PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 49 (63), 1991, 201–214

ON SOME EQUIVALENCE TRANSFORMATIONS OF HILBERT SPACE VALUED STANDARD WIENER PROCESS

Ljiljana Petruševski

Abstract. The process $X(t)$ taking values in real separable Hilbert space H with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$ will be called equivalent to process $Z(t)$ on interval $(0,T)$ if the mapping $A:(u,Z(t))\to(u,X(t))$, $u\in H$, $0<t<T$, can be extended to a bounded linear operator which has a bounded inverse and for which $I-(A^*A)^{1/2}$ (simultaneously $I-A^*A$ and $I-(A^{-1})^*(A^{-1})$) is a Hilbert-Schmidt operator. Let

$$X(t)=\int_0^t Y(s)\,ds+B(t)$$

where $B(t)$ is a Hilbert space valued standard Wiener process and

$$\int_0^T E\|Y(s)\|^2\,ds<\infty$$

(T is finite or infinite). In this paper the following statement is proved: *If the process $Y(t)$ is independent of future increments of the standard Wiener process $B(t)$, then the Hilbert space valued processes $X(t)$ and $B(t)$ are equivalent.* A significant corollary of that statement is the following: *If the process $Y(t)$ is a measurable process well adapted to $\{\mathcal{U}_t(B)\}$ then the Hilbert space valued processes $X(t)$ and $B(t)$ are equivalent.*

1. Introduction. Let H be a real separable Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$ and let $\mathcal{B}$ be the σ-algebra of the Borel sets in H. Let $(\Omega,\mathcal{U},P)$ be a probability space. We shall consider a Hilbert space valued random variable as a measurable mapping from the probability space to the measurable space $(H,\mathcal{B})$. It is well known that the mapping $\xi:\Omega\to H$ is a Hilbert space valued random variable if and only if (u,ξ) is a real random variable for each $u\in H$. A Hilbert space valued stochastic process $Z(t)$, $0<t<T$ is a family of Hilbert space valued random variables $Z(t)$, $0<t<T$. We assume that $E(u,Z(t))=0$ and $E|(u,Z(t))|^2<\infty$ for each $u\in H$ and $t\in(0,T)$. Denote by $\mathcal{U}_t(Z)$ σ-algebra generated by cylindrical sets

$$\{Z(t_1)\in S_1^H,\ Z(t_2)\in S_2^H,\ \dots,\ Z(t_n)\in S_n^H\}$$

AMS Subject Classification (1985): Primary 60G12 where $t_1, t_2, \dots, t_n \leq t$ and $S_1^H, S_2^H, \dots, S_n^H$ are Borel sets in H.

According to well known theorems, $\mathcal{U}_t(Z)$ is the σ-algebra generated by cylindrical sets

$$\{((u_1, Z(t_1)),\ (u_2, Z(t_2)),\ \dots,\ (u_n, Z(t_n))) \in S_n\}$$

where $t_1, t_2, \dots, t_n \leq t$, $u_1, u_2, \dots, u_n \in H$ and S_n, $n = 1, 2, \dots$ are Borel sets in $\mathbf{R}_n$. The σ-algebra $\mathcal{U}_t(Z)$ is a sub-σ-algebra of $\mathcal{U}$ and the family $\{\mathcal{U}_t(Z)\}$ is an increasing family of sub-σ-algebras of $\mathcal{U}$.

Let $H_t(Z)$ be the linear closure (in quadratic mean) of random variables $(u, Z(s))$, $s \leq t$, $u \in H$. $H_t(Z)$ is a subspace of Hilbert space $L^2(\Omega, \mathbf{R})$ of the real random variables $\xi : E\xi^2 < \infty$, $E\xi = 0$ with scalar product $\langle \xi, \eta \rangle = E\xi\eta$ and norm $|||\xi||| = E\xi^2$. Let

$$H(Z) = \overline{\bigcup_{0<t<T} H_t(Z)}.$$

A Hilbert space valued process $X(t)$ will be called equivalent to the process $Z(t)$ on interval $(0, T)$ if the mapping $A : (u, Z(t)) \to (u, X(t))$, $u \in H$, $0 < t < T$, can be extended to an equivalence operator from Hilbert space $H(Z)$ to Hilbert space $H(X)$, which means that the mapping A can be extended to a bounded linear operator which has a bounded inverse and for which $I - (A^*A)^{1/2}$ is a Hilbert-Schmidt operator.

It is well known that the process $Z(t)$ is then equivalent to the process $X(t)$ [9] and the two Gaussian processes $X(t)$ and $Z(t)$ are equivalent if and only if their distributions are equivalent [8].

We also need the concept of the stochastic Ito integral with respect to standard Wiener process $B(t)$.

$$J = \int_0^T F(t)\, dB(t)$$

is the stochastic Ito integral standardly defined [1] for functions $F(s)$ taking values in the Hilbert space $S_2(H)$ of Hilbert-Schmidt operators in H with the scalar product $(F, G) = \operatorname{Sp} G^* F$ and the norm

$$|F|^2 = \operatorname{Sp} F^* F = \sum_{k=1}^{\infty} \|F e_k\|^2$$

where $\{e_k\}_1^\infty$ is an orthonormal base of the separable Hilbert space H:

$$\int_0^T |F(t)|^2\, dt < \infty.$$

The integrad J is a Hilbert space valued random variable and it can be represented as infinite sum of stochastic Pettis integral [5]

$$J = \sum_{k=1}^{\infty} \int_0^T F(t) e_k\, d(e_k, B(t))$$

so,

$$E\|J\|^2 = \int_0^T |F(t)|^2\,dt < \infty$$

and

$$(u, J) = \sum_{k=1}^{\infty} \int_0^T (F^*(t)u, e_k)\,d(e_k, B(t)).$$

The subject of this paper is the sum

$$X(t) = \int_0^t Y(s)\,ds + B(t) \tag{1.1}$$

where $B(t)$ is a Hilbert space valued standard Wiener process and

$$\int_0^T E\|Y(s)\|^2\,ds < \infty. \tag{1.2}$$

(T is finite or infinite.)

The equivalence of process (1.1) to a standard Wiener process for the real-valued case is extended to Hilbert space valued processes. From the assumption (1.2) it follows that the process $Y(t)$ is an integrable $L^2(\Omega, H)$-space valued function on each finite interval $(0,t)$, $t < T$ where $L^2(\Omega, H)$ is the Hilbert space of Hilbert space valued random variables ξ: $E\|\xi\|^2 < \infty$ with scalar product $(\xi,\eta)_1 = E(\xi,\eta)$ and norm $\|\xi\|_1^2 = E\|\xi\|^2$. Then with probability one

$$\int_0^t Y(s)\,ds = \int_0^t Y(s,\omega)\,ds.$$

It is well known that a process $Y(s)$ is integrable $L^2(\Omega, H)$-space valued function on interval $(0,t)$ if and only if $Y(s) = Y(s,\omega)$ is measurable with respect to $dt \times P$ and $\int_0^t \|Y(s)\|_1\,ds < \infty$.

We will follow Rozanov's conception [9] of the equivalence of real process (1.1) to a standard Wiener process. The study of the equivalence in [9] is not completed, and with the help of [2], we will set the conditions under which $X(t)$ and $B(t)$ are equivalent. Eršov [2] sets the conditions under which the measures of the processes $X(t)$ and $B(t)$ are equivalent, and it will be shown that these conditions are also valid for the equivalence of the two processes. In this paper it is proved: *If the process $Y(t)$ is independent of future increments of the standard Wiener process $B(t)$, then Hilbert space valued processes $X(t)$ and $B(t)$ are equivalent.* The significant corollary of that statement is the following: *If the process $Y(t)$ is measurable process well adapted to $\{\mathcal{U}_t(B)\}$ i.e. $Y(t)$ is measurable with respect to $\mathcal{U}_t(B)$ for each t, then Hilbert space valued processes $X(t)$ and $B(t)$ are equivalent.*

The study of equivalence of process (1.1) to standard Wiener process began in [6] and [7]. This paper is a continuation of that study. Theorem 2 and Theorem 3 are new. Part of Theorem 1 is proved in [6] but we give a simpler proof of that part of the statement. The equivalence on finite interval $(0,T)$ is extended to an arbitrary finite or infinite interval $(0,T)$. The equivalence to Wiener process (Example 1) is proved in [7], but we have simplified the proof. The example 2 is an extension of Theorem 2 [4] for real Gaussian processes to Hilbert space valued Gaussian processes.

2. Definition of equivalence of two processes and equivalence to standard Wiener process. In this section we will define the equivalence of the two processes with the help of the equivalence operator which is defined by Feldman [3] and we will cite well known statements [9] which are also needed.

Definition 1 [3]. An operator G from Hilbert space H_1 to Hilbert space H_2 will be called *equivalence operator* if

(1) G is one-to-one onto, bounded, and has a bounded inverse.

(2) $I-(G^*G)^{1/2}$ is Hilbert-Schmidt.

LEMMA 1. *The operator G from Hilbert space H_1 to Hilbert space H_2 is an equivalence operator if and only if*

(1) *G is one-to-one onto, bounded, and has a bounded inverse.*

(2) *$I-G^*G$ is Hilbert-Schmidt.*

Proof. The proof of that statement follows from Definition 1 and from the equality

$$I-G^*G=\big(I-(G^*G)^{1/2}\big)\big(I+(G^*G)^{1/2}\big).$$

If an operator G is bounded, then the self-adjoint operator $I+(G^*G)^{1/2}=F$ is bounded too, and has a bounded inverse. Let G be an equivalence operator. From the Definition 1 it follows that G is one-to-one onto, bounded, and has a bounded inverse, and the operator $I-G^*G$ is Hilbert-Schmidt as the product of the Hilbert-Schmidt operator $I-(G^*G)^{1/2}$ and a bounded operator F.

Conversely, under the conditions of the theorem, G is one-to-one onto, bounded, and has a bounded inverse. The operator $I-(G^*G)^{1/2}$ is a Hilbert-Schmidt operator as the product of the bounded operator F^{-1} and a Hilbert-Schmidt operator $I-G^*G$. So, G is an equivalence operator. The proof is finished.

Definition 2. The Hilbert space valued proces $X(t)$ will be called *equivalent* to the process $Z(t)$ on intrval $(0,T)$ if the mapping

$$A:(u,Z(t))\to(u,X(t)),\qquad u\in H,\ 0\le t\le T \tag{2.1}$$

can be extended to an equivalence operator from Hilbert space $H(Z)$ to Hilbert space $H(X)$.

The correlation function of Hilbert space valued process $Z(t)$, $t\in(0,T)$ is the operator function $\Gamma_z(s,t)$ which fulfills the equality

$$\langle(u,Z(s)),(v,Z(t))\rangle=(\Gamma_z(s,t)u,v),\qquad u,v\in H.$$

Let $B(t)$ be the Hilbert space valued standard Wiener process. The correlation function of the standard Wiener process is the function $\Gamma_W(s,t) = \min\{s,t\} \cdot I$. We also need the following well-known statements [9].

LEMMA 2. *If $\chi_t(x)$ is an indicator function, then the mapping*

$$V : (u, B(t)) \to \chi_t(x)u, \qquad u \in H, \ t \in (0,T)$$

is an isomorphism from $H(B)$ to Hilbert space $L^2((0,T), H)$ of functions taking values in H with scalar product

$$(f_1, f_2) = \int_0^T (f_1(x), f_2(x))\, dx.$$

LEMMA 3. *The Hilbert space valued process $X(t)$ is equivalent to the standard Wiener process $B(t)$ on interval $(0,T)$ if and only if the mapping*

$$A_1 : \chi_t(x)u \to (u, X(t)), \qquad u \in H, \ t \in (0,T) \tag{2.2}$$

can be extended to a bounded linear operator which has a bounded inverse and for which $I - A_1^* A_1$ is Hilbert-Schmidt operator.

Let us notice that, for isomorphism V defined in Lemma 1 and for operators A and A_1 defined by (2.1) and (2.2),

$$A_1 V = A$$

from where it follows that the statement of Lemma 3 is true.

3. The equivalence. In this section the equivalence of process (1.1) to the standard Wiener process for the real-valued case is extended to Hilbert space valued processes.

Let

$$X(t) = \int_0^t Y(s)\, ds + B(t), \qquad 0 < t < T \tag{3.1}$$

where $B(t)$ is the Hilbert space valued standard Wiener process, and $Y(s)$ $(E(u, Y(s)) = 0)$ is $L^2((0,T), L^2(\Omega, H))$ space valued function i.e.

$$\int_0^T E\|Y(s)\|^2\, ds < \infty. \tag{3.2}$$

(T is finite or infinite.)

From equality (3.1) it follows that

$$(\Gamma_B(s,t)u, v) - (\Gamma_X(s,t)u, v) = f(s,t) + g(s,t) + h(s,t) \tag{3.3}$$

where

$$f(s,t) = -\left\langle \int_0^s (u, Y(x))\,dx, \int_0^t (v, Y(y))\,dy \right\rangle$$
$$g(s,t) = -\left\langle \int_0^s (u, Y(x))\,dx, (v, B(t)) \right\rangle$$
$$h(s,t) = -\left\langle (u, B(s)), \int_0^t (v, Y(y))\,dy \right\rangle.$$

The first right-hand-side addend in equality (3.3) can be written in the following form

$$f(s,t) = -\int_0^t \int_0^s (\Gamma_Y(x,y)u, v)\,dxdy \tag{3.4}$$

where $\Gamma_Y(x,y)$ is the correlation function of the Hilbert space valued process $Y(t)$. According to the assumption (3.2), $\Gamma_Y(x,y)$ is a Hilbert-Schmidt operator for almost each $x, y \in (0,T)$ and for the Hilbert-Schmidt norm $|\Gamma_Y(x,y)|$:

$$\int_0^T \int_0^T |\Gamma_Y(x,y)|^2\,dxdy < \infty. \tag{3.5}$$

Denote by P the projection operator from Hilbert space $L^2(\Omega, H)$ onto $H(B)$. Let

$$P(u, Y(x)) = \sum_{k=1}^{\infty} \int_0^T c_k(u,x,y)\,d(e_k, B(y)). \tag{3.6}$$

Let us observe the second addend from the right-hand-side of (3.3) now. It is obvious that

$$g(s,t) = -\int_0^s \left\langle P(u, Y(x)), \sum_{k=1}^{\infty} \int_0^t (v, e_k)\,d(e_k, B(y)) \right\rangle dx.$$

Hence, according to equality (3.6)

$$g(s,t) = -\int_0^s \sum_{k=1}^{\infty} \int_0^t c_k(u,x,y)(e_k, v)\,dydx$$

and finally

$$g(s,t) = \int_0^t \int_0^s (F(x,y)u, v)\,dxdy \tag{3.7}$$

where

$$F(x,y)u = -\sum_{k=1}^{\infty} c_k(u,x,y)e_k. \tag{3.8}$$

From previous equalities it follows that:

$$\int_0^t (F(x,y)u,v)\,dy = \langle (u,Y(x)),(v,B(t))\rangle \tag{3.9}$$

for each $t \in (0,T)$.

Using these equalities we get that $F(x,y)$ is a linear operator for almost each $x \in (0,T)$ and for almost each $y \in (0,T)$. From equality (3.8) it follows:

$$\|F(x,y)e_j\|^2 = \sum_{k=1}^{\infty} |c_k(e_j,x,y)|^2.$$

Hence, according to (3.6)

$$\int_0^T \|F(x,y)e_j\|^2\,dy = E|P(e_j,Y(x))|^2 \leq E|(e_j,Y(x))|^2$$

and

$$\int_0^T \sum_{k=1}^{\infty} \|F(x,y)e_j\|^2\,dy \leq E\|Y(x)\|^2.$$

Now, it is clear that $F(x,y)$ is a Hilbert-Schmidt operator for which

$$\int_0^T \int_0^T |F(x,y)|^2\,dxdy \leq \int_0^T E\|Y(x)\|^2\,dx$$

so, according to (3.2),

$$\int_0^T \int_0^T |F(x,y)|^2\,dxdy < \infty. \tag{3.10}$$

From (3.7) it simply follows that the third right-hand-side addend of equality (3.3) can be represented in the form

$$h(s,t) = \int_0^t \int_0^s (F^*(y,x)u,v)\,dxdy. \tag{3.11}$$

Let us conclude that from equalities (3,3), (3.4), (3.7), and (3.11) it follows that

$$(\Gamma_B(s,t)u,v) - (\Gamma_X(s,t)u,v) = \int_0^t \int_0^s (K(x,y)u,v)\,dxdy \tag{3.12}$$

where

$$K(x,y) = -\Gamma_Y(x,y) + F(x,y) + F^*(x,y)$$

is a Hilbert-Schmidt operator for almost each $x,y \in (0,T)$ with Hilbert-Schmidt norm $|K(x,y)|$ for which, from (3.5) and (3.10) it follows that

$$\int_0^T \int_0^T |K(x,y)|^2\,dxdy < \infty. \tag{3.13}$$

Let us notice that from assumption (3.2) it simply follows that $\langle (u, Y(x)), (v, B(y)) \rangle$ is a bilinear functional in H for almost each $x, y \in (0, T)$ so that

$$\langle (u, Y(x)), (v, B(y)) \rangle = \left(\Gamma_{Y,B}(x, y)u, v \right)$$

where $\Gamma_{Y,B}(x, y)$ (the mutual correlation function of two processes $Y(t)$ and $B(t)$) is a linear bounded operator. From equality (3.9) it follows that

$$\Gamma_{Y,B}(x, y) = \int_0^y F(x, z)\, dz$$

and

$$\frac{d}{dy} \Gamma_{Y,B}(x, y) = F(x, y) \tag{3.14}$$

is a Hilbert-Schmidt operator for which

$$\int_0^T \int_0^T \left| \frac{d}{dy} \Gamma_{Y,B}(x, y) \right|^2 dx dy < \infty.$$

Let $b = b(x)$ be an arbitrary function from $L^2((0, T), H)$. Let G be a Hilbert-Schmidt integral operator in $L^2((0, T), H)$ defined by the kernel

$$(d/dy)\Gamma_{Y,B}(x, y) = F(x, y) \quad \text{i.e.} \quad Gb(y) = \int_0^T F(x, y) b(x)\, dx.$$

THEOREM 1. *If -1 is not the eigenvalue of the Hilbert-Schmidt integral operator in $L^2((0, T), H)$ with kernel $(d/dy)\Gamma_{Y,B}(x, y) = F(x, y)$, then the stochastic process*

$$X(t) = \int_0^t Y(s)\, ds + B(t)$$

is equivalent to the standard Wiener process $B(t)$ on interval $(0, T)$.

Proof. If we write equality (3.12) in the form

$$(\Gamma_B(s, t)u, v) - (\Gamma_X(s, t)u, v) = \int_0^T \int_0^T (K(x, y)\chi_s(x)u, \chi_t(y)v)\, dx dy$$

it is clear that, for operator A_1 defined by (2.2)

$$(I - A_1^* A_1)b(y) = \int_0^T K(x, y) b(x)\, dx \tag{3.15}$$

which, according to well known theorems ([9]) and relation (3.13), is the Hilbert-Schmidt operator.

Let $b = b(x)$ be an arbitrary function from $L^2((0, T), H)$. Using the previous equalities, we get

$$\int_0^T \int_0^T (K(x, y) b(x), b(y))\, dx dy = f + g + h \tag{3.16}$$

where

$$f = -\int_0^T \int_0^T (\Gamma_Y(x,y)b(x), b(y))\,dxdy$$

$$g = \int_0^T \int_0^T (F(x,y)b(x), b(y))\,dxdy$$

$$h = \int_0^T \int_0^T (F^*(y,x)b(x), b(y))\,dxdy$$

The first right-hand-side addend of equality (3.16) can be written in the form

$$f = E\left|\int_0^T (b(x), Y(x))\,dx\right|^2. \tag{3.17}$$

According to the definition of the operator $F(x,y)$ the second right-hand-side addend of equality (3.16) can be written in the form

$$g = -\sum_{k=1}^{\infty} \int_0^T \int_0^T c_k(b(x), x, y)(e_k, b(y))\,dydx$$

from which it simply follows

$$g = -\int_0^T \langle P(b(x), Y(x)), \eta(b)\rangle\,dx \tag{3.18}$$

where

$$\eta(b) = \sum_{k=1}^{\infty} \int_0^T (b(y), e_k)\,d(e_k, B(y))$$

is real random variable for which

$$E|\eta(b)|^2 = \int_0^T \|b(y)\|^2\,dy. \tag{3.20}$$

Finally

$$g = -\left\langle \int_0^T (b(x), Y(x))\,dx, \eta(b)\right\rangle. \tag{3.21}$$

In the same way, we get that the third right-hand-side addend of equality (3.16) is

$$h = -\left\langle \eta(b), \int_0^T (b(x), Y(x))\,dx\right\rangle. \tag{3.22}$$

From equalities (3.16)–(3.22) it follows

$$\int_0^T \int_0^T (K(x,y)b(x), b(y))\,dxdy$$
$$= \int_0^T \|b(y)\|^2\,dy - E\left|\int_0^T (b(x), Y(x))\,dx + \eta(b)\right|^2.$$

Let $b \neq 0$ be an element of $L^2((0,T),H)$ i.e.

$$\int_0^T \|b(y)\|^2 \, dy \neq 0.$$

If we suppose that

$$\int_0^T \int_0^T (K(x,y)b(x), b(y))\, dx dy = \int_0^T \|b(y)\|^2 \, dy,$$

then we have

$$\int_0^T (b(x), Y(x))\, dx + \eta(b) = 0$$

which is an element of the Hilbert space $L^2(\Omega, \mathbf{R})$. Since P is a bounded operator, from this equality it follows that

$$\sum_{j=1}^{\infty} \int_0^T (b(x), e_j) P(e_j, Y(x))\, dx + \eta(b) = 0.$$

Hence,

$$\sum_{k=1}^{\infty} \int_0^T \sum_{j=1}^{\infty} \int_0^T (b(x), e_j) c(e_j, x, y)\, dx\, d(e_k, B(y)) +$$
$$+ \sum_{k=1}^{\infty} \int_0^T (b(y), e_k)\, d(e_k, B(y)) = 0.$$

So, it is obvious that

$$(b(y), e_k) + \sum_{j=1}^{\infty} \int_0^T (b(x), e_j) c(e_j, x, y)\, dx = 0$$

which, according to the definition of the operator $F(x,y)$ means that

$$b(y) + \int_0^T F(x,y) b(x)\, dx = 0$$

i.e. the Hilbert-Schmidt integral operator defined by kernel $(d/dy)\Gamma_{Y,B}(x,y) = F(x,y)$ has eigenvalue -1 which contradicts the assumption of the theorem. Therefore, for any $b \in L^2((0,T),H)$, $b \neq 0$

$$\int_0^T \int_0^T (K(x,y)b(x), b(y))\, dx dy \neq \int_0^T \|b(y)\|^2 \, dy. \tag{3.23}$$

Let us conclude that from equality (3.15) it follows that $I - A_1^* A_1$ is a Hilbert-Schmidt operator. Hence, A is bounded. From (3.23) it simply follows that the operator's greatest eigenvalue is less then one, i.e. A_1 has a bounded inverse.

According to Lemma 3 this means that the processes $X(t)$ and $B(t)$ are equivalent on the (finite or infinite) interval $(0,T)$. This completes the proof.

4. The main theorems. In this section we shall set some additional conditions and, under these conditions, we shall prove the equivalence of the process (1.1) to the standard Wiener process.

Let

$$X(t) = \int_0^t Y(s)\,ds + B(t), \qquad 0 < t < T \tag{4.1}$$

where $B(t)$ is Hilbert space valued standard Wiener process, and

$$\int_0^T E\|Y(s)\|^2\,ds < \infty \tag{4.2}$$

$(E(u, Y(s)) = 0$ and T is finite or infinte).

THEOREM 2. *If a process $Y(t)$ is independent of future increments of the standard Wiener process $B(t)$, then the process*

$$X(t) = \int_0^t Y(s)\,ds + B(t)$$

is equivalent to the standard Wiener process $B(t)$ on interval $(0,T)$.

Proof. The independence of Hilbert space valued random variables $Y(x)$ and $B(t) - B(s)$ (for $x \le s \le t$) will stand for orthogonality of real random variables $(u, Y(x))$ and $(u, B(t) - B(s))$ $(u, v \in H)$. So,

$$P(u, Y(x)) = \sum_{k=1}^{\infty} \int_0^T c_k(u,x,y)\,d(e_k, B(y))$$

where $c_k(u,x,y) = 0$ for $y > x$.

Let $F(x,y)$ be the Hilbert-Schmidt operator defined by (3.8). The Hilbert-Schmidt integral operator with kernel $(d/dy)\Gamma_{Y,B}(x,y) = F(x,y)$ $(F(x,y) = 0$ for $y > x)$ can be represented in the form

$$Gb(y) = \int_y^T F(x,y)b(x)\,dx.$$

The operator G is a Volterra Hilbert-Schmidt integral operator in $L^2((0,T),H)$. Let -1 be an eigenvalue of completely continuous operator G. Then -1 is an eigenvalue of the adjoint operator G^*:

$$G^*b(x) = \int_0^x F^*(x,y)b(y)\,dy$$

which means that

$$b(x) + \int_0^x F^*(x,y)b(y)\,dy = 0 \quad \text{for almost each } x \in (0,T).$$

This is a homogeneous integral equation of the Volterra type and, under our conditions, it follows that its only solution is $b \in L^2((0,T),H)$, $b(x) = 0$ for almost every $x \in (0,T)$. Hence, the operator G has not the eigenvalue -1. According to Theorem 1, the process $X(t)$ is equivalent to standard Wiener process. The proof is completed.

Let $\mathcal{U}_t(B)$ be a σ-algebra generated by cylindrical sets

$$\{B(t_1) \in S_1^H, B(t_2) \in S_2^H, \dots, B(t_n) \in S_n^H\}$$

where $t_1, t_2, \dots, t_n \le t$ and $S_1^H, S_2^H, \dots, S_n^H$ are Borel sets in H. According to well known theorems, $\mathcal{U}_t(B)$ is the σ-algebra generated by cylindrical sets

$$\{((u_1, B(t_1)),\ (u_2, B(t_2)),\ \dots,\ (u_n, B(t_n))) \in S_n\}$$

where $t_1, t_2, \dots, t_n \le t$, $u_1, u_2, \dots, u_n \in H$ and S_n, $n = 1, 2, \dots$ are Borel sets in $\mathbf{R}_n$. The σ-algebra $\mathcal{U}_t(B)$ is a sub-σ-algebra of $\mathcal{U}$ and the family $\{\mathcal{U}_t(B)\}$ is an increasing family of sub-σ-algebras of $\mathcal{U}$. Let $\mathcal{H}(\mathcal{U}_t(B))$ be the set of Hilbert space valued random variables ξ: $E(u,\xi) = 0$, $E|(u,\xi)|^2 < \infty$ measurable with respect to $\mathcal{U}_t(B)$. In the sequel, we assume that $Y = Y(t)$ is measurable process well adapted to $\mathcal{U}_t(B)$, which means that $Y(t)$ is measurable with respect to $\mathcal{U}_t(B)$ for every t ($Y(t) \in \mathcal{H}(\mathcal{U}_t(B))$ for every t).

THEOREM 3. *The Hilbert space valued process*

$$X(s) = \int_0^t Y(s)\,ds + B(t), \qquad 0 < t < T$$

where $Y(t)$ is a measurable process well adapted to $\mathcal{U}_t(B)$ for which

$$\int_0^T E\|Y(s)\|^2\,ds < \infty$$

is equivalent to the standard Wiener process on interval $(0,T)$.

Proof. The independence of Hilbert space valued random variables $B(s)$ and $B(t+\tau) - B(t)$ (for $s \le t$, $\tau > 0$) will stand for independence of $B(t+\tau) - B(t)$ and $\mathcal{H}(\mathcal{U}_t(B))$. Hence, according to the assumptions of the theorem, the process $Y(t)$ is independent of future increments of the standard Wiener process $B(t)$. Using Theorem 2 we get that the processes $X(t)$ and $B(t)$ are equivalent. The proof is finished.

5. Examples. *Example* 1. Let $X(t)$ be the solution of the stochastic integral equation

$$X(t) = \int_0^t A(s)X(s)\,ds + B(t), \qquad t \in (0,T) \tag{5.1}$$

where $A(t)$ is a Hilbert-Schmidt operator for almost each $t \in (0,T)$ and

(5.2) $$\int_0^T |A(t)|^2\,dt < \infty.$$

Let $R(t,s)$ be operator-function which fulfills the integral equation

(5.3) $$R(t,s) = I + \int_s^t A(x)R(x,s)\,dx.$$

It is well known, that $R(t,s)$ is a bounded operator with norm

$$\|R(t,s)\| \le \exp \int_s^t \|A(x)\|\,dx \le \exp \int_0^T \|A(x)\|\,dx.$$

The process $X(t)$ can be represented in the form

(5.4) $$X(t) = \int_0^t V(t,s)\,dB(s) + B(t)$$

where $V(t,s) = R(t,s) - I$ is Hilbert-Schmidt operator with Hilbert-Schmidt norm $|V(t,s)|$ such that

$$\int_0^T |V(t,s)|^2\,ds < \infty.$$

This statement simply follows from the equality

$$\frac{dV(x,s)}{dx} = A(x)(V(x,s)+I).$$

Put

(5.5) $$Y(t) = A(t)X(t) = \int_0^t A(t)R(t,s)\,dB(s).$$

Hence, according to (5.1)

(5.6) $$X(t) = \int_0^t Y(s)\,ds + B(t).$$

From equality (5.5) it follows that the process $Y(t)$ is independent on future increments of standard Wiener process $B(t)$ ($Y(t)$ is measurable well adapted to $\mathcal{U}_t(B)$) and for finite $t < T$

$$\int_0^t E\|Y(s)\|^2\,ds < \infty.$$

Using Theorem 2 we get that the solution of the stochastic integral equation (5.1) is equivalent to the standard Winer process on finite interval $(0,t)$. If for an infinite T

$$\int_0^T \|A(t)\|\,dt < \infty,$$

then the solution of the stochastic integral equation (5.1) is equivalent to the standard Wiener process on infinite interval $(0,T)$.

Example 2. This example is an extension of Theorem 2 [4] for real Gaussian processes to Hilbert space valued Gaussian processes. Let

$$X(t) = B(t) - \int_0^t \left\{ \int_0^s l(s,x)\,dB(x) \right\} ds \tag{5.7}$$

where $B(t)$ is a Hilbert space valued standard Wiener process and $l(s,x) \in L^2((0,T)^2, S_2(H))$ is a Volterra kernel. Let

$$Y(s) = -\int_0^s l(s,x)\,dB(x).$$

Then the process $X(t)$ can be written in the form (4.1). It is clear that the process $X(t)$ is equivalent to the standard Wiener process on interval $(0,T)$.

REFERENCES

[1] Ю. П. Далецкий, С. В. Фомин, *Меры и дифференциальные уравнения в бесконечномерных пространствах*, Наука, Москва, 1983.

[2] M. P. Eršov, *On absolute continuity of measures corresponding to diffusion type processes*, Probab. Theory Appl. **17** (1972), 182–187.

[3] J. Feldman, *Equivalence and perpendicularity of Gaussian processes*, Pacific J. Math. **8** (1958), 699–708.

[4] M. Hitsuda, *Representation of Gaussian process equivalent to Wiener process*, Osaka J. Math. **5** (1968), 299–312.

[5] G. Kallianpur, V. Mandrecar, *Multiplicity and representation theory of purely non-deterministic stochastic processes*, Probab. Theory Appl. **10** (1965), 553–581.

[6] Љ. Петрушевски, *Об эквивалентности одного класса случайных процессов в гильбертовом пространстве и винеровского процесса*, Publ. Inst. Math. (Beograd) **45** (59) (1989) 185–194.

[7] Љ. Петрушевски, *Об эквивалентности одного процесса диффузионого типа в гильбертовом пространстве и винеровского процесса*, Publ. Inst. Math. (Beograd), **45** (59) (1989) 195–201.

[8] Yu. A. Rozanov, *Infinite-dimensional Gaussian Distributions*, Amer. Math. Soc. Providence, 1971.

[9] Ю. А. Розанов, *Теория обновляющих процессов*, Наука, Москва, 1974.

Katedra za matematiku
Arhitektonski fakultet
Bulevar revolucije 73/II
Beograd

(Received 18 05 1989)