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ON A COMMON FIXED POINT THEOREM OF A GREGUŠ TYPE

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Abstract. It is proved that if T and E (E continuous) are two compatible self mappings of a closed subset K of a complete convex metric space X such that the condition:

$$d(Tx,Ty) \leq ad(Ex,Ey) + (1-a)\max\{d(Ex,Tx),d(Ey,Ty)\}$$

holds for all x,y in K, where $0 < a < 1$, and $\mathrm{Co}[T(K)] \subseteq E(K)$, then T and E have a unique common fixed point. This result generalizes a theorem of Fisher and Sessa [2] and a theorem of Mukherjee and Verma [6] and shows that these theorems remain true when the hypotheses of linearity and non-expansivity of E are reduced to the continuity of E.

Let X be a Banach space and C a closed convex subset of X. Greguš [3] proved the following theorem:

THEOREM 1. *Let $T : C \to C$ be a mapping satisfying the inequality*

$$\|Tx - Ty\| \leq a\|x - y\| + b\|Tx - x\| + c\|Ty - y\| \tag{A}$$

for all $x, y \in C$, where $0 < a < 1$, $b \geq 0$, $c \geq 0$ and $a + b + c = 1$. Then T has a unique fixed point.

Fisher and Sessa [2] extended Theorem 1 to a common fixed point theorem of two weakly commuting mappings T and I (Sessa [7]: T and I are weakly commuting iff $\|TIx - ITx\| \leq \|Ix - Tx\|$). They proved the following theorem:

THEOREM 2. *Let T and I be two weakly commuting mappings of C into itself satisfying the inequality*

$$\|Tx - Ty\| \leq a\|Ix - Iy\| + (1-a)\max\{\|Tx - Ix\|, \|Ty - Iy\|\} \tag{B}$$

for all $x, y \in C$, where $0 < a < 1$. If I is linear, non-expansive in C and such that $I(C)$ contains $T(C)$, then T and I have a unique common fixed point in C.

Mukherjee and Verma in [6] gave an improvement of Th. 2, where C, T and I are the same as in Th. 2, except that now I is affine instead of linear ($I : C \to C$ is affine if $I(cx + (1-c)y) = cIx + (1-c)Iy$; $0 \leq c \leq 1$, [6]).

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In this note we will use a new method and show that in the above theorems a map I need not be linear (affine) nor non-expansive. It is enough that I be continuous and $W(Tx, Ty, 1/2) \in I(C)$ (see Definition 2 below). Also, T and I need not be weakly commutative — it is sufficent that they be compatible. We recall the following definitions:

Definition 1. (G. Jungck [4]). Self-maps T and E of a metric space (X, d) are compatible iff $\lim_n (TEx_n, ETx_n) = 0$ when $\{x_n\}$ is a sequence in X such that $\lim_n Tx_n = \lim_n Ex_n = t$ for some t in X.

Clearly, commuting maps are weakly commuting and weakly commuting maps are comaptible, but neither implication is reversible, as examples in [5] and [7] show.

Definition 2. (Takahashi [8]). Let X be a metric space and $I = [0,1]$ be the closed unit interval. A continuous mapping $W : X \times X \times I \to X$ is said to be a convex structure on X if for all x, y in X, λ in I, $d(u, W(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y)$ for all u in X. X together with a convex structure is called a convex metric space. A subset $K \subseteq X$ is convex, if $W(x,y,\lambda) \in K$ wherever x, y in K and λ in I.

Clearly a Banach space, or any convex subset of it, is a convex metric space with $W(x,y,\lambda) = \lambda x + (1-\lambda)y$. More generally, if X is a linear space with a translation invariant metric satisfying $d(\lambda x + (1-\lambda)y, 0) \leq \lambda d(x,0) + (1-\lambda)d(y,0)$, then X is a convex metric space. There are many other examples but we consider these as paradigmatic.

THEOREM 3. *Let K be a closed subset of a complete convex metric space X and $T, E : K \to K$ two compatible mappings satisfying the following condition:*

$$\text{(C)} \qquad d(Tx, Ty) \leq a d(Ex, Ey) + (1-a)\max\{d(Ex, Tx), d(Ey, Ty)\}$$

for all x, y in K, where $0 < a < 1$. If Co$[T(K)] \subseteq E(K)$ *and E (or T) is continuous in K, then T and E have a unique common fixed point in K.*

Proof. Let $x \in K$ be an arbitrary point and let $y_0 = Ex$ and $y_1 = Tx$. Choose points x_1, x_2, x_3 in K such that $Ex_1 = Tx$, $Ex_2 = Tx_1$, $Ex_3 = Tx_2$. This choice can be done since $T(K)$ is contained in $E(K)$. Put $y_2 = Ex_2 = Tx_1$, $y_3 = Ex_3 = Tx_2$. Then by (C)

$$\begin{aligned} d(y_1, y_2) = d(Tx, Tx_1) &\leq a d(Ex, Ex_1) + (1-a)\max\{d(Ex, Tx), d(Ex_1, Tx_1)\} \\ &= a d(y_0, y_1) + (1-a)\max\{d(y_0, y_1), d(y_1, y_2)\}. \end{aligned}$$

Since $0 < a < 1$ we obtain $d(y_1, y_2) \leq d(y_0, y_1)$. Analogously, we can get

$$\text{(1)} \qquad d(y_2, y_3) \leq d(y_1, y_2) \leq d(y_0, y_1).$$

Similarly, by simple calculations and by using (C) and (1) one can show that the following inequality is true:

$$\text{(2)} \qquad d(y_1, y_3) \leq (1+a) d(y_0, y_1).$$

Let $z = W(y_2, y_3, 1/2)$ and choose $u \in K$ such that $z = Eu$. This choice can be done since $\mathrm{Co}[T(K)] \subseteq E(K)$. Since

$$d(y_1, z) = d(y_1, W(y_2, y_3, 1/2)) \leq (1/2)[d(y_1, y_2) + d(y_1, y_3)],$$

using (1) and (2) we obtain

$$d(y_1, z) \leq (1 + a/2)d(y_0, y_1). \tag{3}$$

Similarly we get

$$d(y_2, z) \leq (1/2)d(y_2, y_3) \leq (1/2)d(y_0, y_1). \tag{4}$$

Put $Tu = v$. Then

$$d(v, z) = d(v, W(y_2, y_3, 1/2)) \leq (1/2)[d(y_2, v) + d(y_3, v)]. \tag{5}$$

By (C) we have

$$\begin{aligned} d(y_2, v) = d(Tx_1, Tu) &\leq ad(Ex_1, Eu) + (1-a)\max\{d(Ex_1, Tx_1), d(Eu, Tu)\} \\ &\leq ad(y_1, z) + (1-a)\max\{d(y_1, y_2), d(z, v)\}. \end{aligned}$$

On using (1) and (3) we get

$$d(y_2, v) \leq a(1 + a/2)d(y_0, y_1) + (1-a)\max\{d(y_0, y_1), d(v, z)\}.$$

Similarly, by (C), (1) and (4) we have

$$d(y_3, v) \leq (a/2)d(y_0, y_1) + (1-a)\max\{d(y_0, y_1), d(v, z)\}.$$

Then by (5) we get

$$d(v, z) \leq (1/4)a(3 + a)d(y_0, y_1) + (1-a)\max\{d(y_0, y_1), d(v, z)\}$$

and hence

$$d(z, v) \leq \max\{(1/4)(4 - a + a^2), (1/4)(3 + a)\} \cdot d(y_0, y_1).$$

As $z = Eu$, $v = Tu$, $y_0 = Ex$, $y_1 = Tx$ we have $d(Eu, Tu) \leq \lambda d(Ex, Tx)$, where $0 < \lambda = (1/4)(4 - a + a^2) < 1$. Now by simple considerations we conclude that

$$\inf\{d(Ex, Tx) : x \in K\} = 0. \tag{6}$$

Now we will prove that the infimum is attained. Put

$$A_n = \{x \in K : d(Ex, Tx) \leq 1/n\} \qquad (n = 1, 2, 3, \ldots).$$

From (6) it follows that A_n is non-empty for every $n = 1, 2, 3, \ldots$. Therefore $\overline{TA_n} \neq \emptyset$ and $\overline{TA_1} \supseteq \overline{TA_2} \supseteq \ldots \supseteq \overline{TA_n} \supseteq \ldots$. Since X is complete it follows that $B = \bigcap_{n=1}^{\infty} \overline{TA_n}$ is non-empty. We will show that B is singleton.

Let $x', y' \in TA_n$. Then there exist $x, y \in A_n$ such that $x' = Tx$, $y' = Ty$. So we have

$$\begin{aligned} d(x', y') = d(Tx, Ty) &\le ad(Ex, Ey) + (1-a)\max\{d(Ex, Tx), d(Ey, Ty)\} \\ &\le a[d(Ex, Tx) + d(Tx, Ty) + d(Ty, Ey)] + (1-a)(1/n) \\ &\le ad(x', y') + 2a(1/n) + (1-a)(1/n). \end{aligned}$$

Hence $d(x', y') \le (1+a)/n(1-a)$. Therefore, $\operatorname{diam}(\overline{TA_n}) = \operatorname{diam}(TA_n) \to 0$, as $n \to \infty$. This implies $B = \{u\}$ for some $u \in K$.

As $u \in \overline{TA_n}$ for every $n = 1, 2, 3, \ldots$, it follows that for each n there is $x'_n \in TA_n$ with $d(u, x'_n) < 1/n$. Let $x_n \in A_n$ be such that $x'_n = Tx_n$. Then $d(u, Tx_n) < 1/n$ and we have $d(u, Ex_n) \le d(u, Tx_n) + d(Ex_n, Tx_n) < 2/n$. Hence

$$\lim_n Ex_n = \lim_n Tx_n = u. \tag{7}$$

Then by the continuity of E

$$\lim_n E(Tx_n) = \lim_n E(Ex_n) = Eu. \tag{8}$$

Since T and E are compatible, (7) implies $\lim_n d(E(Tx_n), T(Ex_n)) = 0$. Then by the triangle inequality and (8) we get

$$\lim_n d(Eu, T(Ex_n)) \le \lim_n d(Eu, E(Tx_n)) + \lim_n d(E(Tx_n), T(Ex_n)) = 0. \tag{9}$$

Now by (C)

$$d(T(Ex_n), Tu) \le ad(E(Ex_n), Eu) + (1-a)\max\{d(E(Ex_n), T(Ex_n)), d(Eu, Tu)\}.$$

Letting n tend to infinity we obtain $d(Eu, Tu) \le (1-a)d(Eu, Tu)$. Since $a > 0$ we conclude that $d(Eu, Tu) = 0$, i.e. $Eu = Tu$. Then by (C) we have

$$d(Tx_n, Tu) \le ad(Ex_n, Eu) + (1-a)\max\{d(Ex_n, Tx_n), d(Eu, Tu)\}.$$

Using (7) and letting n tend to infinity we get $d(u, Tu) \le ad(u, Eu) = ad(u, Tu)$. This (and $a < 1$) implies that $d(u, Tu) = 0$. Therefore we have $Tu = Eu = u$, i.e. u is a common fixed point of T and E. The uniqueness of u is a consequence of the condition (C). The proof is complete.

COROLLARY 1. *Let K be as in Theorem 3 and $T : K \to K$ a mapping satisfying*

$$d(Tx, Ty) \le ad(x, y) + (1-a)\max\{d(x, Tx), d(y, Ty)\} \tag{A'}$$

for all $x, y \in K$, where $0 < a < 1$. Then T has a unique fixed point.

Since a Banach space is a convex metric space and (A) implies (A′), Corollary 1 is a generalization of Greguš's Theorem 1.

COROLLARY 2. *Let K be as in Theorem 3 and E a continuous mapping of K onto K which satisfies the following inequality:*

$$d(x, y) \le ad(Ex, Ey) + (1-a)\max\{d(Ex, x), d(Ey, y)\}$$

with $0 < a < 1$. *Then* E *has a unique fixed point.*

COROLLARY 3. *Let* K *be a closed convex subset of a Banach space and* $T, E : K \to K$ *as in Theorem 3. Then* T *and* E *have a unique common fixed point.*

Clearly, Corollary 3 is an extension of Theorem 2 of Fisher and Sessa and the Mukherjee and Verma's theorem [6] and a theorem of Diviccaro, Fisher and Sessa for the case $p = 1$. The following example shows it.

Example 1. Let $K = [0,1]$ be the closed unit interval and $T, E : K \to K$ be defined by $Tx = x/4$ and $Ex = (x)^{1/2}$. Clearly $T(K) \subseteq E(K)$, E is continuous and T and E weakly commute. As

$$d(Tx,Ty) = \frac{|x-y|}{4} \leq \frac{|x-y|}{4}\,\frac{2}{x^{1/2}+y^{1/2}} = \frac{d(Ex,Ey)}{2}$$

for all $x, y \in K$, we conclude that all the hypotheses of Corollary 3 are satisfied and 0 is a unique common fixed point. But E is neither linear nor nonexpansive and so Theorem 2 of Fisher and Sessa is not applicable.

The following example shows that Corollary 1 is an extension of Greguš's theorem.

Example 2. Let $K = [-1,1]$, $Tx = 0$ for $-1 \leq x \leq 1/2$ and $Tx = -1$ for $1/2 < x \leq 1$. Then T satisfies (A′) with $a = 1/3$. But T does not satisfy (A) as, for example, for $x = 0$ and $y = 3/4$:

$$d(Tx,Ty) = 1 > \max\{d(x,y), [d(x,Tx) + d(y,Ty)]/2\} = \max\{3/4, 7/8\}.$$

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