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MARTINGALE APPROACH TO RANDOM EVOLUTION

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Abstract. It is proved that so called random evolution process defined as $X_\alpha(t) = (2\sqrt{\alpha})^{-1}\left[(-1)^{N_\alpha(t)} - 1 + 2\alpha\int_0^t(-1)^{N_\alpha(s)}\,ds\right]$, where $\{N_\alpha(t), t \geq 0\}$ is a Poisson process with intensity α, weakly converges to Brownian motion, when α tends to infinity. This is used to prove Stroock's result on approximating distribution of the solutions of Ito stochastic differential equations through the family of functionals defined on the random evolution. Also, a new martingale characterization of Poisson process in the class of pure point processes is given.

Introduction. In [Str] Stroock has pointed out an interesting method of approximating the probability distribution of the solution of Ito stochastic differential equastions through the distribution of so called random evolution. Namely if $\{B_t, t \geq 0\}$ is standard Brownian motion in $\mathbf{R}$, and $\{N(t), t \geq 0\}$ a Poisson process with intensity 1, then Stroock's theorem can be formulated as:

THEOREM. *Let $a : \mathbf{R} \to \mathbf{R}$ and $b : \mathbf{R} \to \mathbf{R}$ be bounded, smooth functions having bounded derivatives of all orders. Define*

$$X(t,x) = x + \int_0^t a(X(s,x))\,dB_s + \int_0^t b(X(s,x))\,ds$$

and

$$X_\varepsilon(t,x) = x + \varepsilon\int_0^t a(X_\varepsilon(s,x))(-1)^{N(s)}\,ds + \varepsilon^2\int_0^t b(X_\varepsilon(s,x))\,ds.$$

Then the process $\{X_\varepsilon(t\varepsilon^{-2},x), t \geq 0\}$ weakly[1] *converges to the process $\{X(t,x), t \geq 0\}$, as $\varepsilon \to 0$.*

We shall give a fully probabilistic proof of Stroock's theorem, based on the theory of martingales. This approach enabled us to relax the constraints on coefficients a and b, assuming them to be just Lipschitz continuous. Also, the case of time dependent coefficients is included.

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[1] The family of processes $\{Y_\alpha(t)\}$ weaklly converges to the process $\{Y(t)\}$ as $\alpha \to \infty$ if for each $n \in \mathbf{N}$ and $t_1, \dots, t_n \geq 0$ the distribution of $(Y_\alpha(t_1), \dots, Y_\alpha(t_n))$ converges to the distribution of $(Y(t_1), \dots, Y(t_n))$.

Martingale results. Let $\{N_\alpha(t), t \geq 0\}$ be a Poisson procces with intensity α and $\{\mathcal{F}_t, t \geq 0\}$ the natural filtration generated by it. If we define process $\{X_\alpha(t), t \geq 0\}$ as

$$X_\alpha(t) = \frac{1}{2\sqrt{\alpha}}\left[(-1)^{N_\alpha(t)} - 1 + 2\alpha \int_0^t (-1)^{N_\alpha(s)}\, ds\right], \tag{1}$$

then the following theorem holds:

THEOREM 1.1. *Process $\{X_\alpha(t), t \geq 0\}$ is a locally bounded martingale relative to $\{\mathcal{F}_t, t \geq 0\}$, whose predictable quadratic variation process $\{\langle X_\alpha\rangle, t \geq 0\}$ satisfies $\langle X_\alpha\rangle_t = t$ a.s.*

Proof. Elementary calculations show that $\mathrm{E}\{X_\alpha(t)\} = 0$ and $\mathrm{E}\{X_\alpha(t)^2\} = t$. Then, using the fact that Poisson process has homogeneous independent increments, one gets

$$\mathrm{E}\{X_\alpha(t+s) - X_\alpha(t) | \mathcal{F}_t\} = (-1)^{N_\alpha(t)}\mathrm{E}\{X_\alpha(s)\} = 0, \quad \text{and} \tag{2}$$

$$\mathrm{E}\{(X_\alpha(t+s) - X_\alpha(t))^2 | \mathcal{F}_t\} = \mathrm{E}\{(X_\alpha(s))^2\} = s. \tag{3}$$

From (2) if follows that $\{X_\alpha(t), t \geq 0\}$ is a martingale, and by Doob-Meyer decomposition $X_\alpha(t)^2 = N_t + \langle X_\alpha\rangle_t$, where $\{N_t\}$ is a martingale and $\{\langle X_\alpha\rangle_t\}$ predictable quadratic variation. So, by (3) it is $\mathrm{E}\{\langle X_\alpha\rangle_{t+s} | \mathcal{F}_t\} = \langle X_\alpha\rangle_t + s$. Hence, $\{\langle X_\alpha\rangle_t - t\}$ is predictable martingale of bounded variation, what implies that $\langle X_\alpha\rangle_t - t = 0$, [Ell]; so the theorem is proved.

It is interesting that the martingale property of the process similar to the process $\{X_\alpha(t)\}$ characterizes Poisson process in the class of renewal processes. Namely, let $\{\mathcal{E}_n, n \in \mathrm{N}\}$ be a sequence of independent, identically distributed nonnegative random variables with partial sums $S_n = \sum_{k=1}^n \mathcal{E}_k$ and associated renewal process $N(t) = \sum_{n=1}^\infty \mathrm{I}\{S_n \leq t\}$. Then the following theorem holds:

THEOREM 1.2. *If, for some $\alpha > 0$, the process $\{X_\alpha(t), t \geq 0\}$, defined as*

$$X_\alpha(t) = \frac{1}{2\sqrt{\alpha}}\left[(-1)^{N(t)} - 1 + 2\alpha \int_0^t (-1)^{N(s)}\, ds\right],$$

is martingale, then renewal process $\{N(t)\}$ is Poisson process with intensity α.

Proof. Martingale $\{X_\alpha(t)\}$ has paths of locally bounded variation so its projection on the space of continuous martingales is zero, [Ell]. Also $\{X_\alpha(t)\}$ is right continuous, with jumps at the moments $t = S_n$ such that $\Delta X_\alpha(S_n) = (-1)^n/\sqrt{\alpha}$. So, Ito's formula gives

$$f(X(t)) - f(0) = \int_{(0,t]} f'(X_\alpha(s-))\, dX_\alpha(s) + \\ + \sum_{S_n \leq t} \left[f(X_\alpha(S_n)) - f(X_\alpha(S_n-)) - f'(X_\alpha(S_n-))\Delta X_\alpha(S_n)\right].$$

for $f \in \mathbf{C}_2$. Putting $t = S_k$ and taking expectation one gets

$$\mathbf{E}\{f(X_\alpha(S_k))-f(0)\} = \sum_{n=1}^{k} \mathbf{E}\{f(X_\alpha(S_n))-f(X_\alpha(S_n-))-f'(X_\alpha(S_n-))\Delta X_\alpha(S_n)\}.$$

Now, for $k = 1$, $X_\alpha(S_1) = -1/\sqrt{\alpha} + \sqrt{\alpha}\mathcal{E}_1$ and $X_\alpha(S_1-) = \sqrt{\alpha}\mathcal{E}_1$. Then it follows $f(0) = \mathbf{E}\{f(\sqrt{\alpha}\mathcal{E}_1)\} - \mathbf{E}\{f'(\sqrt{\alpha}\mathcal{E}_1)\}/\sqrt{\alpha}$. Taking $f(x) = \exp(-\lambda x/\sqrt{\alpha})$ one obtains $\mathbf{E}\{\exp(-\lambda\mathcal{E}_1)\} = \alpha/(\alpha+\lambda)$, so $\mathbf{P}(\mathcal{E}_1 \in dx) = \alpha e^{-\alpha x}\,dx$. As $\{\mathcal{E}_n, n \in \mathbf{N}\}$ are identically distributed, $\{N(t)\}$ is Poisson process. □

Convergence results. Now, we shall investigate the limit distribution, as $\alpha \to \infty$ of the process $\{X_\alpha(t)\}$, defined in (1), as well as the limit distribution of the process $\{J_\alpha(t)\}$, where $J_\alpha(t) = \sqrt{\alpha}\int_0^t (-1)^{N_\alpha(s)}\,ds$.

THEOREM 2.1. *Processes $\{X_\alpha(t)\}$ and $\{J_\alpha(t)\}$ weakly converge to the standard Brownian motion, as $\alpha \to \infty$.*

Proof. Put $f(t,\lambda,\theta) = \mathbf{E}\{\exp(i\lambda J_\alpha(t)) \cdot \mathbf{I}\{(-1)^{N_\alpha(t)} = \theta\}\}$, for $\lambda \in \mathbf{R}$ and $\theta \in \{-1,1\}$. Then

$$\begin{aligned}
&f(t+\varepsilon,\lambda,\theta) = \mathbf{E}\{\exp(i\lambda J_\alpha(t+\varepsilon)) \cdot \mathbf{I}\{(-1)^{N_\alpha(t+\varepsilon)} = \theta\}\} = \\
&= \mathbf{E}\{\exp(i\lambda J_\alpha(t+\varepsilon)) \cdot \mathbf{I}\{(-1)^{N_\alpha(t)} = \theta\} \cdot \mathbf{I}\{N_\alpha(t+\varepsilon) = N_\alpha(t)\}\} + \\
&+ \mathbf{E}\{\exp(i\lambda J_\alpha(t+\varepsilon)) \cdot \mathbf{I}\{(-1)^{N_\alpha(t)} = -\theta\} \cdot \mathbf{I}\{N_\alpha(t+\varepsilon) = N_\alpha(t)+1\}\} + o(\varepsilon) = \\
&= \exp(i\lambda\theta\sqrt{\alpha}\varepsilon) \cdot \mathbf{P}\{N_\alpha(t+\varepsilon) = N_\alpha(t)\} \cdot f(t,\lambda,\theta) + \\
&+ \mathbf{E}\left\{\exp\left(-i\lambda\theta\sqrt{\alpha}\int_0^\varepsilon (-1)^{N_\alpha(t+s)-N_\alpha(t)}\,ds\right) \cdot \mathbf{I}\{N_\alpha(t+s) = N_\alpha(t)+1\}\right\} \times \\
&\qquad \times f(t,\lambda,-\theta) + o(\varepsilon) = \exp(i\lambda\theta\sqrt{\alpha}\varepsilon - \alpha\varepsilon) f(t,\lambda,\theta) + \\
&+ \mathbf{E}\{\exp(-i\lambda\theta\sqrt{\alpha}(\mathcal{E}_1 - \varepsilon + \mathcal{E}_1)) \cdot \mathbf{E}\{\mathcal{E}_1 \le \varepsilon < \mathcal{E}_1 + \mathcal{E}_2\}\} f(t,\lambda,-\theta) + o(\varepsilon).
\end{aligned}$$

So, we conclude that

$$\frac{\partial}{\partial t} f(t,\lambda,\theta) = (i\lambda\theta\sqrt{\alpha} - \alpha) f(t,\lambda,\theta) + \alpha f(t,\lambda,-\theta). \tag{4}$$

Then, using the above formula one easily gets

$$\frac{\partial^2}{\partial t^2} f(t,\lambda,\theta) + 2\alpha\frac{\partial}{\partial t} f(t,\lambda,\theta) + \lambda^2\alpha f(t,\lambda,\theta) = 0. \tag{5}$$

Having in mind that $f(0,\lambda,\theta) = (1+\theta)/2$, from the relations (4) and (5) it follows that, for $\lambda^2 \ne \alpha$,

$$\begin{aligned}
f(t,\lambda,1) = &\frac{1}{2}\left(1 + \frac{i\lambda/\sqrt{\alpha}}{(1-\lambda^2/\alpha)^{1/2}}\right) \exp(-t\alpha + t(\alpha^2 - \lambda^2\alpha)^{1/2}) \\
&+ \frac{1}{2}\left(1 - \frac{i\lambda/\sqrt{\alpha}}{(1-\lambda^2/\alpha)^{1/2}}\right) \exp(-t\alpha - t(\alpha^2 - \lambda^2\alpha)^{1/2}), \quad \text{and}
\end{aligned}$$

$$f(t,\lambda,-1) = \frac{1}{2}\frac{1}{(1-\lambda^2/\alpha)^{1/2}}\exp\bigl(-t\alpha + t(\alpha^2-\lambda^2\alpha)^{1/2}\bigr)$$
$$-\frac{1}{2}\frac{1}{(1-\lambda^2/\alpha)^{1/2}}\exp\bigl(-t\alpha - t(\alpha^2-\lambda^2\alpha)^{1/2}\bigr).$$

Straightforward calculation shows that $\lim_{\alpha\to\infty} f(t,\lambda,\theta) = e^{-\lambda^2/2t}/2$. Now,

$$\mathbf{E}\bigl\{\exp\bigl(i\lambda J_\alpha(t)\bigr)\bigr\} = f(t,\lambda,1) + f(t,\lambda,-1), \quad \text{and}$$
$$\mathbf{E}\bigl\{\exp\bigl(i\lambda X_\alpha(t)\bigr)\bigr\} = f(t,\lambda,1) + e^{-i\lambda/\sqrt{\alpha}} f(t,\lambda,-1),$$

so one can conclude that

$$\lim_{\alpha\to\infty}\mathbf{E}\bigl\{\exp\bigl(i\lambda J_\alpha(t)\bigr)\bigr\} = \lim_{\alpha\to\infty}\mathbf{E}\bigl\{\exp\bigl(i\lambda X_\alpha(t)\bigr)\bigr\} = \mathbf{E}\bigl\{\exp(i\lambda B_t)\bigr\}.$$

For $0 \le t_1 \le \ldots \le t_n$ and $\lambda_1,\ldots,\lambda_n \in \mathbf{R}$, let us define

$$\mathcal{E}_{t_1,\ldots,t_n}(\lambda_1,\ldots,\lambda_n) = \mathbf{E}\left\{\exp\left(i\sum_{k=1}^n \lambda_k J_\alpha(t_k)\right)\right\}.$$

Then

$$\mathcal{E}_{t_1,\ldots,t_n}(\lambda_1,\ldots,\lambda_n) =$$
$$= \mathbf{E}\left\{\exp\left(i\sum_{k=1}^n \lambda_k J_\alpha(t_1)\right)\cdot\exp\left(i\sum_{k=2}^n \lambda_k\bigl(J_\alpha(t_k) - J_\alpha(t_1)\bigr)\right)\right\} =$$
$$= \mathbf{E}\left\{\exp\left(i\sum_{k=1}^n \lambda_k J_\alpha(t_1)\right)\cdot\exp\left(i\sum_{k=2}^n \int_{t_1}^{t_k}(-1)^{N_\alpha(s)}\,ds\right)\right\} =$$
$$= \mathbf{E}\left\{\exp\left(i\sum_{k=1}^n \lambda_k J_\alpha(t_1)\right)\times\right.$$
$$\left.\times\exp\left(i(-1)^{N_\alpha(t_1)}\sum_{k=2}^n \lambda_k\int_0^{t_k-t_1}(-1)^{N_\alpha(t_1+s)-N_\alpha(t_1)}\,ds\right)\right\}$$
$$= \mathbf{E}\left\{\exp\left(i\sum_{k=1}^n \lambda_k J_\alpha(t_1)\right)\times\right.$$
$$\left.\times\mathbf{E}\left\{\exp\left(i(-1)^{N_\alpha(t_1)}\sum_{k=2}^n \lambda_k\int_0^{t_k-t_1}(-1)^{N_\alpha(t_1+s)-N_\alpha(t_1)}\right)\,\middle|\,\mathcal{F}_{t_1}\right\}\right\} =$$
$$= f\left(t_1,\sum_{k=1}^n \lambda_k,1\right)\cdot\mathcal{E}_{t_2-t_1,\ldots,t_n-t_1}(\lambda_2,\ldots,\lambda_n) +$$
$$+ f\left(t_1,\sum_{k=1}^n,\lambda_k,-1\right)\cdot\mathcal{E}_{t_2-t_1,\ldots,t_n-t_1}(-\lambda_2,\ldots,-\lambda_n).$$

The principle of mathematical induction yields

$$\lim_{\alpha\to\infty} \mathbf{E}\Big\{\exp\Big(i\sum_{k=1}^{n}\lambda_k J_\alpha(t_k)\Big)\Big\} = \mathbf{E}\Big\{\exp\Big(i\sum_{k=1}^{n}\lambda_k B_{t_k}\Big)\Big\}.$$

The proof for $\{X_\alpha(t)\}$ goes along the same lines, having in mind that

$$X_\alpha(t) - X_\alpha(s) =$$
$$= (-1)^{N_\alpha(s)}\frac{1}{2\sqrt{\alpha}}\Big[(-1)^{N_\alpha(t)-N_\alpha(s)} - 1 + 2\alpha\int_0^{t-s}(-1)^{N_\alpha(s+u)-N_\alpha(u)}\,du\Big]. \;\square$$

The Stroock's theorem, and more, follows as a consequence of the theorem from [**G-S**, p. 339].

THEOREM (Gikhman-Skorokhod). *Let $\{U_\alpha(t), t\geq 0\}$ be a process adapted to filtration $\{\mathcal{F}_t, t\geq 0\}$, such that*

(a) $|\mathbf{E}\{U_\alpha(t+s) - U_\alpha(t)|\mathcal{F}_t\}| \leq Cs$, (b) $\mathbf{E}\{(U_\alpha(t+s) - U_\alpha(t))^2|\mathcal{F}_t\} \leq Cs$,

(c) *coefficients $a : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ and $b : \mathbf{R}_+ \times \mathbf{R} \to \mathbf{R}$ satisfy:*

$$|a(t,0)| + |b(t,0)| \leq C,$$
$$|a(t,x) - a(t,y)| + |b(t,x) - b(t,y)| \leq C|x-y|,$$
$$|a(t+s,x) - a(t,x)| + |b(t+s,x) - b(t,x)| \leq g(s)(1+|x|),$$

where $g(s) \to 0$ as $s \to 0$.

(d) *the family of random elements $\{x_\alpha, U_\alpha(t), t \geq 0\}$ weakly converges to $\{x, U(t), t\geq 0\}$ as $\alpha\to\infty$;*

then the process $\{\zeta_\alpha(t), t\geq 0\}$, defined as

$$\zeta_\alpha(t) - x_\alpha = \int_0^t a(s,\zeta_\alpha(s))\,ds + \int_0^t b(s,\zeta_\alpha(s))\,dU_\alpha(s),$$

weakly converges to the process $\{\zeta_\alpha(t), t\geq 0\}$, defined as

$$\zeta(t) - x = \int_0^t a(s,\zeta(s))\,ds + \int_0^t b(s,\zeta(s))\,dU(s). \quad \square$$

Finally, Theorem 2.1 and Gikhman-Skorokhod theorem imply the following theorem.

THEOREM 2.2. *Suppose coefficients a and b satisfy condition* (c) *from the Gihman-Skorohod theorem. Then the process $\{X_\alpha(t), t\geq 0\}$, defined as*

$$X_\alpha(t,x) = x + \sqrt{\alpha}\int_0^t a(s, X_\alpha(s,x))(-1)^{N_\alpha(s)}\,ds + \int_0^t b(s, X_\alpha(s,x))\,ds,$$

weakly converges, as $\alpha\to\infty$, to the process $\{X(t,x), t\geq 0\}$, where

$$X(t,x) = x + \int_0^t a(s,X(s,x))\,dB_s + \int_0^t b(s,X(s,x))\,ds. \quad \square$$

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