PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 49 (63), 1991, 17–20

A PROPERTY OF THE NUMBER OF PERFECT MATCHINGS OF A GRAPH

Ivan Gutman

Abstract. Let $x_1, \dots, x_p, y_1, \dots, y_p$ be independent edges of a graph G. Denote the sets $\{x_1, \dots, x_p\}$ and $\{y_1, \dots, y_p\}$ by X and Y, respectively. We consider mappings $F : X \cup Y \to \{0,1\}$. For a given mapping F and two subsets $X_i \subseteq X$ and $Y_j \subseteq Y$, $i, j \in \{1, 2, \dots, 2^p\}$, we define G_{ij} as the subgraph obtained from G by deleting the edge z if $F(z) = 0$ and by deleting the edge z together with its endpoints if $F(z) = 1$, $z \in X_i \cup Y_j$. We prove that the absolute value of $\det \|k_{ij}\|$ is independent of the mapping F, where k_{ij} is the number of perfect matchings of G_{ij}.

In this paper we consider finite graphs without loops and multiple edges. A perfect matching of a graph G is a set of edges of G, such that every vertex of G is the endpoint of exactly one edge form this set. The number of distinct perfect matchings of the graph G is denoted by $k(G)$.

Let p be a positive integer. In this paper we are concerned with graphs possessing at least $2p$ independent edges, i.e. $2p$ edges no two of which have a common endpoint. Let G be such a graph and let x_i, y_i, $i = 1, \dots, p$, be a set of its independent edges. Let further $X = \{x_1, \dots, x_p\}$ and $Y = \{y_1, \dots, y_p\}$ and denote by $\mathcal{P}(X) = \{X_1, X_2, \dots, X_{2^p}\}$ and $\mathcal{P}(Y) = \{Y_1, Y_2, \dots, Y_{2^p}\}$ the power sets of X and Y respectively. Label the subsets of X and Y so that $x_j \in X_i \iff y_j \in Y_i$. Define a mapping $F : X \cup Y \to \{0,1\}$ i.e. for any $z \in X \cup Y$, $F(z) = 1$ or $F(z) = 0$. Let $f = \sum_{z \in X \cup Y} F(z)$. The set of all mappings of $X \cup Y$ onto $\{0,1\}$ is denoted by $\mathcal{F}_p$. Recall that $|\mathcal{F}_p| = 2^{2p}$.

Define two special mappings from $\mathcal{F}_p$:

$$F_0 \text{ has the property } f = 0, \text{ i.e. } \quad z \in X \cup Y \implies F_0(z) = 0,$$
$$F_1 \text{ has the property } f = 2p, \text{ i.e. } \quad z \in X \cup Y \implies F_1(z) = 1.$$

For a given mapping $F \in \mathcal{F}_p$ and a given set of edges $X_i \cup Y_j$ define a subgraph $G_{ij} = G_{ij}(F)$ of the graph G. Let G be obtained by deleting from G the edges

---

AMS Subject Classification (1985): Primary 05 C 70

$z \in X_i \cup Y_j$ and, in addition, deleting the endpoints of those edges $z \in X_i \cup Y_j$ for which $F(z) = 1$. (If $X_i = Y_i = \varnothing$, then $G_{ij} = G$.)

Denote by $k_{ij} = k_{ij}(F)$ the number of perfect matchings of G_{ij}. Let $\mathbf{K}(F) = \|k_{ij}(F)\|$; note that $\mathbf{K}(F)$ is a square matrix of order 2^p. We say that the subset X_i is associated with the i-th row of $\mathbf{K}(F)$ whereas the subset Y_j is associated with the j-th column of $\mathbf{K}(F)$.

THEOREM 1. (a) *If* $p = 1$, *then for all* $F \in \mathcal{F}_1$

$$\det \mathbf{K}(F) = (-1)^f \det \mathbf{K}(F_0). \tag{1}$$

(b) *If* $p > 1$ *then* $\det \mathbf{K}(F)$ *is independent of* $F \in \mathcal{F}_p$.

Proof. We first observe that Theorem 1 holds in a trivial manner if the graph G has no perfect matchings, $k(G) = 0$. Namely, then none of the subgraphs G_{ij} has perfect matchings. Consequently, for all $F \in \mathcal{F}_p$ $\mathbf{K}(F)$ is a zero matrix and $\det \mathbf{K}(F) = 0$.

Suppose therefore that $k(G) > 0$. Then in order to prove Theorem 1 we need the well-known identity [2]

$$k(G) = k(G - x_{uv}) + k(G - u - v) \tag{2}$$

where x_{uv} denotes an edge of G connecting the vertices u and v.

Let $p \geq 1$. Consider a mapping F, $F \in \mathcal{F}_p$, $F \neq F_1$. Then there exists some $z_0 \in X \cup Y$ such that $F(z_0) = 0$. Define a mapping F^* via

$$F^*(z) = F(z) \quad \text{for } z \in X \cup Y \setminus \{z_0\}; \qquad F^*(z_0) = 1.$$

It is then sufficient to prove that for $p > 1$,

$$\det \mathbf{K}(F) = \det \mathbf{K}(F^*). \tag{3}$$

Recall that from the definition of F^* it immediately follows

$$k_{ij}(F) = k_{ij}(F^*) \quad \text{if} \quad z_0 \notin X_i \cup Y_j. \tag{4}$$

We now have to distinguish between two cases. Either $z_0 \in X$ or $z_0 \in Y$. Suppose first that $z_0 \in Y$.

Let Y_j be a subset of Y containing z_0 and let $Y_j \setminus \{z_0\} = Y_{j_0} \in \mathcal{P}(Y)$. Then a special case of (2) is

$$k(G_{ij_0}(F)) = k(G_{ij}(F)) + k(G_{ij}(F^*)), \quad \text{i.e.}$$
$$k(G_{ij}(F)) - k(G_{ij_0}(F)) = -k(G_{ij}(F^*)).$$

This implies that by subtracting the j_0-th column of $K(F)$ from the j-th column we obtain a matrix of the form

$$\begin{Vmatrix} k_{11}(F) & \dots & k_{1,j-1}(F) & -k_{1j}(F^*) & k_{1,j+1}(F) & \dots & k_{1,2^p}(F) \\ k_{21}(F) & \dots & k_{2,j-1}(F) & -k_{2j}(F^*) & k_{2,j+1}(F) & \dots & k_{2,2^p}(F) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ k_{2^p,1}(F) & \dots & k_{2^p,j-1}(F) & -k_{2^p,j}(F^*) & k_{2^p,j+1}(F) & \dots & k_{2^p,2^p}(F) \end{Vmatrix}$$

Since by such a transformation the value of the determinant is not changed we immediately arrive at

$$\det \mathbf{K}(F) = -\det \mathbf{K}^j(F),$$

where $\mathbf{K}^j(F)$ is the matrix obtained by writing F^* instead of F in the j-th column of $K(F)$.

The above described construction is to be repeated for all columns of $\mathbf{K}(F)$ associated with the subsets of Y containing z_0. It is convenient to start with such subsets of greatest cardinality and to end with subsets of smallest cardinality. Bearing in mind (4) we then finally arrive at

$$\det \mathbf{K}(F) = (-1)^r \det \mathbf{K}(F^*) \tag{5}$$

where r is the number of times the construction has been repeated. Clearly, r is equal to the number of subsets of Y containing z_0, i.e. $r = 2^{p-1}$.

Whence, if $p > 1$ then r is even and formula (3) follows. If $p = 1$ then r is odd and (5) leads to (1).

If $z_0 \in X$ then a fully analogous reasoning can be applied, except that, of course, in this case we have to transform the pertinent rows of $\mathbf{K}(F)$.

This completes the proof of Theorem 1. □

COROLLARY 1.1. *For* $p \geq 1$

$$\det \mathbf{K}(F_0) = \det \mathbf{K}(F_1). \tag{6}$$

Proof. For $p > 1$ the equation (6) is just a special case of Theorem 1 (b). If $p = 1$ then (6) follows from (1) and the fact that for F_1, $f = 2$. □

A result equivalent to Corollary 1.1 was reported (without proof) in a recent paper [**1**].

COROLLARY 1.2. *If $z^* \in X \cup Y$ is an edge contained in all perfect matchings of the graph G then for all $F \in \mathcal{F}_p$, $p \geq 1$, the determinant of* $\mathbf{K}(F)$ *is equal to zero.*

Proof. Without loss of generality we may assume that $z^* \in X$. Choose a mapping F from $\mathcal{F}_p$ for which $F(z^*) = 0$. Then all elements of $\mathbf{K}(F)$ lying on rows associated with the subgraphs of X containing z^* are equal to zero and therefore $\det \mathbf{K}(F) = 0$. Because of Theorem 1 this latter equality holds for all mappings from $\mathcal{F}_p$. □

COROLLARY 1.3. *If $z^* \in X \cup Y$ is an edge not contained in any perfect matching of the graph G then for all $F \in \mathcal{F}_p$, $p \geq 1$, the determinant of* $\mathbf{K}(F)$ *is equal to zero.*

Proof is analogous: choose a mapping F for which $F(z^*) = 1$. □

COROLLARY 1.4. *If G has a unique perfect matching then for all $F \in \mathcal{F}_p$, $p \geq 1$,* $\det \mathbf{K}(F) = 0$.

COROLLARY 1.5. *If G is a forest then for all $F \in \mathcal{F}_p$, $p \geq 1$,* $\det \mathbf{K}(F) = 0$.

If G is a graph containing circuits then $\det \mathbf{K}(F)$ needs not be equal to zero. The simplest example of this kind is provided by the four-membered circuit C_4. In this graph we may choose two independent edges x and y (whence $p = 1$). Then $\mathcal{F}_p$ has four elements. For the mappings $F(x) = F(y) = 0$ and $F(x) = F(y) = 1$ we have $\mathbf{K}(F) = \left\| \begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix} \right\|$, whereas for the mappings $F(x) = 0$, $F(y) = 1$ and $F(x) = 1$, $F(y) = 0$ we have $\mathbf{K}(F) = \left\| \begin{smallmatrix} 2 & 1 \\ 1 & 0 \end{smallmatrix} \right\|$. Consequently, in the case of the graph C_4 the determinant of $\mathbf{K}(F)$ differs from zero for all $F \in \mathcal{F}_p$. This example also illustrates Theorem 1a.

The proof of Theorem 1 is solely based on the recurrrence relation (2). Therefore any other graph invariant $I(G)$ conforming to the recurrence relation

$$I(G) = I(G - x_{uv}) + I(G - u - v) \tag{7}$$

will possess a fully analogous property:

THEOREM 2. *Let $I(G)$ be a graph invariant conforming to eq.* (7). *Then Theorem* 1 *remains valid if the elements of the matrix* $\mathbf{K}(F)$ *are interpreted as* $I(G_{ij})$.

With minor modifications in the proof of Theorem 1 we arrive at another result of this kind.

THEOREM 3. *Let $J(G)$ be a graph invariant such that for each pair of adjacent vertices u, v the recurrence relation* (8) *holds*:

$$J(G) = J(G - x_{uv}) - J(G - u - v). \tag{8}$$

Then for all $p \geq 1$, $\det \mathbf{K}(F)$ *is independent of $F \in \mathcal{F}_p$ provided the elements of the matrix* $\mathbf{K}(F)$ *are interpreted as* $J(G_{ij})$.

REFERENCES

[1] I. Gutman, D. J. Klein, *Generalization of Cyvin's formula*, J. Math. Chem. 6 (1991), 41–50.

[2] L. Lovász, M. D. Plummer, *Matching Theory*, North-Holland, Amsterdam 1986, 335.

Faculty of Science
University of Kragujevac, P.O. Box 60
YU-34000 Kragujevac, Yugoslavia

(Received 28 03 1990)