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ALMOST SURE SAMPLING RECONSTRUCTION OF NON-BAND-LIMITED HOMOGENEOUS RANDOM FIELDS

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Abstract. Almost sure convergence problems of sampling partial (and cardinal) expansion sum sequence are discussed for non-band-limited homogeneous random fields with not necessarily continuous spectral distribution function.

I. Introduction

The reconstruction of a random signal requires only that the sample interval h/π is smaller then the reciprocal of the bandwidth w of the non-negligible frequency components in the spectral distribution function of the signal.

Because the sampling representation (in the mean-square and almost sure sense) is usually not discussed in the case $h = \pi/w$, the first purpose of the paper is to specify the behaviour of the truncation error of band-limited (BL) homogeneous random fields (HRF) with closed sampling interval $\mathcal{S}$. A special importance will be given to the continuity properties of the spectral distribution function of the HRF at the points of $\partial\mathcal{S}$. Also some mean-square (m.s.) and almost sure (a.s.) convergence results are derived in the section III. In the sequel we generalize the results of foregoing sections to the non-band-limited HRF class. With the help of combined aliasing and truncation error we shall prove that for the a.s. convergence of the sampling cardinal series expansion sequence to the intial HRF always holds (section IV). We can also remark that the reference list of the sampling reconstruction of non-BL, HRF is very short [10], [20]. The author didn't find a.s. results in the multidimensional stochastic sampling except for the short remark at the beginning of the page 168 in [22]. So this paper completes the literature of the HRF sampling on the rectangular lattice in BL and non-BL cases in above sense.

The mean-square sampling representation of a weakly stationary (WS), BL stochastic process holds if the spectral distribution function is continuous at the endpoints of the closed sampling interval, [1], [4], [21].

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The case $h = \pi/w$ is avoided (or ignored) in the multidimensional sampling in many papers in few methods. This methods and some related results are listed in short in the following.

(A) Wider bandwidth approach. If the random signal (process or field) is band-limited to the bandwidth w, it is band-limited to any $\tilde{w} > w$. Since the spectral distribution function possesses an at most countable discontinuity set Λ_d the convenient choice of $\tilde{w}$ distinct of Λ_d is always possible. This method is due to Belyaev and was generalized to vectorial BL and non-BL processes in [**18**]. For the sampling of HRF (BL and non-BL case) in a wider bandwidth manner consult [**19**].

(B) Open sampling interval. Gaarder derived periodic sampling results on nonrectangular lattices for fields [**8**]. This approach exploite Petersen-Middleton (2-, and 3-dimensional fields on rectangular lattices) [**15**], [**16**]; Barakat [**2**]; Montgomery [**12**]; Lloyd (open support, non-BL WS processes) [**11**], and Pogány [**17**].

(C) Continuity properties, special function classes. For deterministic signals Parzen supposed that $F \in C(\partial\Pi)$, [**14**], where $\partial\Pi$ consists of all vertex points of the multidimensional rectangle Π. Butzer considers uniformly bounded deterministic signals from $C(\mathbf{R}^n) \cap L(\mathbf{R}^n)$, [**5**], [**6**]. Splettstößer considers uniformly continuous bounded functions on $\mathbf{R}^n$ [**20**].

Unfortunatelly the author doesn't have insight into paper of Miyakawa (in Japanese) listed in [**10**].

Finally there are undefined cases where continuity properties on ∂S are omitted [**9**], [**22**]. Papoulis deals with bounded sampling region but he doesn't specify the closeness of this region [**13**]. We shall show that without this condition on $\partial\Pi$ sampling representations do not hold.

II. Preliminary definitions and results

In his historical paper Belyaev has proved that the partial sampling expansion sum of a band-limited mean-square continuous, wide-sense stationary stochastic process converges with probability 1 (in other words almost surely) to the initial process. He used the method (A).

Following some Belyaev's ideas in the paper [**19**] there are given similar almost sure convergence results for homogeneous random fields in the band-limited and also in the non-band-limited case.

Belyaev's idea was: "Let $\{X(t), t \in \mathbf{R}\}$ be a WSP, BL to the frequency $w > 0$. Then $X(t)$ is BL to frequency $\tilde{w} > w$ too. Hence from the mean-square sampling representation

$$X(t) = \sum_{-\infty}^{\infty} X(n\pi/w)\,\mathrm{sinc}(wt - n\pi)$$

it follows also in the mean-square sense that

$$X(t) = \sum_{-\infty}^{\infty} X(n\pi/\tilde{w})\,\mathrm{sinc}(\tilde{w}t - n\pi)$$

where $\mathrm{sinc}(x) := x^{-1}\sin(x)$.

As the sampling expansion partial sum

$$\tilde{X}_N(t) = \sum_{-N}^{N} X(n\pi/\tilde{w})\,\mathrm{sinc}(\tilde{w}t - n\pi)$$

gives a mean-square truncation error

$$\tilde{e}_N(t) = E|X(t) - \tilde{X}_N(t)|^2 = O(N^{-2})$$

from the convergence of the series $\sum_N \tilde{e}_N(t)$ by the Borel-Cantelli lemma it follows that $P\{\lim_{N\to\infty} \tilde{X}_N(t) = X(t)\} = 1$.

Denote by $\mathbf{N}$, $\mathbf{Z}^q$, $\mathbf{R}^q$ the set of all positive integers, q-tuples of integers, q-tuples of real numbers.

With the aid of the wider bandwidth method has shown that a BL homogeneous random field $\{\xi(x), x \in \mathbf{R}^q\}$ which is BL to the frequency $W = (w_1, \dots, w_q)$ (and also BL to the wider bandwidth $\tilde{W} = (\tilde{w}_1, \dots, \tilde{w}_q)$, $\tilde{w}_j > w_j$, $j = 1, \dots, q$), has a mean-square sampling representation

$$\xi(x) = \sum_n \xi(x^n) \prod_{k=1}^{q} \mathrm{sinc}(\tilde{w}_k x_k - n_k\pi),$$

where $n = (n_1, \dots, n_q) \in \mathbf{Z}^q$, $x = (x_1, \dots, x_q) \in \mathbf{R}^q$, and x^n runs over all points of the lattice $\mathrm{Lat}(\tilde{w}) = \{x^n = (\pi n_1/\tilde{w}_1, \dots, \pi n_q/\tilde{w}_q); n_k \in \mathbf{Z}\}$. Put

$$\tilde{\xi}_N(x) := \sum_{|n|\le N} \xi(x^n) \prod_{k=1}^{q} \mathrm{sinc}(\tilde{w}_k x_k - n_k\pi),$$

then the mean-square truncation error possesses the convergence rate

$$\tilde{\mathfrak{E}}_N(x) = E|\xi(x) - \tilde{\xi}_N(t)|^2 = O(N_+^{-2q}),$$

where $|n| \le N$ means that $|n_i| \le N_i$, $i = 1, \dots, q$, $N_+ = \min_i\{N_i\}$. As

$$\sum_{N_+} \tilde{\mathfrak{E}}_n(x) < \infty,$$

it follows immediately that $\tilde{\xi}_N(x) \to \xi(x)$ almost surely as N_+ tends to inifinity, **[19]**. See also [22, pp. 167–168] for the sampling expansion of a spherically BL, HRF.

III. Derivation of truncation error

Before giving the results on the truncation error upper bound we repeat some results from Fourier-series expansion of the function $\exp(i\lambda t)$.

Denote by $\mathbf{1}\{\cdot\}$ the indicator of the event $\{\cdot\}$.

LEMMA 1. *Let* $\alpha := \mathbf{1}\{\lambda \in (-w,w)\}$, *r arbitrary positive integer,* $N \geq 2$. *Then for all* $\lambda \in [-w,w]$ *and all* $t \in \mathbf{R}$

$$\sum_{-N}^{N} \exp(in\lambda\pi/w)\,\mathrm{sinc}(wt - n\pi) + O(N^{-\alpha(r-1)-1}\ln^{\alpha} N) =$$

$$= \alpha \exp(i\lambda t) + (1-\alpha)\cos wt. \quad (3.1)$$

Proof. Consider a $2w$-periodic, r-times derivable function $f(\lambda)$ such that $|f^{(r)}(\lambda)| \leq M_r$. Then the remainder $R_N(f)$ of the symmetric complex Fourier-expansion of $f(\lambda)$ on $(-w,w)$ is bounded above, namely:

$$|R_N(f)| \leq AM_r(w/\pi)^r N^{-r} \ln N. \quad (3.2)$$

Here A is an absolute constant. This result of Bernstein is treated in detail in [7]. For instance it is suggested that $A = 2 + (1 + \ln\pi)/\ln 2$.

The remainder of the Fourier-series of $e^{i\lambda t}$ on $[-w,w]$ is $R_N(e^{i\lambda t}) = \sum_{|n|>N} \exp(in\pi\lambda/w)\,\mathrm{sinc}(wt - n\pi)$. From (3.2) it follows:

$$|R_N(e^{i\lambda t})| \leq A(w|t|/\pi)^r N^{-r} \ln N \quad (3.3)$$

for all $\lambda \in (-w,w)$.

It is obvious that $\cos(wt) = \sum_{-\infty}^{+\infty}(wt - n\pi)^{-1}\sin(wt)$. Since

$$\sum_{-N}^{N} \exp(in\lambda\pi/w)\,\mathrm{sinc}(wt - n\pi) = \sum_{-N}^{N}(wt - n\pi)^{-1}\sin(wt)$$

at the points $\lambda = \pm w$, and for sufficiently large N

$$\left|\cos(wt) - \sum_{-N}^{N}(wt - n\pi)^{-1}\sin(wt)\right| \leq \left|\sum_{|n|>N}(wt - n\pi)^{-1}\right|$$

$$\leq 2w|t|\pi^{-2}\sum_{N+1}^{\infty}|(n\pi)^2 - (wt)^2|^{-1} < 2w|t|\pi^{-2}\sum_{N+1}^{\infty} n^{-2} < 2w|t|\pi^{-2}/N,$$

by (3.3) follows the assertion of the Lemma 1. ■

Remark 1. By the result (3.1) it is easy to prove that

$$\sum_{|n|\leq N} \exp(in\pi\lambda/w)\,\mathrm{sinc}(wt - n\pi) =$$

$$= \left(e^{it\lambda} + O(N^{-r}\ln N)\right)^{\alpha}\left(\cos(wt) + O(N^{-1})\right)^{1-\alpha}. \quad (3.4)$$

This form of relation (3.1) is more useful for the derivation of the q-dimensional variant of Lemma 2. □

Denote by $\langle a, b\rangle = \sum_1^q a_j b_j$ the inner product of the vectors $a, b \in \mathbf{R}^q$. Without changing the order of coordinates of the vector $a = (a_1, \dots, a_q)$ we choose p coordinates from a, $0 \le p \le q$. Let a^p denotes the new vector which consists of such p coordinates, and let $\underline{a}^{q-p}$ denotes its "complement" with respect to a. So $\langle a^p, b^p\rangle = \sum_1^p a_{k_j} b_{k_j}$, $k_j \in \{1, \dots, q\}$. A closed cube and a rectangular lattice in $\mathbf{R}^q$ are denoted by $\Pi := \mathrm{X}_1^q[-w_k, w_k]$ and $\mathrm{Lat}_N(W) = \{x^n = (\pi n_1/w_1, \dots, \pi n_q/w_q), |n| \le N\}$ respectively, where $W = (w_1, \dots, w_q)$ is the q-dimensional sampling frequency. There will be no difficulties to recognize the difference between partial time-vector x^p and the lattice point x^n.

LEMMA 2. *Let $r \in \mathbf{N}$, $\alpha_k := 1\{\lambda_k \in (-w_k, w_k)\}$, $k = \overline{1,q}$, $\beta := 1\{r = 1\}$, $N_+ := \min\{N_1, \dots, N_q\} \ge [\exp(1/r)] + 1$. Then we have*

$$\sum_{|n| \le N} e^{i\langle\lambda, x^n\rangle} \prod_1^q \operatorname{sinc}(w_k x_k - n_k\pi) + O(N_+^{-\beta(r-1)-1} \ln^\beta N_+) =$$

$$= e^{i\langle\lambda, x\rangle} \prod_1^q \alpha_k + \prod_1^q (1-\alpha_k)\cos(w_k x_k) +$$

$$+ \sum_{p=1}^{q-1} \sum_{k_1 < k_2 < \dots < k_q} e^{i\langle\lambda^p, x^p\rangle} \left(\prod_1^p \alpha_{k_j}\right) \prod_{m=p+1}^q (1-\alpha_{k_m})\cos(w_{k_m} x_{k_m}) \tag{3.5}$$

where $x^n \in \mathrm{Lat}_N(W)$ and $x^p := (x_{k_1}, \dots, x_{k_p})$. Relation (3.5) holds for all $x \in \mathbf{R}^q$ and all $\lambda \in \Pi$.

Proof. By (3.4) we obtain:

$$\sum_{|n| \le N} e^{i\langle\lambda, x^n\rangle} \prod_1^q \operatorname{sinc}(w_k x_k - n_k\pi) =$$

$$= \prod_1^q \left(\exp(i\lambda_k x_k) + O(N_k^{-r} \ln N_k)\right)^{\alpha_k} \left(\cos(w_k x_k) + O(N_k^{-1})\right)^{1-\alpha_k} =: E_q.$$

In special cases if $\alpha_k = 1$ ($\alpha_k = 0$), $0 \le k \le q$ we get

$$E_q = e^{i\langle\lambda, x\rangle} + O(N_+^{-r} \ln N_+) \left(= \prod_1^q \cos(w_k x_k) + O(N_+^{-1})\right)$$

respectively.

The general case gives us

$$\sum_{p=1}^{q-1} \sum_{k_1 < \dots < k_q} e^{i\langle\lambda^p, x^p\rangle} \left(\prod_1^p \alpha_{k_j}\right) \prod_{m=p+1}^q (1-\alpha_{k_m})\cos(w_{k_m} x_{k_m}) +$$

$$+ O\left(\bigvee_1^q N^{-\alpha_k(r-1)-1} \ln^{\alpha_k} N_k\right).$$

This completes the proof of the lemma. ■

Let $\{\xi(x), x \in \mathbf{R}^q\}$ be a BL to W, HRF with $E\xi(x) = 0$; the correlation function of the observed field is $K_\xi(x-y) = E\xi(x)\xi^*(y)$ with the variance $\mathfrak{S}^2 = K_\xi(0)$. The spectral representations of $\xi(x)$ and $K_\xi(\tau)$ are

$$\xi(x) = \int_{-W}^{W} e^{i(\lambda,x)}\, dZ(\lambda); \quad K_\xi(\tau) = \int_{-W}^{W} e^{i(\lambda,\tau)}\, dF(\lambda)$$

respectively. Here $\int_{-W}^{W} := \int_{-w_1}^{w_1} \cdots \int_{-w_q}^{w_q}$; $Z(\lambda)$ is the spectral process and $F(\lambda)$ the spectral distribution function of the HRF $\xi(x)$. Suppose $F(\lambda)$ is left continuous with respect to λ_i, $i = \overline{1,q}$. Put

$$\xi_N(x) = \sum_{|n|\le N} \xi(x^n) \prod_{k=1}^{q} \operatorname{sinc}(w_k x_k - n_k \pi) \tag{3.6}$$

where $x^n \in \mathrm{Lat}_N(W)$.

At first we are interested in the mean-square sampling truncation error. From a general point of view the masses of $F(\lambda)$ at all points of $\partial\Pi$ have an important role in the evaluation of the truncation error. Namely we shall show that if $F(\lambda)$ possesses discontinuities on $\partial\Pi$ the truncation error does not vanish if $N_+ \to \infty$. For example suppose that $F(\lambda)$ is discontinuous at $\pm w_1$. Then $F(\pm w_1; \underline{\lambda}^{q-1}) = F(w_1+; \underline{\lambda}^{q-1}) - F(w_1; \underline{\lambda}^{q-1}) + F(-w_1+; \underline{\lambda}^{q-1}) - F(-w_1; \underline{\lambda}^{q-1})$.

THEOREM 1. *Under the conditions of the Lemma 2 it follows that the mean-square truncation error* $\mathfrak{E}_N(x) = E|\xi(x) - \xi_N(x)|^2$ *for all* $x \in \mathbf{R}^q$ *is*

$$\mathfrak{E}_N(x) = \sum_{p=0}^{q} \sum_{k_1<k_2<\ldots<k_q} \left| 1 - e^{i(\underline{W}^{q-p}, \underline{x}^{q-p})} \prod_{p+1}^{q} \cos(w_{k_j} x_{k_j}) \right|^2 \times$$
$$\times \operatorname*{Var}_{\mathfrak{W}^p} F(\lambda^p; \pm \underline{W}^{q-p}) + O(N_+^{-\beta(r-1)-1} \ln^\beta N_+). \tag{3.7}$$

where $\mathfrak{W}^p := \mathrm{X}_1^p(-w_{k_j}, w_{k_j})$.

Proof. Put $\Delta_N(\lambda) := e^{i(\lambda,x)} - \sum_{|n|\le N} e^{i(\lambda,x^n)} \prod_1^q \operatorname{sinc}(w_k x_k - n_k\pi)$. Then it follows clearly by Lemma 2 that

$$\Delta_N(\lambda) = \left(1 - \prod_1^q \alpha_k\right) e^{i(\lambda,x)} - \prod_1^q (1-\alpha_k)\cos(w_k x_k) -$$
$$- \sum_{p=1}^{q} \sum_{k_1<\ldots<k_q} e^{i(\lambda^p, x^p)} \left(\prod_1^p \alpha_{k_j}\right) \prod_{p+1}^{q} (1-\alpha_{k_m})\cos(w_{k_m} x_{k_m}) +$$
$$+ O(N_+^{-\beta(r-1)-1} \ln^\beta N_+). \tag{3.8}$$

Fixing p and $(k_1, \ldots, k_q)$ we specify the vector $\underline{\lambda}^{q-p}$ at $\pm\underline{W}^{q-p}$. That means $\lambda^p \in \mathfrak{W}^p$; $\underline{\lambda}^0 := \lambda^q$; $\underline{\lambda}^q := \pm W$; $\alpha_{k_1} = \ldots = \alpha_{k_p} = 1$ and $\alpha_{k_{p+1}} = \ldots = \alpha_{k_q} = 0$.

Therefore (3.8) becomes

$$[\Delta_N(\lambda)]_{\pm \underline{W}^{q-p}} = e^{i(\lambda^p, x^p)} \left(e^{i(\pm \underline{W}^{q-p}, \underline{x}^{q-p})} - \prod_{p+1}^{q} \cos(w_{k_m} x_{k_m}) \right) + \\ + O(N_+^{-\beta(r-1)-1} \ln^\beta N_+). \quad (3.9)$$

By direct computation we obtain

$$\mathfrak{E}_N(x) = \int_{-W}^{W} |\Delta_N(\lambda)|^2 \, dF(\lambda) = \int_{-W+}^{W-} |\Delta_N(\lambda)|^2 \, dF(\lambda) + [|\Delta_n|^2 F]_{\lambda = \pm W} + \\ + \sum_{p=1}^{q-1} \sum_{k_1 < k_2 < \ldots < k_q} \int_{-w_{k_1}+}^{w_{k_1}-} \cdots \int_{-w_{k_p}+}^{w_{k_p}-} [|\Delta_N(\lambda)|^2 \, dF(\lambda)]_{\underline{\lambda}^{q-p} = \pm \underline{W}^{q-p}}.$$

As

$$[|\Delta_N(\lambda)|^2]_{\pm \underline{W}^{q-p}} = \begin{cases} O(N_+^{-2r} \ln^2 N_+) & p = q \\ \left|1 - e^{i(\pm \underline{W}^{q-p}, \underline{x}^{q-p})} \prod_{p+1}^{q} \cos(w_{k_j} x_{k_j})\right|^2 + & \\ \quad + O(N_+^{-\beta(r-1)-1} \ln^\beta N_+) & 0 \le p < q \end{cases} \quad (3.10)$$

we can see that $[|\Delta_n(\lambda)|^2]_{\pm \underline{W}^{q-p}}$ does not depend of the sign of $\pm \underline{W}^{q-p}$. Therefore we can assume:

$$\mathfrak{E}_N(x) = \sum_{p=1}^{q} \sum_{k_1 < \ldots < k_q} \left| 1 - e^{i(\underline{W}^{q-p}, \underline{x}^{q-p})} \prod_{1}^{p} \cos(w_{k_j} x_{k_j}) \right|^2 \times \\ \times \operatorname*{Var}_{\mathfrak{W}^p} F(\lambda^p; \pm \underline{W}^{q-p}) + O(N_+^{-\beta(r-1)-1} \ln^\beta N_+) + O(N_+^{-2r} \ln^2 N_+). \quad (3.11)$$

Finally, it is not hard to show that (3.11) proves the assertion of Theorem 1. ■

We can write:

$$\mathfrak{E}_N(x) = \int_{\Pi} |\Delta_N(\lambda)|^2 \, dF(\lambda) = \int_{\mathfrak{W}} |\Delta_N(\lambda)|^2 \, dF(\lambda) + \\ + \sum_{p=1}^{q} \int_{\mathfrak{W}^p} [|\Delta_N(\lambda)|^2 \, dF(\lambda)]_{\pm \underline{W}^{q-p}}.$$

Since $|\Delta_N(\lambda)|^2$ does not depend of λ we get from (3.10) that

$$\mathfrak{E}_N(x) = O(N_+^{-2r} \ln^2 N_+) + \sum_{p=1}^{q} \sum_{k_1 < \ldots < k_q} [|\Delta_N(\lambda)|^2]_{\underline{W}^{q-p}} \times \\ \times \operatorname*{Var}_{\mathfrak{W}^p} F(\lambda^p; \pm \underline{W}^{q-p}). \quad (3.12)$$

Suppose F is continuous at all points of $\partial \Pi$. So $\mathrm{Var}_{\mathfrak{W}^p} F \equiv 0$. For the case $q = 1$ consult **[1]**, **[21]**. Finally we have

$$\mathfrak{E}_N(x) = O(N_+^{-2r} \ln^2 N_+). \quad (3.13)$$

So we complete the proof of

PROPOSITION 1. *If* $F \in C(\partial\Pi)$ *then* $\operatorname{l.i.m.}_{N_+\to\infty} \xi_N(x) = \xi(x)$. ■

The related a.s. convergence result is established in

PROPOSITION 2. *If* $F \in C(\partial\Pi)$, *then* $P\{\lim_{N_+\to\infty} \xi_N(x) = \xi(x)\} = 1$.

Proof. As by (3.13) $\sum_{N_+} \mathfrak{E}_N(x)$ converges, from the Borel-Cantelli lemma the assertion follows. ■

When $F(\lambda)$ is discontinuous in at least one of the points of Π then $\mathfrak{E}_N(x)$ contains a nonzero term. In the sequel denote by $\mathfrak{U}_q$ the constant term in (3.11), i.e.

$$\mathfrak{U}_q := \sum_{p=1}^{q} \sum_{k_1<\dots<k_q} \left|1 - e^{i\langle \underline{W}^{q-p}, \underline{x}^{q-p}\rangle} \prod_{p+1}^{q} \cos(w_{k_j} x_{k_j})\right|^2 \times \\ \times \operatorname*{Var}_{\mathfrak{W}^p} F(\lambda^p; \pm \underline{W}^{q-p}). \quad (3.14)$$

Examples. For a WS, BLPs is $\mathfrak{U}_1 = \sin^2(wt)\big(F(w+) - F(w) + F(-w+) - F(-w)\big)$. This result was shown by Balakrishnan and by Wong, but they did not evaluate the convergence rate of the truncation error [1], [21].

Similarly a 2-dimensional case gives us

$$\begin{aligned}\mathfrak{U}_2 = {} & \sin^2(w_1x_1)(F(w_1+, w_2-) - F(w_1, w_2-) - F(w_1+, -w_2+) - \\ & - F(w_1, -w_2+)) + \sin^2(w_2x_2)(F(w_1-, w_2+) - F(w_1-, w_2) - \\ & - F(-w_1+, w_2+) + F(-w_1+, w_2)) + |1 - \exp(i\langle W, x\rangle) \times \\ & \times \Pi_1^2 \cos(w_kx_k)|^2 (F(w_1+, w_2+) - F(w_1, w_2+) - F(w_1+, w_2) + \\ & + F(w_1, w_2) + \dots + F(-w_1+, -w_2+) - F(-w_1, -w_2+) - \\ & - F(-w_1+, -w_2) + F(-w_1, -w_2)). \quad \square\end{aligned}$$

IV. Non-band-limited homogeneous random fields

A non-BL, HRF $\{\xi(x), x \in \mathbf{R}^q\}$ possesses the spectral representation $\xi(x) = \int_{\mathbf{R}^q} e^{i\langle\lambda,x\rangle}\, dZ(\lambda)$. Its correlation function $K_\xi(\tau) = E\xi(x)\xi^*(y)$, $\tau = x - y$, has the spectral representation $K_\xi(x) = \int_{\mathbf{R}^q} e^{i\langle\lambda,x\rangle}\, dF(\lambda)$.

Consider the set Λ_d of all discontinuity points of the spectral distribution function $F(\lambda)$. It is well-known that Λ_d is at most countable and $\sum_{\Lambda_d} F \le \mathfrak{S}^2$.

Consider a q-dimensional closed cube $\Pi = \mathsf{X}_{j=1}^q[-w_j, w_j]$. We consider the indicator function $\mathbf{1}_\Pi := \mathbf{1}\{\lambda \in \Pi\}$ as a spectral characteristic of a filter $\mathcal{L}$ of a HRF $\xi(x)$. So we get

$$\mathcal{L}\xi(x) = \mathcal{L}\int_{\mathbf{R}^q} e^{i\langle\lambda,x\rangle}\, dZ(\lambda) = \int_{\mathbf{R}^q} e^{i\langle\lambda,x\rangle}\, \mathbf{1}_\Pi\, dZ(\lambda) = \int_{-W}^{W} e^{i\langle\lambda,x\rangle}\, dZ(\lambda). \quad (4.1)$$

A positive, monotonous increasing sequence of real numbers $\{w_{jk}\}_1^\infty$, $j = \overline{1,q}$, divergent to ∞ gives us a sequence of monotonous increasing q-dimensional cubes $\Pi(k) := \mathrm{X}_{j=1}^q[-w_{jk}, w_{jk}]$, $k \in \mathbf{N}$. With the help of $\{\Pi(k)\}_1^\infty$ we define

(i) a sequence of spectral characteristics $\mathbf{1}_{\Pi(k)} \to \mathbf{1}_{\mathbf{R}^q} \equiv 1$ if $k \to \infty$.

(ii) a sequence of filters $\{\mathcal{L}_k\}_1^\infty$, such that

$$\mathcal{L}_k\xi(x) = \int_{\Pi(k)} e^{i\langle\lambda,x\rangle}\, dZ(\lambda).$$

If $\xi(x)$ is defined on the probability space $(\Omega, \mathcal{F}, P)$, $\mathcal{L}_k\xi(x)$, $k \in \mathbf{N}$, is defined on $(\Omega, \mathcal{F}, P)$ too.

(iii)
$$E|\xi(x) - \mathcal{L}_k\xi(x)|^2 = \int_{\Pi'(k)} dF(\lambda),\ \Pi'(k) := \mathbf{R}^q \setminus \Pi(k).$$

Denote by $\mathcal{B}_{\mathbf{R}^q}$ the σ-algebra of Borel sets of $\mathbf{R}^q$. As $F(\cdot)$ is a finite Borel measure on $\mathbf{R}^q$ and $F(B) = \int_B dF(\lambda)$, $B \in \mathcal{B}_{\mathbf{R}^q}$, by putting $P\{\cdot\} = \mathfrak{S}^{-2}F(\cdot)$ we get

$$E|\xi(x) - \mathcal{L}_k\xi(x)|^2 = \mathfrak{S}^2 P\{\Pi'(k)\}. \tag{4.2}$$

(iv)
$$\operatorname*{l.i.m.}_{k\to\infty} \mathcal{L}_k\xi(x) = \xi(x) \text{ for all } x \in \mathbf{R}^q.$$

The proof of (iv) is obvious.

In this way we get a sequence of BL, HRFs $\{\mathcal{L}_k\xi(x)\}_1^\infty$. Therefore we can apply the results of the foregoing sections to $\{\mathcal{L}_k\xi(x)\}_1^\infty$ elementwise.

Since Λ_d is at most countable, then $\{w_{jk}\}_1^\infty$, $j = \overline{1,q}$ can be chosen distinct from Λ_d. This method was used in **[17]**, **[18]**.

Remark 2. In current considerations we do not suppose that $\Lambda_d \cap \{w_{jk}\} = \varnothing$. However, it is convenient to use such $\{w_{jk}\}$ that Λ_d is a subset of $\{\pm w_{jk}\}$ and $\{w_{jk}\}$ is an increasing positive sequence which diverges to ∞, whenever it is possible. □

Finally, let $\xi_{N,k}(x)$ be just the sampling partial expasion sum of the BL, HRF $\mathcal{L}_k\xi(x)$, namely:

$$\xi_{N,k}(x) = \sum_{|n|\le N} \mathcal{L}_k\xi(x^n) \prod_{j=1}^q \operatorname{sinc}(w_j x_j - n_j\pi) \tag{4.3}$$

where x^n runs over $\mathrm{Lat}_N(W_k) = \{(\pi n_1/w_{1k}, \ldots, \pi n_q/w_{qk}),\ |n_i| \le N_i,\ j = \overline{1,q}\}$. From Theorem 1 it follows that

$$E|\mathcal{L}_k\xi(x) - \xi_{N,k}(x)|^2 = \mathfrak{U}_{qk} + O(N_+^{-\beta(r-1)-1}\ln^\beta N_+), \tag{4.4}$$

where $N_+ = \min(N_1, \ldots, N_q)$, $\beta = \mathbf{1}\{r = 1\}$ and $\mathfrak{U}_{qk}$ is the term like (3.14) with respect to the sampling interval $\Pi(k)$.

LEMMA 3. *For all $x \in \mathbf{R}^q$; $k, N \in \mathbf{N}$ the following holds*

$$\xi(x) - \mathcal{L}_k\xi(x) \perp \mathcal{L}_k\xi(x) - \xi_{N,k}(x). \tag{4.5}$$

Proof. Since $Z(\lambda)$ is a process with orthogonal increments and because

$$E\big(\xi(x)-\mathcal{L}_k\xi(x)\big)\big(\mathcal{L}_k\xi(x)-\xi_{N,k}(x)\big)^* =$$
$$= \int_{\mathbf{R}^q}\int_{\mathbf{R}^q} e^{i\langle\lambda,x\rangle}\Big(e^{-i\langle\lambda,x\rangle} - \sum_{|n|\le N} e^{-i\langle\lambda,x\rangle}\prod_1^q \cos(w_{jk}x_{jk}-n_j\pi)\Big)\times$$
$$\times\, \mathbf{1}_{\Pi'(k)}\mathbf{1}_{\Pi(k)}E\,dZ(\lambda)\,dZ^*(\lambda)\equiv 0 \quad (4.6)$$

we have that (4.5) follows. ■

THEOREM 2. *The mean-square truncation error of the sampling expansion of a non-BL, HRF $\xi(x)$ is*

$$\mathfrak{E}_{N,k}(x) = E|\xi(x)-\xi_{N,k}(x)|^2 = \mathfrak{S}^2 P\{\Pi'(k)\} + \mathfrak{U}_{qk} + O(N_+^{-\beta(r-1)-1}\ln^\beta N_+). \quad (4.7)$$

Proof. Since $\mathfrak{E}_{N,k}(x) = E|\xi(x)-\mathcal{L}_k\xi(x)|^2 + E|\mathcal{L}_k\xi(x)-\xi_{N,k}(x)|^2$ from (4.4) and (4.5) the assertion follows. ■

Remark 3. The relation (4.4) shows that $\xi_{N,k}(x)$ does not converge to $\mathcal{L}_k\xi(x)$ in the mean-square or almost surely as $N_+\to\infty$ when $F\notin C(\partial\Pi(k))$, because then $\mathfrak{U}_{qk}\neq 0$, $k\in\mathbf{N}$.□

The cardinal sampling expansion series $\eta_a(x)$ of a BL, HRF $\eta(x)$ is the expression

$$\sum_n \eta(x^n)\prod_1^q \operatorname{sinc}(w_j x_j - n_j\pi), \quad (4.8)$$

where n runs over $\mathbf{Z}^q$ and $x^n\in \operatorname{Lat}(W)$.

THEOREM 3. *There exists a convenient choice of $\{w_{jk}\}_1^\infty$, $j=\overline{1,q}$ such that the sequence of cardinal series expansions $\{\xi_{a,k}(x)\}_1^\infty$ converges almost surely to the initial non-BL, HRF $\xi(x)$ if $k\to\infty$.*

Proof. Let $\{w_{jk}\}_1^\infty$, $j=\overline{1,q}$, be a sequence from the Remark 2. If it is not possible to construct such a $\{w_{jk}\}$, then suppose $\Lambda_d\cap\{w_{jk}\}$ is nonempty (such a sequence always exists).

Since the probability $P_k = P\{|\xi(x)-\xi_{a,k}(x)|\ge\varepsilon\}$ is bounded above by $\varepsilon^{-2}\mathfrak{U}_{qk} + (\mathfrak{S}/\varepsilon)^2 P\{\Pi'(k)\}$ it is sufficient to prove that the series $\sum_{k=1}^\infty \mathfrak{U}_{qk}$ and $\sum_{k=1}^\infty P\{\Pi'(k)\}$ converges. But,

$$\sum_{k=1}^\infty \mathfrak{U}_{qk} = \sum_{k=1}^\infty\sum_{p=1}^q\sum_{k_1<\ldots<k_q}\Big|1-e^{i\langle W_k^p,x^p\rangle}\prod_{j=1}^p\cos(w_{k_j}x_{k_j})\Big|^2\times$$
$$\times \operatorname*{Var}_{\mathfrak{W}_k^p} F(\lambda^p;\pm\underline{W}_k^{q-p}) \le 4\sum_k\sum_{p=1}^q \operatorname*{Var}_{\mathfrak{W}_k^p} F(\lambda^p;\pm\underline{W}_k^{q-p}) \le 4\mathfrak{S}^2<\infty.$$

On the other hand one can clearly choose such $\{w_{jk}\}$ that $\sum_{k=1}^{\infty} P\{\Pi'(k)\} < \infty$, since $\{\mathfrak{S}^{-2}F(B), B \in \mathcal{B}_{\mathbb{R}^q}\}$ is a probability measure. Finally

$$\sum_{k=1}^{\infty} P_k \leq (\mathfrak{S}/\varepsilon)^2 \Big(4 + \sum_{k=1}^{\infty} P\{\Pi'(k)\}\Big) < \infty,$$

and by the Borel-Cantelli lemma we get $P\{\lim_{k\to\infty} \xi_{a,k}(x) = \xi(x)\} = 1.$ ■

V. Conclusions and final remarks

In this way we have shown that for non-BL, HRFs the continuity of the spectral distribution function is not a necessary condition for the almost sure convergence of the sampling series expansion sequence in the foregoing sense.

The author must also remark that the purpose of this paper was not to give an exact truncation error upper bounds.

The result on $\mathfrak{U}_1$ as an example can be found in [1]. But Balakrishnan did not consider non-band-limited processes; also he needed a continuity property of the spectral distribution function for his investigations. Lloyd in his famous paper discussed the almost sure convergence in a different approach. He gave very interesting results, but he used Λ_d of the spectral distribution function distinct of the sampling point set, namely he decomposed the entire real axis into a union of disjoint open intervals [11]. In the papers[6], [20] non-band-limited processes and their m.s. sampling representations were considered in a different context.

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