphi(x+\mu)\,dx\right\}.$$

THEOREM 1. *Let the sequence $\{X(t), t \in D\}$ satisfy Assumption A and let the spectral density function f be continuously-differentiable. Then, the finite-dimensional distributions of the random process $\tilde{\xi}_N(\lambda), -\infty < \lambda < +\infty$, converge weakly to those of Gaussian process $Z(\lambda)$, $-\infty < \lambda < +\infty$.*

THEOREM 2. *Let the sequence $\{X(t), t \in D\}$ be Gaussian and its spectral density function f bounded. Then, there exist the constants $K > 0$, $\eta > 0$ and $\varepsilon > 0$ such that the inequality*

$$E|\tilde{\xi}_N(\lambda) - \tilde{\xi}_N(\mu)|^\eta \le K|\lambda - \mu|^{1+\varepsilon}$$

holds for every N, λ and μ.

Let $C = C[a, b]$ be the space of all real continuous functions defined on $[a, b], -\infty < a < b < +\infty$, with the uniform metric $\rho(x, y) = \sup_{a \le t \le b}|x(t) - y(t)|$ and let $\mathcal{C}$ be the clas of Borel sets in C. Denote by P_N and P the probability measures on $(C, \mathcal{C})$ generated by the random process $\tilde{\xi}_N(\lambda), a \le \lambda \le b$, and $Z(\lambda), a \le \lambda \le b$, respectively. Then, we have the following

THEOREM 3. *Let the sequence $\{X(t), t \in D\}$ be Gaussian and its spectral density function f continuously-differentiable. Then, P_N converges weakly to P, when $N \to \infty$.*

3. Proofs. *Proof of Theorem* 1. By Theorem 2 from the paper [3], the inequality (3) and $E\tilde{\xi}_N(\lambda) = 0$ it follows that all cumulants of the random process $\tilde{\xi}_N(\lambda), -\infty < \lambda < \infty$, converge to those of Gaussian process $Z(\lambda), -\infty < \lambda < \infty$.

Let us denote $F_N(x) = \sin(Nx/2)\sin^{-1}(x/2)$. To prove Theorems 2 and 3 we need several lemmas:

LEMMA 1. *Let $\{X(t), t \in D\}$ be a Gaussian process. Then, for $\lambda, \mu \in [-\pi B_N^{-1}, \pi B_N^{-1}]$ one has*

$$E\,\tilde{\xi}_N(\lambda)\tilde{\xi}_N(\mu) = NB_N \int_{\Pi^2} \varphi_N(x - \lambda B_N)\varphi_N(y - \mu B_N) G_N(x, y)\, dx\, dy, \tag{4}$$

where

$$G_N(x) = \left\{ \frac{1}{2\pi N} \int_{\Pi} f(\alpha) F_N(\alpha - x) F_N(\alpha + y)\, d\alpha \right\}^2 + \left\{ \frac{1}{2\pi N} \int_{\Pi} f(\alpha) F_N(\alpha - x) F_N(\alpha - y)\, d\alpha \right\}^2 .$$

Proof. For the Gaussian vector (X_1, X_2, X_3, X_4) we have

$$E(X_1X_2X_3X_4) = E(X_1X_2)E(X_3X_4) + E(X_1X_3)E(X_2X_4) + E(X_1X_4)E(X_2X_3)$$

and consequently we get

$$\begin{aligned} E\, I_N(x) I_N(y) &= \left(\frac{1}{2\pi N}\right)^2 \sum_{t_1,t_2,t_3,t_4=1}^{N} E\, X(t_1)X(t_2)X(t_3)X(t_4) e^{i(t_1-t_2)x + i(t_3-t_4)y} \\ &= \left(\frac{1}{2\pi N}\right)^2 \int_{\Pi} f(\alpha) \left| \sum_{t=1}^{N} e^{i(\alpha - x)t} \right|^2 d\alpha \int_{\Pi} f(\beta) \left| \sum_{t=1}^{N} e^{i(\beta - y)t} \right|^2 d\beta \\ &\quad + \left| \int_{\Pi} \frac{e^{iN(\alpha - x)} - 1}{e^{i(\alpha - x)} - 1} \frac{e^{iN(\alpha + y)} - 1}{e^{i(\alpha + y)} - 1} f(\alpha)\, d\alpha \right|^2 \\ &\quad + \left| \int_{\Pi} \frac{e^{iN(\alpha - x)} - 1}{e^{i(\alpha - x)} - 1} \frac{e^{iN(\alpha - y)} - 1}{e^{i(\alpha - y)} - 1} f(\alpha)\, d\alpha \right|^2 . \end{aligned} \tag{5}$$

We also have

$$EI_N(x) = \frac{1}{2\pi N} \int_{\Pi} \left| \sum_{t=1}^{N} e^{it(\alpha - x)} \right|^2 f(\alpha)\, d\alpha, \tag{6}$$

$$\frac{e^{iNx} - 1}{e^{ix} - 1} = F_N(x) \exp \frac{i(N-1)x}{2}, \tag{7}$$

$$E\tilde{\xi}_N(\lambda)\tilde{\xi}_N(\mu) = NB_N[E\,\hat{f}_N(\lambda B_N)\hat{f}_N(\mu B_N) - E\hat{f}(\lambda B_N)E\hat{f}_N(\mu B_N)] \tag{8}$$

$$= NB_N \int_{\Pi^2} \varphi_N(x-\lambda B_N)\varphi_N(y-\mu B_N)\times$$
$$\times\,[E\,I_N(x)I_N(y) - EI_N(x)EI_N(y)]\,dx\,dy$$

and then (4) follows from (5)–(8).

LEMMA 2. *Let $\{X(t), t\in D\}$ be a Gaussian process and $\sup_\lambda |f(\lambda)| = C_3 < +\infty$. Then, there exists a constant $c_2 \in (0,+\infty)$, such that the following inequality holds for every N, λ and μ:*

$$E|\tilde{\xi}_N(\lambda) - \tilde{\xi}_N(\mu)|^2 \le C_2|\lambda-\mu|. \tag{9}$$

Proof. Using Lemma 2 and (2) we obtain

$$E|\tilde{\xi}_N(\lambda) - \tilde{\xi}_N(\mu)|^2 = E\tilde{\xi}_N^2(\lambda) + E\tilde{\xi}_N^2(\mu) - 2E\,\tilde{\xi}_N(\lambda)\tilde{\xi}_N(\mu)$$
$$= NB_N \int_{\Pi^2} \varphi_N(x-\lambda B_N)[\varphi_N(y-\lambda B_N) - \varphi_N(y-\mu B_N)]G_N(x,y)\,dx\,dy$$
$$+ NB_N \int_{\Pi^2} \varphi_N(y-\mu B_N)[\varphi_N(x-\lambda B_N) - \varphi_N(x-\mu B_N)]G_N(x,y)\,dx\,dy$$
$$\le NH|\lambda-\mu|\left\{\int_{\Pi^2}[|\varphi_N(x-\lambda B_N)| + |\varphi_N(y-\mu B_N)|]G_N(x,y)\,dx\,dy\right\}$$
$$\le NH|\lambda-\mu|(A_1+A_2+A_3+A_4),$$
$$A_1 = \int_{\Pi^2} |\varphi_N(x-\lambda B_N)|\left\{\frac{1}{2\pi N}\int_\Pi F_N(\alpha-x)F_N(\alpha+y)f(\alpha)\,d\alpha\right\}^2 dx\,dy$$
$$A_2 = \int_{\Pi^2} |\varphi_N(x-\lambda B_N)|\left\{\frac{1}{2\pi N}\int_\Pi F_N(\alpha-x)F_N(\alpha-y)f(\alpha)\,d\alpha\right\}^2 dx\,dy$$

where A_3 and A_4 are similar integrals with $\varphi_N(y-\mu B_N)$ instead of $\varphi_N(x-\lambda B_N)$. It follows from the equality

$$\int_\Pi F_N(x-t)F_N(y-t)\,dt = 2\pi F_N(x-y)$$

that

$$NA_1 \le C_3\int_{\Pi^2} |\varphi_N(x-\lambda B_N)|\frac{1}{N}F_N^2(x+y)\,dx\,dy$$
$$\le 2\pi C_3 \int_\Pi |\varphi_N(x-\lambda B_N)|\,dx\frac{1}{2\pi N}\int_\Pi F_N^2(z)\,dz \le k_1 < +\infty$$

and than, the inequality (9) follows easily.

LEMMA 3. *Let the sequence $\{X(t), t\in D\}$ be Gaussian, $\sup_\lambda |f(\lambda)| = C_3 < +\infty$ and let χ_n be the n-th cumulant of the random variable $\tilde{\xi}_N(\lambda) - \tilde{\xi}_N(\mu)$. Then for every $n \ge 2$ there exists a constant c_n such that the inequality*

$$|\chi_n| \le c_n|\lambda-\mu|^{n-1} \tag{10}$$

is valid for every N, λ *and* μ.

Proof. In the case $n = 2$ the inequality (10) follows from Lemma 2. Suppose $n \geq 3$. The random variable $\tilde{\xi}_N(\lambda) - \tilde{\xi}_N(\mu)$ can be represented in the form

$$\tilde{\xi}_N(\lambda) - \tilde{\xi}_N(\mu) = (AX, X) - E(AX, X),$$

where $X = (X(1), \ldots, X(N))$ and A is $N \times N$-matrix whose element A_{ts} is given by

$$A_{ts} = \frac{1}{2\pi}\sqrt{\frac{B_N}{N}} \int_\Pi [\varphi_N(x - \lambda B_N) - \varphi_N(x - \mu B_N)] \cos(t - s)x \, dx.$$

Since X is a Gaussian vector, the characteristic function of the random variable $\tilde{\xi}_N(\lambda) - \tilde{\xi}_N(\mu)$ has the following form

$$(11) \qquad g_N(t, \lambda, \mu) = g_N(t) = \exp\Big(-it \sum_{t=1}^{N} \mu_j^{(N)}\Big) \prod_{j=1}^{N} |1 - 2it\mu_j^{(N)}|^{-1/2},$$

where $\mu_j = \mu_j^{(N)}, j = 1, 2, \ldots, N$ are the eigenvalues of the matrix MA and M is the covariance matrix of the random vector X. It follows from (11) that

$$\chi_n = 2^{n-1} n! \sum_{j=1}^{N} \mu_j^n,$$

and for $n \geq 2$ we obtain

$$(12) \qquad |\chi_n| \leq 2^{n-2} n! \max_{1 \leq j \leq N} |\mu_j|^{n-2}\, 2 \sum_{j=1}^{N} \mu_j^2 = 2^{n-2} n! \chi_2 \max_{1 \leq j \leq N} |\mu_j|^{n-2}.$$

Notice that $\max|\mu_j| \leq \|MA\| \leq \|M\|\|A\|$. Let $y = (y_1, \ldots, y_N)$ be a unit vector in R^N. Using the fact that M is the covariance matrix of the random vector $(X(1), \ldots, X(N))$ we get

$$\|M\| = \sup_{\|y\|=1} |(My, y)| = \sup_{\|y\|=1} \sum_{t,s=1}^{N} y_t y_s \int_\Pi e^{i\lambda(t-s)} f(\lambda)\, d\lambda$$
$$= \sup_{\|y\|=1} \int_\Pi \left| \sum_{t=1}^{N} y_t e^{i\lambda t} \right|^2 f(\lambda)\, d\lambda.$$

Since

$$\sup_{\|y\|=1} \int_\Pi \left| \sum_{t=1}^{N} y_t e^{i\lambda t} \right|^2 d\lambda = \sup_{\|y\|=1} \int_\Pi \Big(\sum_{t=1}^{N} y_t^2 + 2 \sum_{t \neq s} y_t y_s \cos(t - s)x \Big)\, dx = 2\pi$$

it follows that $\|M\| \le 2\pi C_3$. Using (2) we get

$$\|A\| = \sup_{\|y\|=1} |(A(y,y)| = \sup_{\|y\|=1} \sum_{t,s=1}^{N} \frac{y_t y_s}{2\pi} \sqrt{\frac{B_N}{N}} \times$$

$$\times \int_{\Pi} [\varphi_N(x-\lambda B_N) - \varphi_N(x-\mu B_N)] \cos(t-s)x\, dx$$

$$\le \sup_{\|y\|=1} \frac{1}{2\pi} \sqrt{\frac{B_N}{N}} \int_{\Pi} |\varphi_N(x-\lambda B_N) - \varphi_N(x-\mu B_N)| \left|\sum_{t=1}^{N} y_t e^{ixt}\right|^2 dx$$

$$\le \frac{H|\lambda-\mu|}{\sqrt{NB_N}}.$$

and than we obtain (10) easily.

Proof of Theorem 2. Let $n \in \{1, 2, \dots\}$. Then we have

$$E|\tilde{\xi}_N(\lambda) - \tilde{\xi}_N(\mu)|^{2n} = \sum K_{j_1 j_2 \dots j_{2n}} \chi_1^{j_1} \chi_2^{j_2} \cdots \chi_{2n}^{j_{2n}},$$

where the sum is carried out over all vectors $(j_1, j_2, \dots, j_{2n})$ for which the equation $j_1 + 2j_2 + 3j_3 + \dots + 2nj_{2n} = 2n$ holds. Since $\chi_1 = 0$ all addends for which $j_1 > 0$ vanish.

For the addend corresponding to the vector $(0, j_2, \dots, j_{2n})$ we have

$$|\chi_2^{j_2} \chi_3^{j_3} \cdots \chi_{2n}^{j_{2n}}| \le K_n |\lambda - \mu|^a,$$

where $a = \sum_{s=2}^{2n} (s-1)j_s$. The constant K_n is the same for every N, λ and μ. Since $\sum_{s=2}^{2n} (s-1)j_s \ge 2$ for $n \ge 3$, the desired result follows if we put $\alpha = 6$ and $\varepsilon = 1$.

Proof of Theorem 3. We shall use the following assertion (e.g. Bilingsley [2]): P_N converges weakly to P, when $N \to \infty$, if the finite dimensional distributions of P_N converge weakly to those of P and the family of probability measures $\{P_N, N = 1, 2, \dots\}$ is tight. (The family $\{P_N\}$ is tight if for every positive ε there exists a compact set $S \subset C$, such that $P(S) > 1 - \varepsilon$ for all N.) The tightness of sequence $\{P_N\}$ follows from Theorem 2 and then the desired result follows by Theorem 1.

REFERENCES

[1] R. Bentkus, *On cumulants of the spectrum estimate of a stationary time series*, Lietuvos matematikos rinkinys **16** N4 (1976), 37–61.

[2] P. Bilingsley, *Convergence of Probability Measures*, Wiley, New York, 1968.

[3] P. Mladenović, *On covariance of spectral estimates of stationary random sequence*, Publ. Inst. Math. (Beograd) **49 (63)** (1991), 239–245.

[4] I. Žurbenko, *The Spectral Analysis of Time Series*, North-Holland, Amsterdam, 1986.

Matematički fakultet
Studentski trg 16,
11000 Beograd, Yugoslavia

(Received 03 11 1989)