PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 49 (63), 1991, 189–200

ON THE ZEROS OF OSCILLATORY SOLUTIONS OF LINEAR SECOND-ORDER DIFFERENTIAL EQUATIONS

M. Hačik and E. Omey

Abstract. Let $y(x)$ be a real non-trivial solution of the differential equation $(*)$ $y''+fy=0$ and let $\{x_n\}_{\mathrm{N}}$ denote the sequence of zeros of y. In this paper we examine the relationship between the asymptotic behaviour of f and that of $\{x_n\}_{\mathrm{N}}$ and sequences related to it. We also compare the zeros of solutions of $(*)$ with those of solutions of $Y''+gY=0$ where f and g do not differ too much.

1. Introduction.

Let $y(x)$ be a real non-trivial solution of the differential equation

$$y''(x)+f(x)y(x)=0 \qquad x\geq 0 \tag{1.1}$$

such that y has an infinite number of zeros $\{x_n\}_{\mathrm{N}}$. Here and in the sequel $f:\mathbf{R}^+\to\mathbf{R}^+$ is a *continuous* function of x. In this note we examine the relationship between the asymptotic behaviour of the zeros of $y(x)$ and that of $f(x)$.

In this section we first define the kind of asymptotic behaviour we are interested in. In sections 2 and 3 we state our results. For further use we define the following classes of functions.

Definition 1.1. A positive and measurable function $\varphi:\mathbf{R}^+\to\mathbf{R}^+$ belongs to the class of *Beurling slowly varying* functions (denoted also $\varphi\in\mathrm{B}$) if

$$\lim_{x\to\infty}\frac{\varphi(x+t\varphi(x))}{\varphi(x)}=1, \qquad \text{for each } t\in\mathbf{R}. \tag{1.2}$$

If φ satisfies (1.2) uniformly on compact t-sets of $\mathbf{R}$ (denoted $\varphi\in\mathrm{Bu}$), then φ is called *self-neglecting*, see [7].

Keywords: Oscillatory differential equations; Beurling-varying functions; regularly varying functions.

AMS Subject Classification (1985): Primary 60F05

Bloom [4] showed that if $\varphi \in$ B is continuous, then $\varphi \in$ Bu. Whether measurability is enough remains an open question. Bloom also showed that $\varphi \in$ Bu if and only if it has the representation

$$\varphi(x) = c(x) \int_0^x \varepsilon(s)\, ds \tag{1.3}$$

where $\lim_{x\to\infty} c(x) = 1$ and $\varepsilon(x)$ is continuous with $\lim_{x\to\infty} \varepsilon(x) = 0$.

It follows from (1.3) that any $\varphi \in$ Bu satisfies

$$\lim_{x\to\infty} \frac{\varphi(x)}{x} = 0. \tag{1.4}$$

A class of functions related to B is defined in

Definition 1.2. A positive and measurable function $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ belongs to the class Γ with auxiliary function $h : \mathbf{R}^+ \to \mathbf{R}^+$ if ($\varphi \in \Gamma(h)$)

$$\lim_{x\to\infty} \frac{\varphi(x + th(x))}{\varphi(x)} = e^t, \qquad \text{for each } t \in \mathbf{R}.$$

As has been proved by de Haan [9], the auxiliary function h can be chosen in such a way that h has the representation (1.3) and hence $h \in$ Bu. Also if $\varphi \in \Gamma(h)$ then $\varphi^{-1/2} \in$ Bu.

In the following we will also need the class of regularly varying functions.

Definition 1.3. A positive and measurable function $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ is called regularly varying with index $\alpha \in \mathbf{R}$ (denoted $\varphi \in \mathrm{RV}_\alpha$) if

$$\lim_{x\to\infty} \frac{\varphi(xt)}{\varphi(x)} = t^\alpha, \qquad \text{for each } t > 0. \tag{1.5}$$

When $\alpha = 0$, φ is said to be slowly varying. If (1.5) holds, then it is well known that it holds uniformly on compact t-sets of $(0, \infty)$. Hence (1.4) and (1.5) imply that $\varphi \in$ Bu. Note that (1.4) holds when $\alpha < 1$.

A subclass of the slowly varying functions is defined as follows.

Definition 1.4. A slowly varying function φ is called Π-varying with auxiliary function $L \in \mathrm{RV}_0$ (denoted $\varphi \in \Pi(L)$) if

$$\lim_{x\to\infty} \frac{\varphi(xt) - \varphi(x)}{L(x)} = \log t, \qquad \text{for each } t > 0.$$

For sequences we also give the following

Definition 1.5. A sequence $\{a_n\}_{\mathbf{N}}$ of positive numbers belongs to any of the classes defined above if the function $a(t) := a_{[t]}$ does.

All these classes of functions have proved to be very useful in all kind of problems that deal with the asymptotic behaviour of functions and sequences. Applications in the theory of differential equations may be found in Marić and Tomić [11], Wiman [16] or Omey [12]. For probabilistic applications of these classes of functions we refer to de Haan [8, 9], or Feller [6]. For a survey of the theory of regular variation and related subjects we refer to Seneta [13], de Haan [8] or Bingham et al. [1, 3].

2. Results

2.1. First order results. Let us start with the following result of Eastham [5, p. 74].

LEMMA 2.1. (i) *Suppose $y(x)$ has n zeros in $[a,b]$ and suppose $f(x) \le M$ for all $x \in [a,b]$. Then $n \le (b-a)\Pi^{-1}\sqrt{M}+1$.*

(ii) *Suppose $y(x)$ has n zeros in (a,b) and suppose $m \le f(x)$ for all $x \in [a,b)$. Then $n \ge (b-a)\Pi^{-1}\sqrt{m}-1$.* □

From Lemma 2.1 we now obtain a result which is crucial for the rest of the paper. Let $\{x_n\}_{\mathrm{N}}$ denote the sequence of zeros of a non-trivial solution of (1.1).

LEMMA 2.2. *For $n = 1, 2, \dots$ we have*

$$\Pi/\sqrt{f(\theta_s)} \le x_{n+1} - x_n \le \Pi/\sqrt{f(\theta_i)} \tag{2.1}$$

where $f(\theta_i) = \inf_I f(s)$, $f(\theta_s) = \sup_I f(s)$ and $I = [x_n, x_{n+1}]$. Also

$$\theta_i - \Pi/\sqrt{f(\theta_i)} \le \theta_s \le \theta_i + \Pi/\sqrt{f(\theta_i)}. \tag{2.2}$$

Furthermore, if $1/\sqrt{f} \in Bu$, then $x_{n+1} \sim x_n$ $(n \to \infty)$ and

$$\lim_{n\to\infty} \sqrt{f(x_n)}(x_{n+1} - x_n) = \Pi. \tag{2.3}$$

Proof. Inequality (2.1) follows from Lemma 2.1 by taking $a = x_n$ and $b = x_{n+1}$. Recall that we assume f to be continuous so that θ_s and θ_i are well-defined. Now (2.2) follows from (2.1) and the inequality $|\theta_s - \theta_i| \le x_{n+1} - x_n$. Finally, if $f^{-1/2} \in$ Bu it follows from (2.2) and the uniform convergence in (1.2) that $f(\theta_i) \sim f(\theta_s)$ $(n \to \infty)$. Since $f(\theta_i) \le f(x_n) \le f(\theta_s)$, (2.3) follows at once. To see that $x_{n+1} \sim x_n$ $(n \to \infty)$ use (2.3) and $x^2 f(x) \to \infty$ $(x \to \infty)$ (cf. (1.4)). □

To state our next result, let $N(A)$ denote the number of zeros in a set A.

THEOREM 2.3. *Suppose $1/\sqrt{f} \in$ Bu; then*

(i) $N((0,x]) \sim \Pi^{-1}\int_0^x \sqrt{f(s)}\,ds \qquad (x \to \infty)$

(ii) $N((x, x + t/\sqrt{f(x)}]) = t\Pi^{-1} + O(1)$ *for each $t > 0$.*

Proof. (i) From (2.3) we have

$$\sum_{n=1}^{m} \sqrt{f(x_n)}(x_{n+1} - x_n) \sim \Pi m \sim \Pi N((0, x_m]) \qquad (m \to \infty). \tag{2.4}$$

Using a standard argument (2.4) gives

$$\int_0^{x_m} \sqrt{f(s)}\, ds \sim \Pi N((0, x_m]) \qquad (m \to \infty). \tag{2.5}$$

Using (2.3) the result now easily follows from (2.5).

(ii) To prove (ii), use Lemma 2.1 to obtain

$$\frac{t}{\Pi\sqrt{f(x)}}\sqrt{f(\theta_i)} - 1 \le N\left(\left(x, x + t\frac{1}{\sqrt{f(x)}}\right]\right) \le \frac{t}{\Pi\sqrt{f(x)}}\sqrt{f(\theta_s)} + 1,$$

where $f(\theta_i) = \inf_I f(s)$, $f(\theta_s) = \sup_I f(s)$ and $I = [x, x + t/\sqrt{f(x)}]$. Since $1/\sqrt{f} \in$ Bu it follows that $f(x) \sim f(\theta_i) \sim f(\theta_s)$ $(x \to \infty)$ and (ii) follows.

Remarks 2.4. 1. The condition $f^{-1/2} \in$ Bu implies that $x^2 f(x) \to \infty$ $(x \to \infty)$ (cf. (1.4)). For the case where $x^2 f(x) \to C$ $(x \to \infty)$, $C < \infty$, we refer to Section 3.

2. If the functions f and h are such that $h^2(x)f(x) \to \infty$ $(x \to \infty)$ and

$$\lim_{x\to\infty} \frac{f(x + th(x))}{f(x)} = 1$$

unformly in compact t-intervals of $\mathbf{R}$, then the same arguments as in the proof of Theorem 2.3 yield

$$\lim_{x\to\infty} \frac{N((x, x + th(x)])}{h(x)\sqrt{f(x)}} = \frac{t}{\Pi}, \qquad \text{for each } t > 0. \tag{2.6}$$

In the special case when $h \equiv 1$ we get from (2.6) and definition 1.4 that $N(]0, \log x]) \in \Pi(\Pi^{-1}\sqrt{f(\log x)})$. Note that (2.6) implies that

$$N((0, x + th(x)]) \sim N((0, x]) \qquad (x \to \infty). \tag{2.7}$$

Also (2.6) gives the rate of convergence in (2.7).

A natural extension of the class Bu is the class of so-called selfcontrolled functions, i.e. measurable functions $\varphi : \mathbf{R}^+ \to \mathbf{R}^+$ for which there exist Δ, $G > 0$, $x_0 \ge 0$ such that

$$\varphi(x + t\varphi(x)) \le G\varphi(x), \qquad |t| \le \Delta,\ x \ge x_0. \tag{2.8}$$

Using a similar technique as in the proof of Theorem 2.3 we now obtain the following extension of that theorem.

THEOREM 2.5. *If f is continuous such that $\int_0^\infty \sqrt{f(s)}\,ds = \infty$ and if $\varphi :=$ $^{1/2}$ satisfies* (2.8) *with $\Delta \geq \Pi$, then*

$$\frac{1}{G} < \liminf_{x\to\infty} \frac{N((0,x])}{\Pi^{-1}\int_0^x \sqrt{f(s)}\,ds} \leq \limsup_{x\to\infty} \frac{N((0,x])}{\Pi^{-1}\int_0^x \sqrt{f(s)}\,ds} < G. \quad \square$$

For further properties of the class of self-controlled functions we refer to Goldie and Smith [7].

As a corollary we now obtain the following result of which part (i) is well known, see e.g. Hartman [10, XI.5].

COROLLARY 2.6. *Supose f has a measurable derivative f'.*
(i) *If $f'(x) = o(f^{3/2}(x))$ $(x \to \infty)$, then the results of Theorem* 2.3 *hold.*
(ii) *If $|f'(x)| = O(f^{3/2}(x))$ $(x \to \infty)$, then the results of Theorem* 2.5 *hold.*

Proof. Let $\varphi(x) := f^{-1/2}(x)$. Then we have

$$\left|\frac{\varphi(x + t\varphi(x)) - \varphi(x)}{\varphi(x)}\right| \leq \int_0^t |\varphi'(x + \theta\varphi(x))|\,d\theta. \tag{2.9}$$

In case (i) we have $\varphi'(x) = o(1)$ $(x \to \infty)$ and using (2.9) this implies $\varphi \in \mathrm{Bu}$ and Theorem 2.3 applies. In case (ii) we have $|\varphi'(x)| = O(1)$ so that using (2.9)

$$\left|\frac{\varphi(x + t\varphi(x))}{\varphi(x)} - 1\right| \leq At$$

where $|\varphi'(x)| \leq A$ (say). Hence (2.8) holds.

If we now restrict attention to regularly varying functions or to functions in the class Γ we obtain that if $f^{-1/2} \in \mathrm{Bu}$, then $N((0,x]) \in \mathrm{RV}_b$ (resp. $N((0,x]) \in \Gamma(h)$ if and only if $\Pi^{-1}\int_0^x \sqrt{f(s)}\,ds \in \mathrm{RV}_b$ (resp. $\Gamma(h)$).

When it is known that f is regularly varying or that f is in the class Γ we obtain

COROLLARY 2.7. (i) *If $f \in \mathrm{RV}_a$, $a > -2$, then $N((0,x]) \in \mathrm{RV}_{(a+2)/2}$ and*

$$N((0,x]) \sim 2\Pi^{-1}(a+2)^{-1}x\sqrt{f(x)} \qquad (x \to \infty).$$

(ii) *If $f \in \Gamma(h(x))$, then $N((0,x]) \in \Gamma(2h(x))$ and*

$$N((0,x]) \sim 2\Pi^{-1}h(x)\sqrt{f(x)} \qquad (x \to \infty).$$

The proof of this corollary follows from Theorem 2.3 and elementary properties of the classes RV and Γ (see de Haan [8] or Seneta [13]).

Remark 2.8. If in Corollary 2.7 (i) $a = -2$ and $x^2 f(x) \to \infty$ $(x \to \infty)$, it remains true that $N((0,x]) \in \mathrm{RV}_0$. For a refinement of this result we refer to Theorem 2.11 (ii).

Next we obtain a result for the sequence of zeros $\{x_n\}_{\mathbf{N}}$.

COROLLARY 2.9. (i) *If* $f \in \mathrm{RV}_a$, $a > -2$, *then* $x_n\sqrt{f(x_n)} \sim n\Pi(a+2)/2$ $(n \to \infty)$ *and*

$$n\frac{x_{n+1} - x_n}{x_n} \to \frac{2}{a+2} \qquad (n \to \infty).$$

(ii) *If* $f \in \Gamma(h(x))$, *then* $h(x_n)\sqrt{f(x_n)} \sim n\Pi/2$ $(n \to \infty)$ *and*

$$n\frac{x_{n+1} - x_n}{h(x_n)} \to 2 \qquad (n \to \infty).$$

Proof. Use Corollary 2.7 with x replaced by x_n and then use (2.3).

Remarks 2.10. 1. A somewhat weaker result than that of Corollary 2.9 can be obtained as follows. If we define $F(x) := \Pi^{-1}\int_0^x \sqrt{f(s)}\,ds$, then F is non-decreasing and F^{-1} is well-defined. Theorem 2.3 now states that

$$F(x_n) \sim n \qquad (n \to \infty). \tag{2.10}$$

a) If $F \in \mathrm{RV}_b$ $(b > 0)$, then (2.10) implies $x_n \sim F^{-1}(n)$ $(n \to \infty)$ and this in turn implies that $\{x_n\}_{\mathbf{N}} \in \mathrm{RV}_{1/b}$ in the sense of Definition 1.5. The additional assumption that $f \in \mathrm{RV}_a$ $(a > -2)$ ($\Longrightarrow$ $F \in \mathrm{RV}_b$ with $b = (a+2)/2$) gives $\{x_n\}_{\mathbf{N}} \in \mathrm{RV}_{1/b}$ but also that $\{x_{n+1} - x_n\}_{\mathbf{N}} \in \mathrm{RV}_{1/b-1}$.

b) If we assume $F \in \Gamma(h)$, then (2.10) implies that $|x_n - F^{-1}(n)| = O(h(F^{-1}(n)))$ $(n \to \infty)$ (cf. de Haan [9]). Hence also $|x_n - F^{-1}(n)| = O(h(x_n))$, and $\{x_n\}_{\mathbf{N}} \in \Pi(h(x_n))$ in the sense of Definition 1.5.

For a refinement of these results we refer to Theorem 2.13.

2. Since in (1.1) $y''(x)$ has the opposite sign of $y(x)$, y has exactly one local extreme value between two consecutive zeros of y. Hence the values at which $|y|$ reaches a local maximum have the same asymptotic behaviour as the zeros $\{x_n\}_{\mathbf{N}}$.

2.2. Second-order behaviour. In Theorem 2.3 we obtained a result for $N((0,x])$. Next we state a second order result in which we estimate the difference between $N((0,x])$ and $\Pi^{-1}\int_0^x \sqrt{f(s)}\,ds$. A classical result of Hartman [10, XI] states that if f is continuous and of bounded variation in every interval $[0,x]$, then

$$\left|N((u,v]) - \frac{1}{\Pi}\int_u^v \sqrt{f(s)}\,ds\right| \le 1 + \frac{1}{4\Pi}\int_u^v \frac{|d\sqrt{f(s)}|}{\sqrt{f(s)}}, \qquad 0 \le u \le v < \infty$$

and in general this estimate is best possible. If f is monotone this inequality reduces to

$$(2.11)\qquad \left|N((u,v]) - \frac{1}{\Pi}\int_u^v \sqrt{f(s)}\,ds\right| \le 1 + \frac{1}{8\Pi}\left|\log\frac{f(v)}{f(u)}\right|, \qquad 0 < u \le v < \infty.$$

Now we prove

THEOREM 2.11. (i) *If $f \in \mathrm{RV}_a$, $a \ge -2$, and if f is monotone, then*

$$(2.12)\qquad \left|N((u,x]) - \frac{1}{\Pi}\int_u^x \sqrt{f(s)}\,ds\right| = O(\log x) \qquad (x \to \infty).$$

(ii) *If $f \in \mathrm{RV}_{-2}$ and if f is monotone with $x^2 f(x) \to \infty$ $(x \to \infty)$, then* $N((0,x]) \in \Pi(\Pi^{-1}x\sqrt{f(x)})$.

Proof. (i) If $f \in \mathrm{RV}_a$, then it follows from the representation theorem of regularly varying functions, that $|\log f(x)| = O(\log x)$ $(x \to \infty)$, (see e.g. [**13**], [**1**, II]). Hence (2.12) follows from (2.11).

(ii) If $f \in \mathrm{RV}_{-2}$ we have for every $t \ge 1$ that

$$\lim_{x\to\infty} \frac{\int_x^{xt} \sqrt{f(s)}\,ds}{x\sqrt{f(x)}} = \lim_{x\to\infty}\int_1^t \sqrt{\frac{f(xs)}{f(x)}}\,ds = \int_1^t \frac{1}{s}\,ds = \log t.$$

Hence part (ii) follows from (2.11) using $x^2 f(x) \to \infty$ $(x \to \infty)$.

Remarks 2.12. 1. If $x^2 f(x) \to C$, $C < \infty$ $(x \to \infty)$, then Theorem 2.11 (i) only gives $N((u,x]) = O(\log x)$ $(x \to \infty)$ since in this case $\int_u^x \sqrt{f(s)}\,ds = O(\log x)$ $(x \to \infty)$. For a refinement of this result we refer to section 3, Corollary 3.2.

2. If the conditions of Theorem 2.11 (ii) hold, then it follows from de Haan [9] that the sequence $\{x_n\}_{\mathrm{N}}$ belongs to the class $\Gamma(\Pi^{-1}x_n\sqrt{f(x_n)})$. If for example $f(x) = x^{-2}\log^2 x$ $(x > 0)$, then $N((0,x]) \in \Pi(\Pi^{-1}\log x)$ and $\{x_n\}_{\mathrm{N}} \in \Gamma(\Pi^{-1}\log x_n)$.

Since (cf. 2.11) $|n - \Pi^{-1}\log^2 x_n| = O(\log x_n)$ $(n \to \infty)$ it follows that $\{x_n\}_{\mathrm{N}} \in \Gamma(\sqrt{2n/\Pi})$ and that $\{x_n\}_{\mathrm{N}}$ is of the order of growth $e^{\sqrt{2\Pi n}}$. If $f(x) = x^{-2}(\log^4 x)$ $(x > 0)$ then in a similar way it follows that $x_n \sim \exp\big((3\Pi n)^{1/3}\big)$ $(n \to \infty)$.

Using the previous results we now refine the results of remark 2.10.1. If $f \in \mathrm{RV}_a$, $a > -2$, we get from (2.12) that

$$(2.13)\qquad |n - F(x_n)| = O(\log x_n) \qquad (n \to \infty).$$

Since in this case $\{x_n\}_{\mathrm{N}} \in \mathrm{RV}_{a/(a+2)}$ it follows from Seneta [**13**], that $\log x_n = O(\log n)$ and hence (2.13) can be replaced by

$$(2.14)\qquad |n - F(x_n)| = O(\log n) \qquad (n \to \infty).$$

If $f \in \Gamma(h)$, then (2.11) gives $|n - F(x_n)| = O(\log f(x_n))$ $(n \to \infty)$. Since $\{h(x_n)\}_{\mathrm{N}} \in \mathrm{RV}_0$ (cf. Remark 2.10.1), it follows from Corollary 2.9 (ii) and **[13]** that also here (2.14) holds. Now (2.14) implies

$$F^{-1}(n - B\log n) \le x_n \le F^{-1}(n + B\log n) \qquad (n \ge n_0) \tag{2.15}$$

for some $B > 0$ and $n \ge n_0$. Since $(F^{-1})'(x) = \Pi/\sqrt{f(F^{-1}(x))}$ it follows that

$$|F^{-1}(n \pm B\log n) - F^{-1}(n)| = \frac{\Pi}{\sqrt{f(F^{-1}(\theta))}} B\log n, \tag{2.16}$$

where $|\theta - n| \le B\log n$. If $f \in \mathrm{RV}_a$, $a > -2$, this implies that $f(F^{-1}(\theta)) \sim f(F^{-1}(n)) \sim f(x_n)$ $(n \to \infty)$. Combining this with (2.15), (2.16) and Corollary 2.9 we obtain

$$|x_n - F^{-1}(n)| = O\big((x_n \log n)/n\big) \qquad (n \to \infty).$$

If we assume $f \in \Gamma(h)$, we have $|F^{-1}(\theta) - F^{-1}(n)| = o(h(x_n))$ $(n \to \infty)$. Using Remark 2.10.1.b this implies $f(F^{-1}(\theta)) \sim f(x_n)$ $(n \to \infty)$. Combining this with (2.15), (2.16) and Corollary 2.9 we obtain

$$|x_n - F^{-1}(n)| = O\big((h(x_n)\log n)/n\big) \qquad (n \to \infty).$$

Hence we have proved

THEOREM 2.13. *Assume f is a monotone function and define* $F(x) := \Pi^{-1}\int_0^x \sqrt{f(s)}\,ds$.

(i) *If $f \in \mathrm{RV}_a$, $a > -2$, then* $|x_n - F^{-1}(n)| = O\big((x_n \log n)/n\big)$ $(n \to \infty)$

(ii) *If $f \in \Gamma(h)$, then* $|x_n - F^{-1}(n)| = O\big((h(x_n)\log n)/n\big)$ $(n \to \infty)$. □

Our next application is devoted to comparing the zeros $\{x_n\}_{\mathrm{N}}$ of any nontrivial solution of (1.1) with the zeros $\{y_n\}_{\mathrm{N}}$ of any non-trivial solution of $Y''(x) + gY(x) = 0$ $(x \ge 0)$. For convenience we define $G(x) := \Pi^{-1}\int_0^x \sqrt{g(s)}\,ds$.

THEOREM 2.14. (i) *If $f, g \in \mathrm{RV}_a$, $a > -2$, are monotone and if $f(x) \sim g(x)$ $(x \to \infty)$, then $x_n \sim y_n$ $(n \to \infty)$. Furthermore, if $|F(x) - G(x)| = O(\log x)$ $(x \to \infty)$, then $|y_n - x_n| = O\big((x_n \log n)/n\big)$ $(n \to \infty)$.*

(ii) *If $f, g \in \Gamma(h)$ are monotone and if $f(x) \sim g(x)$ $(x \to \infty)$, then $|y_n - x_n| = O(h(x_n))$ $(n \to \infty)$. Furthermore, if $|F(x) - G(x)| = O(\log f(x))$ $(x \to \infty)$, then $|y_n - x_n| = O\big((h(x_n)\log n)/n\big)$ $n \to \infty$.*

Proof. (i) The proof of the first part follows from Remark 2.10.1.a and $F^{-1}(x) \sim G^{-1}(x)$ $(x \to \infty)$. The proof of the second part follows from Theorem 2.13 and $|F^{-1}(x) - G^{-1}(x)| = O(\log x)$ $(x \to \infty)$.

(ii) To prove the first part, note that the assumptions imply that $F(x) \sim G(x)$ $(x \to \infty)$ so that (cf. de Haan [9])

$$\frac{F^{-1}(x) - G^{-1}(x)}{h(F^{-1}(x))} \to 0 \qquad (x \to \infty).$$

Now use remark 2.10.1.b to obtain $|y_n - x_n| = o(h(x_n))$ $(n \to \infty)$. The proof of the second part follows from Theorem 2.13 since $|F(x) - G(x)| = O(\log f(x))$ implies

$$|F^{-1}(n) - G^{-1}(n)| = O\big((h(x_n)\log n)/n\big) \qquad (n \to \infty). \quad \square$$

Our next result is devoted to the differences $\Delta x_n = x_{n+1} - x_n$. In Corollary 2.9 we obtained a result for Δx_n. Next we seek a result for the second-order differences $\Delta^2 x_n = \Delta x_{n+1} - \Delta x_n$. We prove that if $1/\sqrt{f} \in \mathrm{Bu}$ with some known second-order behaviour then $\tau(x_n)\Delta^2 x_n = O(1)$ or $o(1)$ $(n \to \infty)$ for some normalizing function τ. For convenience we only prove a result for nondecreasing f.

LEMMA 2.15. *If f is non-decreasing and if $g := 1/\sqrt{f} \in \mathrm{Bu}$, such that for some function $\tau(x)$,*

$$\tau(x)|g(x + \Pi g(x)) - g(x)| = O(1) \quad or \quad o(1) \qquad (x \to \infty), \tag{2.16}$$

then

$$\tau(x_n)|x_{n+1} - x_n - \Pi g(x_n)| = O(1) \quad or \quad o(1) \qquad (x \to \infty). \tag{2.17}$$

Furthermore, if $\tau(x_n) = O(\tau(x_{n+1}))$ $(n \to \infty)$, then

$$\tau(x_n)\Delta^2 x_n = O(1) \quad or \quad o(1) \qquad (n \to \infty). \tag{2.18}$$

Proof. From (2.1) and the monotoncity of g we have that

$$\Pi(g(x_n + \Pi g(x_n)) - g(x_n)) \le \Pi(g(x_{n+1}) - g(x_n)) \le \Delta x_n - \Pi g(x_n) \le 0. \tag{2.19}$$

Hence (2.16) implies (2.17). To prove (2.18), combine (2.19), (2.17) and the identity

$$\tau(x_n)\Delta^2 x_n = \frac{\tau(x_n)}{\tau(x_{n+1})}\tau(x_{n+1})\big(\Delta x_{n+1} - \Pi g(x_{n+1})\big) + \tau(x_n)\Pi\big(g(x_{n+1}) - g(x_n)\big) - \tau(x_n)\big(\Delta x_n - \Pi g(x_n)\big). \quad \square$$

As an example let us assume that f has a measurable derivative f' such that $f' = o(f^{3/2}(x))$ and such that

$$\lim_{x \to \infty} \frac{f'(x + t/\sqrt{f(x)})}{f'(x)} = 1 \tag{2.20}$$

uniformly in compact t-intervals of $\mathbf{R}$. Then we have

$$|g(x+tg(x))-g(x)| \leq \int_0^t |g'(x+\theta g(x))| g(x)\, d\theta. \tag{2.21}$$

Now $g'(x) = f'(x)f^{-3/2}(x) \to 0$ so that $g \in \mathrm{Bu}$. Hence using (2.20) we have

$$\frac{g'(x+\theta g(x))}{g'(x)} = \frac{f'(x+\theta/\sqrt{f(x)})}{f'(x)} \cdot \left(\frac{g(x+\theta g(x))}{g(x)}\right)^3 \to 1 \qquad (x \to \infty)$$

uniformly for $\theta \in [0,1]$. But then it follows from (2.21) that

$$|g(x+tg(x))-g(x)| = O(|g'(x)g(x)|) \qquad (x \to \infty)$$

and (2.16) holds with $\tau(x) = \dfrac{1}{|g'(x)|g(x)} = \dfrac{f^2(x)}{|f'(x)|}$.

If e.g. $f' \in \mathrm{RV}_a$ ($a > -1$), then (2.20) holds and (2.16) holds with $\tau(x) \sim xf(x)/(a+1)$ $(x \to \infty)$. Using Corollary 2.9 (i), (2.18) becomes $n^2\Delta^2 x_n/x_n = O(1)$ $(n \to \infty)$. If e.g. $f' \in \Gamma(h)$, then [9] $f(x) \sim h(x)f'(x)$ $(x \to \infty)$ and (2.20) holds. Hence (2.16) holds with $\tau(x) \sim h(x)f(x)$ $(x \to \infty)$. Using Corollary 2.9 (ii), (2.18) becomes $n^2\Delta^2 x_n/h(x_n) = O(1)$ $(n \to \infty)$.

3. The behaviour in case $x^2 f(x) \to C$ $(x \to \infty)$

Up to now we always assumed that $f^{-1/2} \in \mathrm{Bu}$ which implies $x^2 f(x) \to \infty$ $(x \to \infty)$. When $x^2 f(x) \to C \geq 1/4$ $(x \to \infty)$ however (1.1) may remain oscillatory. To deal with this kind of equations we transform (1.1) into a more suitable form. Generally, consider the differential equation

$$(gy')' + fy = 0, \qquad g > 0,\ x \geq a \geq 0 \tag{3.1}$$

where $g \in C^1[a,\infty)$ and $f \in G[a,\infty)$. If we choose $\Psi \in G^2[a,\infty)$ we can define

$$\xi(x) := \int_a^x \frac{ds}{g(x)\Psi^2(x)}, \qquad \eta(\xi) = \frac{y(x)}{\Psi(x)}. \tag{3.2}$$

With this transformation (3.1) becomes

$$d^2\eta/\delta\xi^2 + F(\xi)\eta(\xi) = 0, \qquad 0 \leq \xi \leq \xi(\infty) \tag{3.3}$$

where $F(\xi) = [(g\Psi')' + f\Psi]\Psi^3 g$ (see [15, p. 597]). Now x_n is a zero of y if and only if $\xi_n \equiv \xi(x_n)$ is a zero of η. Hence the number of zeros of y less than or equal t is the same as the number of zeros of η less than $\xi(t)$. Now Theorem 2.3 yields

THEOREM 3.1. *Suppose $F(\xi)$ satisfies the conditions of Theorem 2.3. Then for equation* (3.1) *we have*

$$N((0,x]) \sim \frac{1}{\Pi} \int_a^x \sqrt{F(\xi(s))}\xi'(s)\, ds \qquad (x \to \infty). \quad \square \tag{3.4}$$

In the case of (1.1) we have $g \equiv 1$. If we choose $\Psi(x) = \sqrt{x}$, $a = 1$, then $\xi = \log x$ and $F(\xi) = x^2 f(x) - 1/4$. Hence Theorem 3.1 yields

COROLLARY 3.2. *If* $F(x) = e^{2x} f(e^x) - 1/4 \in \mathrm{RV}_a$, $a > -2$, *then*

$$N((0,x]) \sim \Pi^{-1} \log x \sqrt{x^2 f(x) - 1/4}\, \frac{2}{a+2} \qquad (x \to \infty).$$

If $F \in \Gamma(h(x))$, *then*

$$N((0,x]) \sim \Pi^{-1} 2h(\log x) \sqrt{x^2 f(x) - 1/4} \qquad (x \to \infty). \quad \square$$

Remark 3.3. Corollary 3.2 of course remains valid when $x^2 f(x) \to \infty$.

As an example let us consider the following generalized Euler equation. For $n \geq 0$ define

$$\begin{aligned} l_0(x) &= x, & l_{n+1}(x) &= \log(l_n(x)); \\ M_0(x) &= 1, & M_{n+1}(x) &= M_n(x) l_n(x); \\ K_{-1}(x) &= -1, & K_n(x) &= [K_{n-1}(x) + 1] l_n^2(x); \\ e^{-2} &= \infty, & e^{-1} &= 0, \quad e^n = \exp(e^{n-1}). \end{aligned}$$

Also let

$$f_n(x) = \frac{4k^2 + 1 + K_{n-1}(x)}{4M_n^2(x)}, \qquad n \geq 0,\ x \geq e^{n-2},$$

where k is an arbitrary positive constant. The differential equation

$$y'' + f_n y = 0, \qquad x \geq e^{n-2} \tag{3.5}$$

is called a generalized Euler equation. Its fundamental system of solutions is given by

$$y_1(x) = M_n^{1/2}(x) \sin(k l_n(x)), \qquad y_2(x) = M_n^{1/2}(x) \cos(k l_n(x)). \tag{3.6}$$

For $n = 1$ we have $f_1(x) = (4k^2 + 1)/(4x^2)$ so that, applying the transformation $\Psi(x) = \sqrt{x}$, $\xi(x) = \log x$, we obtain $F_1(\xi) = k^2 = f_0(x) \in \mathrm{RV}_0$. Hence Corollary 3.2 gives

$$N_1((0,x]) \sim \Pi^{-1}(\log x) \cdot k = \Pi^{-1} l_1(x) k \qquad (x \to \infty).$$

For $n = 1, 2, \ldots$ we obtain by induction that

$$F_{n+1}(\xi) = x^2 f_{n+1}(x) - 1/4 = f_n(\xi).$$

The result of n applications of the transformation as before reduces equation (3.5) to equation (3.5) with $n = 1$.

Hence the zeros of the nontrivial solutions of (3.5) satisfy

$$N_n((0,x]) \sim k\Pi^{-1} l_n(x) \qquad (x \to \infty)$$

a result which is obvious from (3.6).

REFERENCES

[1] N. H. Bingham and C. M. Goldie, *Extensions of regular variation. I: Uniformity and quatifiers. II: Representations and indices.* Proc. London Math. Soc. (3) **44** (1982), 473–496 and 497–534.

[2] N. H. Bingham and C. M. Goldie, *On one-sided Tauberian conditions*, Analysis **3** (1983), 159–188.

[3] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation*, Cambridge University Press, 1987.

[4] S. Bloom, *A characterization of B-slowly varying functions*, Proc. Am. Math. Soc. **54** (1976), 243–250.

[5] M. S. P. Eastham, *Theory of Ordinary Differential Equations*, Van Nostrand Reinhold, London, 1970.

[6] W. Feller, *An Introduction to Probabilty Theory and its Applications. Vol. II.* John Wiley & Sons, New York, 1966.

[7] C. M. Goldie and R. L. Smith, *Slow variation with remainder: a survey of theory and its applications*, Quart. J. Math. (3) **38** (1987), 1–27.

[8] L. de Haan, *On Regular Variation and its Application to the Weak Convergence of Sample Extremes*, Math. Centre Tract 32, Amsterdam, 1970.

[9] L. de Haan, *Equivalence classes of regularly varying functions.* Stoch. Proc. Appl. **2** (3) (1974), 243–259.

[10] Ph. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, New York, 1964.

[11] V. Marić and M. Tomić, *Asymptotics of solutions of a generalized Thomas-Fermi equation*, J. Diff. Equations **35** (1980), 36–44.

[12] E. Omey, *Regular variation and applications to second order linear differential equations*, Bull. Soc. Math. Belg. **33** (B) (1981), 207–229.

[13] E. Seneta, *Regularaly Varying Functions*, Lecture Notes in Math. **508**, Springer-Verlag, 1976.

[14] J. Vosmanský, *Distribution of zeros of solutions of linear second order differential equations*, Differentsial. Urav. **15** (1979), 2115–2124.

[15] D. Willett, *Classification of second order linear diffiniall equations with respect to oscillation*, Adv. in Math. **3** (1969), 594–623.

[16] A. Wiman, *Über eine Stabilitätsfrage in der Theorie der linearen Differentialgleichungen*, Acta Math. **66** (1936), 121–145.

Ehsal, Stormstraat 2, B-1000 Brussels, Belgium (Received 12 06 1989)

Transport College, Žilina, Czechoslovakia

Added in proof. After finishing this paper M. Hačik deceased. Therefore the second author dedicates this paper to M. Hačik, a nice colleague and friend.