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RESOLUTIONS OF SPACES ARE STRONG EXPANSIONS

Sibe Mardešić

Abstract. Strong expansions of spaces are morphisms of pro-Top from spaces to inverse systems, which satisfy a stronger version of the homotopy conditions of K. Morita. In the present paper it is proved that resolutions of spaces are always strong expansions. This strengthens the known result that resolutions are always expansions.

1. Introduction

Let $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ be an inverse system of spaces over a directed set Λ and let $\boldsymbol{p} : X \to \boldsymbol{X}$ be a morphism of pro-Top, i.e., $\boldsymbol{p}$ is a collection of mappings $p_\lambda : X \to X_\lambda$, $\lambda \in \Lambda$, such that $p_{\lambda\lambda'}p_{\lambda'} = p_\lambda$, for $\lambda \leq \lambda'$. $\boldsymbol{p}$ is called an expansion of X if the following conditions of K. Morita are satisfied [7]:

(M1) If P is an ANR (for metric spaces) and $f : X \to P$ is a mapping, then there exist a $\lambda \in \Lambda$ and a mapping $h : X_\lambda \to P$ such that

$$hp_\lambda \simeq f. \tag{1}$$

(M2) If $\lambda \in \Lambda$, P is an ANR and $f_0, f_1 : X_\lambda \to P$ are mappings such that

$$f_0 p_\lambda \simeq f_1 p_\lambda, \tag{2}$$

then there exists a $\lambda' \geq \lambda$ such that

$$f_0 p_{\lambda\lambda'} \simeq f_1 p_{\lambda\lambda'}. \tag{3}$$

The following notion of strong expansion was defined in [2] and [5].

Definition 1. A morphism $\boldsymbol{p} : X \to \boldsymbol{X}$ of pro-Top is a strong expansion provided it satisfies condition (M1) and the following condition:

(SM2) Let P be an ANR, $\lambda \in \Lambda$, let $f_0, f_1 : X_\lambda \to P$ be maps and let $F : X \times I \to P$ be a homotopy such that

$$F(x,0) = f_0 p_\lambda(x), \qquad x \in X, \tag{4}$$

$$F(x,1) = f_1 p_\lambda(x), \qquad x \in X. \tag{5}$$

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Then there exist a $\lambda' \geq \lambda$ and a homotopy $H : X_{\lambda'} \times I \to P$, such that

$$H(z,0) = f_0 p_{\lambda\lambda'}(z), \qquad z \in X_{\lambda'}, \tag{6}$$

$$H(z,1) = f_1 p_{\lambda\lambda'}(z), \qquad z \in X_{\lambda'}, \tag{7}$$

$$H(p_{\lambda'} \times 1) \simeq F(\mathrm{rel}(X \times \partial I)). \tag{8}$$

Remark 1. (SM2) differs from (M2) only in the additional requirement (8). Therefore, every strong expansion is an expansion.

Recall that a resolution [4] (also see [6]) is a morphism $\boldsymbol{p} : X \to \boldsymbol{X}$ of pro-Top, where the following two conditions are satisfied.

(R1) If $P \in$ ANR, $\mathcal{V}$ is an open covering of P and $f : X \to P$ is a mapping, then there exist a $\lambda \in \Lambda$ and a mapping $h : X_\lambda \to P$ such that f and hp_λ are $\mathcal{V}$-near mappings, denoted by

$$(hp_\lambda, f) \leq \mathcal{V}. \tag{9}$$

(R2) For $P \in$ ANR and $\mathcal{V}$ an open covering of P, there exists an open covering $\mathcal{V}'$ of P such that, whenever $\lambda \in \Lambda$ and $f_0, f_1 : X_\lambda \to P$ are mappings such that

$$(f_0 p_\lambda, f_1 p_\lambda) \leq \mathcal{V}', \tag{10}$$

then there is a $\lambda' \geq \lambda$ such that

$$(f_0 p_{\lambda\lambda'}, f_1 p_{\lambda\lambda'}) \leq \mathcal{V}. \tag{11}$$

The main result of this paper is the folloowing theorem.

THEOREM 1. *Every resolution $\boldsymbol{p} : X \to \boldsymbol{X}$ of a space X is a strong expansion of X.*

CORROLARY 1. *Every resolution is an expansion.*

This known fact (see [6, I, 6.1, Theorem 2]) follows from Theorem 1, by Remark 1. Another consequence of Theorem 1 is the following corollary.

COROLLARY 2. *Every space X admits a cofinite strong ANR-expansion, i.e. a strong expansion $\boldsymbol{p} : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$, where all the X_λ's are ANR's and Λ is cofinite.*

This is a consequence of [3, II, Theorem 1], where the analogous statement for resolutions is proved.

2. Main lemma

LEMMA 1. *Let $\boldsymbol{p} : X \to \boldsymbol{X}$ be a resolution and let λ, P, f_0, f_1 and F be as in (SM2). Then for every open covering $\mathcal{U}$ of P, there exist a $\lambda' \geq \lambda$ and a homotopy $H : X_{\lambda'} \times I \to P$ such that*

$$H(y,0) = f_0 p_{\lambda\lambda'}(y), \quad y \in X_{\lambda'}, \tag{1}$$

$$(2) \qquad H(y,1) = f_1 p_{\lambda\lambda'}(y), \quad y \in X_{\lambda'},$$
$$(3) \qquad (F, H(1 \times p_{\lambda'})) \leq \mathcal{U}.$$

The proof of this key lemma is deferred to Section 5.

LEMMA 2. *Let P be an ANR and let $\mathcal{U}$ be an open covering of P. Then there exists an open covering $\mathcal{V}$ of P such that, whenever for a space Z two mappings $h_0, h_1 : Z \to P$ are $\mathcal{V}$-near, then there exists a $\mathcal{U}$-homotopy $H : Z \times I \to P$ which connects h_0 with h_1. Moreover, if for a subset $A \subseteq Z$, $h_0|A = h_1|A$, then one can achieve that H is a homotopy* rel A.

This lemma is proved in [6, I, 3.2, Theorem 6].

Proof of Theorem 1. We will now derive Theorem 1 from Lemmas 1 and 2.

Proof of (M1). Take for $\mathcal{U}$ any open covering of P and choose $\mathcal{V}$ as in Lemma 2. By (R1), applied to $\mathcal{V}$, there is a $\lambda \in \Lambda$ and a mapping $h : X_\lambda \to P$ such that $(hp_\lambda, f) \leq \mathcal{V}$. Then, by the choice of $\mathcal{V}$, $hp_\lambda \simeq f$.

Proof of (SM2). Let $\lambda, P, f_0, f_1 : X_\lambda \to P$ and $F : X \times I \to P$ be as in (SM2). Take any open covering $\mathcal{U}$ of P and choose $\mathcal{V}$ according to Lemma 2. Now apply Lemma 1 to the covering $\mathcal{V}$. One obtains a $\lambda' \geq \lambda$ and a homotopy $H : X_{\lambda'} \times I \to P$, which satisfies 1.(6), 1.(7) and

$$(4) \qquad (F, H(1 \times p_{\lambda'})) \leq \mathcal{V}.$$

Put $Z = X \times I$, $A = X \times \partial I$, $h_0 = F$, $h_1 = H(p_{\lambda'} \times 1)$ and note that

$$(5) \qquad h_0|A = h_1|A.$$

This is so because, by (1.4) and (1.6),

$$(6) \qquad \begin{aligned} h_0(x,0) &= F(x,0) = f_0 p_\lambda(x) = f_0 p_{\lambda\lambda'} p_{\lambda'}(x) \\ &= H(p_{\lambda'}(x), 0) = h_1(x,0), \qquad x \in X. \end{aligned}$$

Similarly, by (1.5) and (1.7),

$$(7) \qquad \begin{aligned} h_0(x,1) &= F(x,1) = f_1 p_\lambda(x) = f_1 p_{\lambda\lambda'} p_{\lambda'}(x) \\ &= H(p_{\lambda'}(x), 1) = h_1(x,0), \qquad x \in X. \end{aligned}$$

Also note that $(h_0, h_1) \leq \mathcal{V}$, because of (4). Consequently, by the choice of $\mathcal{V}$ (Lemma 2), there exists a homotopy rel$(X \times \partial I)$, which connects F and $H(p_{\lambda'} \times I)$. This establishes (1.8).

3. A lemma on homotopy equalizers

The following lemma is needed in Section 5 in the proof of Lemma 1.

LEMMA 3. *Let X be a space, let P, P' be ANR's, let $f : X \to P'$, $g_0, g_1 : P' \to P$ be maps and let $F : X \times I \to P$ be a homotopy such that*

$$(1) \qquad F(x,0) = g_0 f(x), \qquad x \in X,$$

$$F(x,1) = g_1 f(x), \qquad x \in X. \tag{2}$$

Then there exist an ANR P'', maps $f' : X \to P''$, $g : P'' \to P'$ and a homotopy $G : P'' \times I \to P$ such that

$$gf' = f, \tag{3}$$
$$G(z,0) = g_0 g(z), \qquad z \in P'', \tag{4}$$
$$G(z,1) = g_1 g(z), \qquad z \in P'', \tag{5}$$
$$G(f' \times 1) = F. \tag{6}$$

This lemma strengthens [6, I, 4.1, Lemma 1] by the additional requirement (6). In the proof we use ideas from the proof given in [6] together with some improvements taken from the proof of an analogous result in [1, Lemma 5].

Proof of Lemma 3. Let P^I be the space of paths in P (compact-open topology). Define mappings $h : X \to P^I$ and $f' : X \to P' \times P^I$ by

$$(h(x))(t) = F(x,t), \qquad x \in X, \ t \in I, \tag{7}$$
$$f'(x) = (f(x), h(x)), \qquad x \in X. \tag{8}$$

Let $g : P' \times P^I \to P'$ denote the first projection, i.e.

$$g(y,\omega) = y, \qquad y \in P', \ \omega \in P^I. \tag{9}$$

Then f, f' and g satisfy (3).

Now define $P'' \subseteq P' \times P^I$ by

$$P'' = \{(y,\omega) \in P' \times P^I : \omega(0) = g_0(y), \omega(1) = g_1(y)\}. \tag{10}$$

Furthermore, let $G : P'' \times I \to P$ be given by

$$G((y,\omega),t) = \omega(t), \qquad (y,\omega) \in P'', \ t \in I. \tag{11}$$

Note that G satisfies (4) and (5). Moreover, by (8), (11) and (7),

$$\begin{aligned} G(f' \times 1)(x,t) &= G((f(x), h(x)), t) \\ &= (h(x))(t) = F(x,t), \qquad x \in X, \ t \in I, \end{aligned} \tag{12}$$

which shows that (6) is also satisfied.

In order to complete the proof we will show that P'' is an ANR.

Let Z be a metric space, $A \subseteq Z$ a closed subset and $\varphi : A \to P'' \subseteq P' \times P^I$ a mapping. We must find a neighborhood V of A in Z and an extension $\tilde{\varphi} : V \to P''$ of φ.

Consider $q\varphi : A \to P^I$, where $q : P' \times P^I \to P^I$ denotes the second projection.

Let $\Phi : A \times I \to P$ be given by

$$\Phi(a,t) = (q\varphi(a))(t), \qquad a \in A, \ t \in I. \tag{13}$$

Since $\varphi(a) \in P''$ and $g : P' \times P^I \to P'$ is the first projection, (10) implies

$$(q\varphi(a))(0) = g_0 g\varphi(a), \qquad a \in A, \tag{14}$$

$$(q\varphi(a))(1) = g_1 g\varphi(a), \qquad a \in A. \tag{15}$$

Formulae (13)–(15) show that

$$\Phi(a,0) = g_0 g\varphi(a), \qquad a \in A, \tag{16}$$

$$\Phi(a,1) = g_0 g\varphi(a), \qquad a \in A. \tag{17}$$

Since P' is an ANR, $g\varphi : A \to P'$ admits an extension $\psi : U \to P'$ to a neighborhood U of A in Z

$$\psi|A = g\varphi. \tag{18}$$

Clearly $g_0\psi, g_1\psi : U \to P$ are extensions of $g_0 g\varphi$ and $g_1 g\varphi$ respectively. Since also P is an ANR, there exist a neigborhood V of A in Z, $V \subseteq U$, and an extension $\tilde{\Phi} : V \times I \to P$ of Φ, which connects $g_0\psi|V$ with $g_1\psi|V$ (see [6, I, 3.2, Theorem 8]), i.e.,

$$\tilde{\Phi}(z,0) = g_0\psi(z), \qquad z \in V, \tag{19}$$

$$\tilde{\Phi}(z,1) = g_1\psi(z), \qquad z \in V, \tag{20}$$

$$\tilde{\Phi}(a,t) = \Phi(a,t), \qquad a \in A,\ t \in I. \tag{21}$$

We now define mapings $\chi : V \to P^I$ and $\tilde{\varphi} : V \to P' \times P^I$ by

$$(\chi(z))(t) = \tilde{\Phi}(z,t), \qquad z \in V,\ t \in I, \tag{22}$$

$$\tilde{\varphi}(z) = (\psi(z), \chi(z)), \qquad z \in V. \tag{23}$$

Note that (21) and (13) imply

$$\chi|A = q\varphi. \tag{24}$$

Since g and q are the two projections of $P' \times P^I$, it follows from (18) and (23) that $\tilde{\varphi}$ extends φ.

To complete the proof it suffices to note that $\tilde{\varphi}(z) \in P''$, for $z \in V$. Indeed, (22), (19) and (20) yield

$$(\chi(z))(0) = g_0\psi(z), \tag{25}$$

$$(\chi(z))(1) = g_1\psi(z). \tag{26}$$

4. A lemma on stacked coverings

Let $\mathcal{W}$ be an open covering of a space Z and, for each $W \in \mathcal{W}$, let $\mathcal{J}_W$ be an open covering of the unit interval I. Then the sets $W \times J$, where $W \in \mathcal{W}$ and

$J \in \mathcal{J}_W$, form an open covering $\mathcal{S}$ of $Z \times I$. We call such a covering $\mathcal{S}$ a *stacked* covering.

LEMMA 4. *Let Z be a normal space and let $\mathcal{S}$ be a stacked covering of $Z \times I$, where $\mathcal{W}$ is locally finite and each $\mathcal{J}_W$, $W \in \mathcal{W}$, is finite. Moreover, for each $W \in \mathcal{W}$, let α_W be a real number, $\alpha_W > 0$. Then there exists a continuous function $\varphi : Z \to I$ with the property that every $z \in Z$ admits a $W \in \mathcal{W}$ such that*

$$z \in W, \tag{1}$$

$$0 < \varphi(z) \leq \alpha_W. \tag{2}$$

Proof. Let $(\varphi_W, W \in \mathcal{W})$ be a partition of unity on Z subordinated to $\mathcal{W}$, i.e., $\varphi_W : Z \to I$, $W \in \mathcal{W}$, are continuous functions such that

$$\varphi_W(z) \neq 0 \implies z \in W, \tag{3}$$

$$\sum_W \varphi_W(z) = 1. \tag{4}$$

Put

$$\varphi = \sup\{\alpha_W \varphi_W : W \in \mathcal{W}\}. \tag{5}$$

By local finiteness of $\mathcal{W}$, every point $z_0 \in P$ admits a neighborhood U such that $\varphi_W|U = 0$, except for a finite collection of indexes $\{W_1, \dots, W_n\}$. Then

$$\varphi|U = \{\alpha_{W_1}\varphi_{W_1}|U, \dots, \alpha_{W_n}\varphi_{W_n}|U\}, \tag{6}$$

which implies continuity of φ at z_0.

For a given $z \in Z$ one cannot have $\varphi_W(z) = 0$, for all $W \in \mathcal{W}$, because of (4). Therefore, $\varphi(z) > 0$, for every $z \in Z$. By (5), every $z \in Z$ admits a $W \in \mathcal{W}$ such that

$$\varphi(z) = \alpha_W \varphi_W(z) \leq \alpha_W. \tag{7}$$

Since $\varphi(z) > 0$, also $\varphi_W(z) > 0$, and therefore, (3) implies that $z \in W$.

5. Proof of Lemma 1

Let $\boldsymbol{p} : X \to \boldsymbol{X}$ be a resolution and let λ, P, f_0, f_1 and F be as in (SM2), i.e., satisfy (1.4) and (1.5). Moreoover, let $\mathcal{U}$ be an open covering of P.

We first choose an open star-refinement $\mathcal{U}'$ of $\mathcal{U}$. Then we choose an open covering $\mathcal{V}$ of P such that the assertions of Lemma 2 hold for $\mathcal{U}'$ and $\mathcal{V}$. We also assume that $\mathcal{V}$ is a star-refinement of $\mathcal{U}'$. Next, we choose $\mathcal{V}'$ so that $\mathcal{V}'$ is a star-refinement of $\mathcal{V}$ and (R2) holds for P, $\mathcal{V}$ and $\mathcal{V}'$.

Put $P' = P \times P$ and denote by $g_0, g_1 : P' \to P$ the two projections. Then define $f : X \to P'$ as the only map for which

$$g_0 f = f_0 p_\lambda, \qquad g_1 f = f_1 p_\lambda. \tag{1}$$

By (1.4) and (1.5), we have

$$F_0 = g_0 f, \qquad F_1 = g_1 f. \tag{2}$$

We now apply Lemma 3 to X, P, P', f, g_0, g_1 and F and conclude that there exist an ANR P'', maps $f' : X \to P''$, $g : P'' \to P'$ and a homotopy $G : P'' \times I \to P$ such that

$$gf' = f, \tag{3}$$

$$G_0 = g_0 g, \qquad G_1 = g_1 g, \tag{4}$$

$$G(f' \times 1) = F. \tag{5}$$

We now consider the open covering $G^{-1}(\mathcal{V}')$ of $P'' \times I$ and choose a refinement, which is a stacked covering $\mathcal{S}$ of $P'' \times I$, given by a locally finite open covering $\mathcal{W}$ of P'' and by finite open coverings $\mathcal{J}_W$, $W \in \mathcal{W}$, of I.

Applying (R1) to $f' : X \to P''$ and $\mathcal{W}$, choose a $\lambda'' \geq \lambda$ and a mapping $h : X_{\lambda''} \to P''$ such that

$$(f', hp_{\lambda''}) \leq \mathcal{W}. \tag{6}$$

Note that for any $W \in \mathcal{W}$, $W \times 0 \subseteq W \times J$, for some $J \in \mathcal{J}_W$ and $W \times J$ is contained in $G^{-1}(V')$ for some $V' \in \mathcal{V}'$. Hence, by (4),

$$g_0 g(W) = G_0(W) = G(W \times 0) \subseteq G(W \times J) \subseteq V',$$

i.e. $g_0 g(\mathcal{W})$ refines $\mathcal{V}'$. Consequently, (6) implies

$$(g_0 g f', g_0 g h p_{\lambda''}) \leq \mathcal{V}'. \tag{7}$$

Now note that (3) and (1) yield

$$g_0 g f' = g_0 f = f_0 p_\lambda = f_0 p_{\lambda\lambda''} p_{\lambda''}, \tag{8}$$

so that (7) becomes

$$(g_0 g h p_{\lambda''}, f_0 p_{\lambda\lambda''} p_{\lambda''}) \leq \mathcal{V}'. \tag{9}$$

Analogously we obtain

$$(g_1 g h p_{\lambda''}, f_1 p_{\lambda\lambda''} p_{\lambda''}) \leq \mathcal{V}'. \tag{10}$$

By the choice of $\mathcal{V}'$ (see (R2)), there is a $\lambda' \geq \lambda''$ such that

$$(g_0 g h p_{\lambda''\lambda'}, f_0 p_{\lambda\lambda'}) \leq \mathcal{V}, \tag{11}$$

$$(g_1 g h p_{\lambda''\lambda'}, f_1 p_{\lambda\lambda'}) \leq \mathcal{V}. \tag{12}$$

By the choice of $\mathcal{V}$, we conclude that there exist $\mathcal{U}'$-homotopies $K, L : X_{\lambda'} \times I \to P$ such that

$$K_0 = f_0 p_{\lambda\lambda'}, \qquad K_1 = g_0 g h p_{\lambda''\lambda'}, \tag{13}$$

$$L_0 = f_1 p_{\lambda\lambda'}, \qquad L_1 = g_1 g h p_{\lambda''\lambda'}. \tag{14}$$

Furthermore, by (6), for any $t \in I$, both points $(f'(x), t)$ and $(hp_{\lambda''}(x), t)$ belong to some member of $\mathcal{S}$ and therefore also to $G^{-1}(V')$ for some $V' \in \mathcal{V}'$. Consequently, $G(f' \times 1)$ and $G(hp_{\lambda''} \times 1)$ are $\mathcal{V}'$-near and a fortiori $\mathcal{V}$-near.

$$\big(G(f' \times 1), G(hp_{\lambda''} \times 1)\big) \leq \mathcal{V}. \tag{15}$$

We will now use the homotopies K, L and $G(hp_{\lambda''\lambda'} \times 1)$ to construct the desired homotopy $H : X_{\lambda'} \times I \to P$.

For every $W \in \mathcal{W}$ choose a number α_W, $0 < \alpha_W \leq 1/3$, which is smaller than the Lebesgue number of the covering $\mathcal{J}_W$. Note that if $t, t' \in I$, $|t - t'| \leq \alpha_W$, then there is a $J \in \mathcal{J}_W$, such that $t, t' \in J$. Consequently, if $z \in W$ and $|t - t'| \leq \alpha_W$, then $(z,t), (z,t') \in W \times J \in \mathcal{S}$, for some $J \in \mathcal{J}_W$, and therefore, both $G(z,t)$ and $G(z,t')$ belong to some $V' \in \mathcal{V}'$.

We now apply Lemma 4 and obtain a continuous function $\varphi : P'' \to I$ such that, for every $z \in P''$, there is a $W \in \mathcal{W}$ with

$$z \in W, \tag{16}$$

$$0 < \varphi(z) \leq \alpha_W \leq 1/3. \tag{17}$$

Note that $0 < \varphi(z) < 1 - \varphi(z) < 1$, for every $z \in P''$.

We define $H(y,t)$, $y \in X_{\lambda'}$, by

$$H(y,t) = \begin{cases} K\left(y, \dfrac{t}{\varphi(z)}\right), & 0 \leq t \leq \varphi(z), \\ G\left(z, \dfrac{t - \varphi(z)}{1 - 2\varphi(z)}\right), & \varphi(z) \leq t \leq 1 - \varphi(z), \\ L\left(y, \dfrac{1-t}{\varphi(z)}\right), & 1 - \varphi(z) \leq t \leq 1, \end{cases} \tag{18}$$

where

$$z = hp_{\lambda''\lambda'}(y). \tag{19}$$

$H : X_{\lambda'} \times I \to P$ is well-defined, because of (4), (13) and (14). Moreover, (13) and (14) show that H satisfies (2.1) and (2.2). It remains to prove (2.3), i.e., to show that, for every $(x,t) \in X \times I$, there exists a $U \in \mathcal{U}$ such that

$$F(x,t), H(p_{\lambda'}(x), t) \in U. \tag{20}$$

Put $y = p_{\lambda'}(x)$ and $z = hp_{\lambda''\lambda'}(y) = hp_{\lambda''}(x)$. We will first consider the case when $\varphi(z) \leq t \leq 1 - \varphi(z)$. Then, by (18) and (5),

$$H(p_{\lambda'}(x), t) = H(y,t) = G\left(z, \frac{t - \varphi(z)}{1 - 2\varphi(z)}\right), \tag{21}$$

$$F(x,t) = G(f'(x), t). \tag{22}$$

By (6), there is a $W \in \mathcal{W}$ such that

$$f'(x), z \in W. \tag{23}$$

Moreover, by (16) and (17), there is a $W' \in \mathcal{W}$ such that $z \in W'$ and $0 < \varphi(z) \leq \leq \alpha_{W'}$. Also note that $\varphi(z) \leq t \leq 1 - \varphi(z)$ implies $|1 - 2t| \leq 1 - 2\varphi(z)$, so that

$$\left|\frac{1-2t}{1-2\varphi(z)}\right| \leq 1, \tag{24}$$

and therefore

$$\left|t - \frac{t-\varphi(z)}{1-2\varphi(z)}\right| \leq \varphi(z) \leq \alpha_{W'} . \tag{25}$$

By the choice of the numbers α_W, this and $z \in W'$ imply the existence of some $V_1' \in \mathcal{V}'$ for which

$$G(z,t), G\left(z, \frac{t-\varphi(z)}{1-2\varphi(z)}\right) \in V_1'. \tag{26}$$

On the other hand, by (23), $(f'(x), t)$ and (z, t) both belong to some member of $\mathcal{S}$ and, therefore, there is a $V_2' \in \mathcal{V}'$ such that

$$G(f'(x), t), G(z,t) \in V_2'. \tag{27}$$

Since $\mathcal{V}'$ is a star-refinement of $\mathcal{V}$, (22), (21), (26) and (27) show that there is a $V \in \mathcal{V}$, which contains the points $F(x,t)$, $H(p_{\lambda'}(x), t)$. Since $\mathcal{V}$ refines $\mathcal{U}$, this establishes (20) in the case $\varphi(z) \leq t \leq 1 - \varphi(z)$.

We now consider the case when $0 \leq t \leq \varphi(z)$. Choose $W \in \mathcal{W}$ so that (16) and (17) hold. Then there is a $J \in \mathcal{J}_W$ such that $0, t \in J$, because $|t - 0| \leq \varphi(z) \leq \alpha_W$. Consequently, there is a $V' \in \mathcal{V}'$ such that

$$G(z,t), G(z,0) \in V'. \tag{28}$$

On the other hand, by (6), there is a $W' \in \mathcal{W}$ which contains both points $f'(x)$ and $z = hp_{\lambda''}(x)$. Therefore, for any $t \in I$, $(f'(x), t)$ and (z, t) belong to $W' \times J_t$, for some $J_t \in \mathcal{J}_{W'}$. Consequently, there is a $V_t' \in \mathcal{V}'$ such that

$$G(f'(x), t), G(z,t) \in V_t'. \tag{29}$$

In particular, for $t = 0$ we have $V_0' \in \mathcal{V}'$ and

$$G(f'(x), 0), G(z,0) \in V_0'. \tag{30}$$

Since $\mathcal{V}'$ is a star-refinement of $\mathcal{V}$, we conclude, by (22), (29), (30) and (28), that there is a $V \in \mathcal{V}$ such that

$$F(x,t), F(x,0) \in V. \tag{31}$$

Since K is a $\mathcal{U}'$-homotopy, there is a $U' \in \mathcal{U}'$, which contains both points $K(y,0)$ and $K(y,t/\varphi(z))$. However, by (13), (1), (3), (4) and (5),

$$K(y,0) = f_0 p_{\lambda\lambda'}(y) = f_0 p_\lambda(x) = g_0 f(x) \\ = g_0 g f'(x) = G(f'(x),0) = F(x,0). \tag{32}$$

Furthermore, by (18),

$$K(y,t/\varphi(z)) = H(y,t) = H(p_{\lambda'}(x),t). \tag{33}$$

We thus have

$$H(p_{\lambda'}(x),t), F(x,0) \in U'. \tag{34}$$

Since $\mathcal{V}$ refines $\mathcal{U}'$ and $\mathcal{U}'$ is a star-refinement of $\mathcal{U}$, (31) and (34) show that there is a $U \in \mathcal{U}$ such that (20) holds.

The last case, when $1-\varphi(z) \le t \le 1$, is symmetric to the case $0 \le t \le \varphi(z)$ (use L instead of K). This completes the proof of Lemma 1 and Theorem 1.

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Matematički odjel
Sveučilište u Zagrebu
41001 Zagreb, p.p. 187
Jugoslavija

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