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FRENET FORMULAE IN RECURRENT LAGRANGE SPACES WITH d CONNECTION

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Abstract. Recurrent Lagrange spaces with d connection are defined. Using ideas of Miron [3] and Moór [4] the connection coefficients and their laws of transformation are determined. The Frenet formulae for horizontal and vertical curves are obtained.

1. Recurrent Lagrange spaces with d connection. Let M be an n-dimensional and E a $2n$-dimensional differentiable manifold and let (E, π, M) be a vector bundle such that $\pi(E) = M$. If $u \in E$, then in some chart, u has coordinates (x^i, y^i), $i, j, k, \dots = 1, 2, \dots, n$. If $(x^{i'}, y^{i'})$ are the coordinates of the same point u in the new coordinate system

$$x^{i'} = x^{i'}(x^1, \dots, x^n), \quad y^{i'} = \frac{\partial x^{i'}}{\partial x^i} y^i, \quad \operatorname{rank}\left[\frac{\partial x^{i'}}{\partial x^i}\right] = n, \tag{1.1}$$

then

$$\text{(a) } \dot{\partial}_i = \frac{\partial}{\partial y^i} = \frac{\partial x^{i'}}{\partial x^i} \dot{\partial}_{i'}, \quad \text{(b) } \partial_i = \frac{\partial}{\partial x^i} = \frac{\partial x^{i'}}{\partial x^i} \partial_{i'} + \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j} y^j \dot{\partial}_{i'}. \tag{1.2}$$

Any vector field $X \in T(E)$ is given by

$$X = X^i \partial_i + \tilde{X}^i \dot{\partial}_i. \tag{1.3}$$

From (1.2b) we can see that ∂_i $(i = 1, 2, \dots, n)$ do not transform as vectors, so we introduce a family of functions $N^i_j(x, y)$ called the nonlinear connection which have the following law of transformation:

$$N^i_j(x, y) = N^{i'}_{j'}(x', y') \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^i}{\partial x^{i'}} = \frac{\partial^2 x^i}{\partial x^j \partial x^{k'}} \frac{\partial x^{j'}}{\partial x^j} y^{k'}. \tag{1.4}$$

We define

$$\delta_i = \partial_i - N^j_i \dot{\partial}_j, \tag{1.5}$$

AMS Subject Classification (1985): Primary 53 B 40 then (1.3) has the form

$$X = X^i \delta_i + \bar{X}^i \dot{\partial}_i, \tag{1.6}$$

where

$$\bar{X}^i = \tilde{X}^i + X^j N_j^i. \tag{1.7}$$

From (1.2)–(1.7) we obtain

$$X^{i'} = \frac{\partial x^{i'}}{\partial x^i} X^i, \quad \bar{X}^{i'} = \frac{\partial x^{i'}}{\partial x^i} \bar{X}^i, \quad \delta_{i'} = \frac{\partial x^i}{\partial x^{i'}} \delta_i, \quad \dot{\partial}_{i'} = \frac{\partial x^i}{\partial x^{i'}} \dot{\partial}_i. \tag{1.8}$$

Any 1-form $\omega \in T^*(E)$ may be written in the form

$$\omega = \tilde{\omega}_i \, dx^i + \bar{\omega}_i \, dy^i. \tag{1.9}$$

As dy^i do not transform as tensors we define

$$\delta y^i = dy^i + N_j^i(x,y) \, dx^j. \tag{1.10}$$

From (1.9) and (1.10) we get

$$\omega = \omega_i \, dx^i + \bar{\omega}_i \, \delta y^i, \tag{1.11}$$

where

$$\omega_i = \tilde{\omega}_i - \bar{\omega}_j N_i^j(x,y). \tag{1.12}$$

We have the following law of transformation:

$$\omega_{i'} = \omega_i \frac{\partial x^i}{\partial x^{i'}}, \quad \bar{\omega}_{i'} = \bar{\omega}_i \frac{\partial x^i}{\partial x^{i'}}, \quad dx^{i'} = dx^i \frac{\partial x^{i'}}{\partial x^i}, \quad \delta y^{i'} = \delta y^i \frac{\partial x^{i'}}{\partial x^i}. \tag{1.13}$$

The vectors δ_i $i = 1, 2, \dots, n$, span $T_H(E)$, the vectors $\dot{\partial}_i$, $i = 1, 2, \dots, n$, span $T_V(E)$ and $T(E) = T_H \oplus T_V$. The 1-forms dx^i, $i = 1, 2, \dots, n$, span $T_H^*(E)$, the 1-forms δy^i, $i = 1, 2, \dots, n$, span $T_V^*(E)$ and $T^*(E) = T_H^*(E) \oplus T_V^*(E)$.

We shall consider the covariant metric tensor $G \in T^*(E) \otimes T^*(E)$, where

$$G = g_{ij} \, dx^i \otimes dx^j + \bar{g}_{ij} \, \delta y^i \otimes \delta y^j, \tag{1.14}$$

$$g_{ij} = g_{ji}, \quad \bar{g}_{ij} = \bar{g}_{ji}, \quad \operatorname{rank}[g_{ij}] = n, \quad \operatorname{rank}[\bar{g}_{ij}] = n. \tag{1.15}$$

If $[g^{ij}]$, $[\bar{g}^{ij}]$ are the inverse matrices of $[g_{ij}]$ and $[\bar{g}_{ij}]$ respectively then the lowering and raising the indices is given by

$$X_i = g_{ij} X^j, \quad \bar{X}_i = \bar{g}_{ij} \bar{X}^j, \quad \omega^i = g^{ij} \omega_j, \quad \bar{\omega}^i = \bar{g}^{ij} \bar{\omega}_j. \tag{1.16}$$

g_{ij} and $\bar{g}_{ij}$ transform as tensors of type $(0,2)$, g^{ij} and $\bar{g}^{ij}$ as tensors of type $(2,0)$. X_i, $\bar{X}_i$, ω^i, $\bar{\omega}^i$ determined by (1.16) transform respectively as ω_i, $\bar{\omega}_i$, dx^i, δy^i (in 1.13). (See also 1.10–1.12). We introduce a linear connection ∇ in $T(E)$ by

$$\nabla_{\delta_i} \delta_j = F_{ji}^k \delta_k, \quad \nabla_{\delta_i} \dot{\partial}_j = \bar{F}_{ji}^k \dot{\partial}_k, \quad \nabla_{\dot{\partial}_i} \delta_j = C_{ji}^k \delta_k, \quad \nabla_{\dot{\partial}_i} \dot{\partial}_j = \bar{C}_{ji}^k \dot{\partial}_k. \tag{1.17}$$

If

(1.18) $$X = X^i\delta_i + \bar{X}^i\dot{\partial}_i, \qquad Y = Y^j\delta_j + \bar{Y}^j\dot{\partial}_j$$

are two vector fields in $T(E)$, then we have

(1.19) $$\nabla_X Y = (Y^j_{|i}X^i + Y^j|_i\bar{X}^i)\delta_j + (\bar{Y}^j_{|i}X^i + \bar{Y}^j|_i\bar{X}^i)\dot{\partial}_j,$$

where

(1.20) $$\begin{aligned} Y^j_{|i} &= \delta_i Y^j + F^j_{ki}Y^k, & \bar{Y}^j_{|i} &= \delta_i\bar{Y}^j + \bar{F}^j_{ki}\bar{Y}^k,\\ Y^j|_i &= \dot{\partial}_i Y^j + C^j_{ki}Y^k, & \bar{Y}^j|_i &= \dot{\partial}_i\bar{Y}^j + \bar{C}^j_{ki}\bar{Y}^k. \end{aligned}$$

If we want $Y^j_{|i}$, $Y^j|_i$, $\bar{Y}^j_{|i}$ and $\bar{Y}^j|_i$ given by (1.20) to transform as tensors then F^k_{ji} and C^k_{ji} must have the following law of transformation:

(1.21) $$F^k_{ji} = F^{k'}_{j'i'}\frac{\partial x^{j'}}{\partial x^i}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^k}{\partial x^{k'}} + \frac{\partial^2 x^{k'}}{\partial x^j\partial x^i}\frac{\partial x^k}{\partial x^{k'}}, \quad C^k_{ji} = C^{k'}_{j'i'}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial x^{i'}}{\partial x^i}.$$

The same formulae are valid if in (1.21) F is substituted by $\bar{F}$ and C by $\bar{C}$.

The linear connection ∇ defined on $T(E)$ by (1.17) introduces in the usual way a connection ∇^* (denoted also by ∇) in $T^*(E)$ where

(1.22) $$\begin{gathered} \nabla_{\delta_i} dx^j = -F^j_{ki}\,dx^k, \quad \nabla_{\delta_i}\delta y^j = -\bar{F}^j_{ki}\delta y^k, \quad \nabla_{\dot{\partial}_i} dx^j = -C^j_{ki}\,dx^k,\\ \nabla_{\dot{\partial}_i}\delta y^j = -\bar{C}^j_{ki}\,\delta y^k, \quad \delta y^i = dy^i + N^i_j\,dx^j. \end{gathered}$$

If X is given by (1.18) and G by (1.14) then we have

(1.23) $$\nabla_X G = (g_{ij|k}X^k + g_{ij}|_k\bar{X}^k)\,dx^i\otimes dx^j + (\bar{g}_{ij|k}X^k + \bar{g}_{ij}|_k\bar{X}^k)\,\delta y^i\otimes\delta y^j,$$

where

(1.24) $$\begin{aligned} &\text{(a)} \quad g_{ij|k} = \partial_k g_{ij} - F^h_{ik}g_{hj} - F^h_{jk}g_{ih},\\ &\text{(b)} \quad g_{ij}|_k = \dot{\partial}_k g_{ij} - C^h_{ik}g_{hj} - C^h_{jk}g_{ih}. \end{aligned}$$

If in (1.24) we substitute g by $\bar{g}$, F by $\bar{F}$, C by $\bar{C}$, then (1.24) is valid. If X is a vector between the points (x^i, y^i) and $(x^i + dx^i, y^i + dy^i)$ i.e.

(1.25) $$X = dx^i\partial_i + dy^i\dot{\partial}_i = dx^i\delta_i + \delta y^i\dot{\partial}_i$$

then (1.19) may be written in the form

(1.26) $$\nabla_X Y = DY = (DY^j)\delta_j + (\bar{D}\bar{Y}^j)\dot{\partial}_j,$$

where

(1.27) $$DY^j = Y^j_{|i}\,dx^i + Y^j|_i\,\delta y^i, \qquad \bar{D}\bar{Y}^j = \bar{Y}^j_{|i}\,dx^i + \bar{Y}^j|_i\,\delta y^i.$$

From (1.23), (1.24) and (1.25) we obtain

$$DG = (Dg_{ij})\, dx^i \otimes dx^j + (\bar{D}\bar{g}_{ij})\, \delta y^i \otimes \delta y^j, \tag{1.28}$$

where

$$Dg_{ij} = g_{ij|k}\, dx^k + g_{ij}|_k\, \delta y^k, \qquad \bar{D}\bar{g}_{ij} = \bar{g}_{ij|k}\, dx^k + \bar{g}_{ij}|_k\, \delta y^k. \tag{1.29}$$

We shall determine the connection coefficients F, $\bar{F}$, C, $\bar{C}$ under conditions

$$g_{ij|k} = \lambda_k g_{ij}, \quad g_{ij}|_k = \mu_k g_{ij}, \quad \bar{g}_{ij|k} = \bar{\lambda}_k \bar{g}_{ij}, \quad \bar{g}_{ij}|_k = \bar{\mu}_k \bar{g}_{ij}, \tag{1.30}$$

where $\lambda = \lambda_k\, dx^k + \bar{\lambda}_k\, \delta y^k$, $\mu = \mu_k\, dx^k + \bar{\mu}_k\, \delta y^k$ are covariant vector fields. From the conditions (1.29) and (1.30) we obtain

$$\begin{aligned} &\text{(a)}\ F_{ijk} = (\delta_i g_{jk} + \delta_k g_{ij} - \delta_j g_{ik}) - (\lambda_i g_{jk} + \lambda_k g_{ij} - \lambda_j g_{ik}) \\ &\qquad + (\tilde{F}_{ijk} - \tilde{F}_{jki} + \tilde{F}_{kij}), \\ &\text{(b)}\ C_{ijk} = (\partial_i g_{jk} + \partial_k g_{ij} - \partial_j g_{ik}) - (\mu_i g_{jk} + \mu_k g_{ij} - \mu_j g_{ik}) \\ &\qquad + (\tilde{C}_{ijk} - \tilde{C}_{jki} + \tilde{C}_{kij}), \end{aligned} \tag{1.31}$$

where

$$\begin{aligned} &\text{(a)}\ F_{ijk} = g_{jh} F^h_{ik}, \qquad C_{ijk} = g_{jh} C^h_{ik}, \\ &\text{(b)}\ \tilde{F}^j_{ik} = F^j_{ik} - F^j_{ki}. \qquad \tilde{C}^j_{ik} = C^j_{ik} - C^j_{ki}. \end{aligned} \tag{1.32}$$

The formulae (1.31) and (1.32) are valid if we write $\bar{F}$, $\bar{g}$, $\bar{\lambda}$, $\tilde{\bar{F}}$, $\bar{C}$, $\bar{\mu}$, $\tilde{\bar{C}}$, instead of F, g, λ, $\tilde{F}$, C, μ, $\tilde{C}$ respectively.

Definition 1.1. A $2n$-dimensional differentiable manifold supplied with the metric tensor G (1.14), (1.15) and the connection coefficients $F, C, \bar{F}, \bar{C}$ determined by (1.31) (1.32) we call recurrent Lagrange space with d-connection and denote by RLd.

If we take $\lambda_i = 0$, $\tilde{F}_{ijk} = 0$, $\mu_i = 0$, $\tilde{C}_{ijk} = 0$ in (1.31) and (1.32) we obtain Cartan connection coefficients in $T_H(E)$ and if we take $\bar{\lambda}_i = 0$, $\tilde{\bar{F}}_{ijk} = 0$, $\bar{\mu}_i = 0$, $\tilde{\bar{C}}_{ijk} = 0$ then we obtain Cartan connection coefficients in $T_V(E)$.

2. Frenet formulae in RLd for the horizontal curve. Let us consider the curve

$$x^i = x^i(t) \quad y^i = y^i(t) \qquad i = 1, 2, \ldots, n, \tag{2.1}$$

where t is a real parameter. The tangent vector to the curve (2.1) is the vector T, where

$$\text{(a)}\ T = \frac{dx^i}{dt}\partial_i + \frac{dy^i}{dt}\dot{\partial}_i = T^i \delta_i + \bar{T}^i \dot{\partial}_i, \quad \text{(b)}\ \bar{T}^i\, dt = \delta y^i = dy^i + N^i_j\, dx^j. \tag{2.2}$$

Now we shall consider only a horizontal curve i.e. a curve for which the tangent vector is in $T_H(E)$, i.e. (2.2a) reduces to (2.3), where

$$(2.3)\quad T(t) = T^i(t)\delta_i,\ \bar{T}^i(t) = 0 \quad \text{iff} \quad \frac{\delta y^i}{dt} = 0 \quad \text{iff} \quad dy^i + N^i_j(x(t), y(t))\, dx^j = 0.$$

Let us introduce the notation $'DQ = \nabla_X Q$ (Q is any tensor), where

$$(2.4)\qquad X = T^i(t)\delta_i = \frac{dx^i}{dt}\delta_i, \qquad \delta y^i = 0.$$

If we denote by T_0 the unit vector in the direction of T then the following relations are true:

$$(2.5)\qquad T^i(t) = k(t)T^i_0(t), \qquad g_{ij}(x(t), y(t))T^i(t)T^j(t) = k^2(t),$$

$$(2.6)\qquad g_{ij}(x(t), y(t))T^i_0(t)T^j_0(t) = 1.$$

In the following all the vectors and tensors will be considered on the curve (2.1), so we shall not denote that they are functions of t.

From (1.27) we have

$$(2.7)\qquad \frac{'DT_0}{dt} = \left(\frac{'DT^i_0}{dt}\right)\delta_i = T^i_{0|j}\frac{dx^j}{dt}\delta_i.$$

As the vector $'DT_0/dt$ is in $T_H(E)$, it can be decomposed in the direction of T_0 and T_1 where T_0 and T_1 are unit mutualy normal vectors in $T_H(E)$. We may write

$$(2.8)\qquad (a)\ g_{ij}T^i_0T^j_1 = 0, \qquad (b)\ g_{ij}T^i_1T^j_1 = 1,$$

$$(2.9)\qquad 'DT^i_0/dt = K^0_0T^i_0 + K^1_0T^i_1.$$

On the other hand, if we denote by t_0, ($t_0 = t^i_0\delta_i$) the unit vector in the direction of $'DT_0/dt$ determined by (2.7), we have

$$(2.10)\qquad (a)\ 'DT^i_0/dt = k_0t^i_0, \qquad (b)\ g_{ij}t^i_0t^j_0 = 1.$$

From $\delta y^k = 0$ and (1.29) we obtain $'Dg_{ij} = g_{ij|k}\,dx^k$ and from (1.30) we have

$$(2.11)\qquad 'Dg_{ij}/dt = \lambda_k g_{ij}\,dx^k/dt = 2Kg_{ij}, \qquad 2K = \lambda_k\,dx^k/dt.$$

From (2.4) and (2.5) we obtain

$$(2.12)\qquad \lambda_k\,dx^k/dt = \lambda_i T^i_0 k(t) = 2K.$$

From (2.6) we get

$$(2.13)\qquad \frac{'Dg_{ij}}{dt}T^i_0T^j_0 + 2g_{ij}\frac{'DT^i_0}{dt}T^j_0 = 0.$$

Substituting (2.11), (2.4) (2.12), (2.9) into (2.13) we obtain

$$(2.14)\qquad K^0_0 = -K.$$

From (2.10a) and (2.10b) we have

$$g_{ij}\frac{'DT_0^i}{dt}\frac{'DT_0^j}{dt}=k_0^2 g_{ij}t_0^i t_0^j=k_0^2. \tag{2.15}$$

On the other hand using (2.8) and (2.9) we have

$$g_{ij}\frac{'DT_0^i}{dt}\frac{'DT_0^j}{dt}=(K_0^0)^2+(K_0^1)^2. \tag{2.16}$$

From (2.15) and (2.16) we get

$$k_0^2=(K_0^0)^2+(K_0^1)^2. \tag{2.17}$$

Using (2.14), from (2.17) we have

$$(K_0^1)^2=k_0^2-K^2. \tag{2.18}$$

Substituting (2.14) and (2.18) into (2.9) we obtain

$$'DT_0^i/dt=-KT_0^i\pm\sqrt{k_0^2-K^2}T_1^i. \tag{2.19}$$

Let us denote by t_1 the unit vector in the direction of DT_1/dt, i.e.

$$\text{(a) } 'DT_1^i/dt=k_1t_1^i, \qquad \text{(b) } g_{ij}t_1^i t_1^j=1. \tag{2.20}$$

Let T_2 be a unit vector in $T_H(E)$ normal to T_0 and T_1 determined by

$$'DT_1^i/dt=K_1^0T_0^i+K_1^1T_1^i+K_1^2T_2^i=K_1^\alpha T_\alpha^i \qquad \alpha=0,1,2, \tag{2.21}$$

$$g_{ij}T_\alpha^iT_2^j=\delta_{\alpha 2} \qquad \alpha=0,1,2. \tag{2.22}$$

Differentiating (2.8a), (2.8b) and using (2.9) and (2.21) we obtain

$$\text{(a) } K_0^1+K_1^0=0, \qquad \text{(b) } K_1^1=-K. \tag{2.23}$$

From (2.23a) and (2.18) we have

$$K_1^0=\mp\sqrt{k_0^2-K^2}. \tag{2.24}$$

From (2.21), (2.23b), (2.22) and (2.20b) we obtain

$$k_1^2=(K_1^0)^2+(K_1^1)^2+(K_1^2)^2. \tag{2.25}$$

Substituting (2.23a) and (2.23b) into (2.25) we get

$$(K_1^2)^2=k_1^2-k_0^2. \tag{2.26}$$

Substituting (2.23b), (2.24) and (2.26) into (2.21) we have

$$'DT_1^i/dt=\mp\sqrt{k_0^2-K^2}T_0^i-KT_1^i\pm\sqrt{k_1^2-k_0^2}T_2^i. \tag{2.27}$$

In the similar way we obtain

$$'DT_2^i/dt = \pm\sqrt{k_1^2 - k_0^2}T_1^i - KT_2^i \mp \sqrt{k_2^2 - k_1^2 + k_0^2 - K^2}T_3^i, \tag{2.28}$$

where

$$'DT_2^i/dt = k_2 t_2^i, \quad g_{ij}t_2^i t_2^j = 1, \quad g_{ij}T_\alpha^i T_3^j = \delta_{\alpha 3}, \qquad \alpha = 0,1,2,3.$$

The general formula we shall obtain by induction. Let us suppose that the following formulae hold:

$$\text{(a) } 'DT_\alpha^i/dt = K_\alpha^{\alpha-1}T_{\alpha-1}^i + K_\alpha^\alpha T_\alpha^i + K_\alpha^{\alpha+1}T_{\alpha+1}^i, \quad \text{(b) } 'DT_\alpha^i/dt = k_\alpha t_\alpha^i, \tag{2.29}$$

where

$$\begin{aligned}
&\text{(a) } g_{ij}T_\alpha^i T_\beta^j = \delta_{\alpha\beta}, \qquad \alpha,\beta \in \{0,1,\dots,m+1\},\\
&\text{(b) } g_{ij}t_\alpha^i t_\alpha^j = 1, \qquad \alpha \in \{0,1,\dots,m\},\\
&\text{(c) } K_\alpha^{\alpha-1} = -K_{\alpha-1}^\alpha, \qquad \text{(d) } K_\alpha^\alpha = -K,\\
&\text{(e) } (K_\alpha^{\alpha+1})^2 = k_\alpha^2 - (K_\alpha^{\alpha-1})^2 - (K_\alpha^\alpha)^2,\\
&\text{(f) } (K_\alpha^{\alpha+1})^2 = k_\alpha^2 - k_{\alpha-1}^2 + k_{\alpha-2}^2 - \dots + (-1)^\alpha k_0^2 + \delta_i K^2,\\
&\qquad \delta_i = \begin{cases} 0 & \text{for } i \text{ odd} \\ -1 & \text{for } i \text{ even.} \end{cases}
\end{aligned} \tag{2.30}$$

For $\alpha = 0$, $\alpha = 1$, $\alpha = 2$ (2.29) (a) has the form (2.19), (2.27), (2.28). Formulae (2.30) for $\alpha = 0$, $\alpha = 1$, $\alpha = 2$ have the form (2.17), (2.18), (2.23) and (2.25).

Let us denote by $T_{H(\alpha+2)}$ the $(\alpha+2)$-dimensional subspace of $T_H(E)$ spanned by $T_0, T_1, \dots, T_{\alpha+1}$. Let T_{m+2} be the unit vector normal to $T_{H(m+2)}$ determined by the relations

$$\begin{aligned}
&\text{(a) } 'DT_{m+1}^i/dt = K_{m+1}^0 T_0^i + K_{m+1}^1 T_1^i + \cdots\\
&\qquad\qquad + K_{m+1}^m T_m^i + K_{m+1}^{m+1}T_{m+1}^i + K_{m+1}^{m+2}T_{m+2}^i,\\
&\text{(b) } g_{ij}T_\alpha^i T_{m+1}^j = \delta_{\alpha,m+1}, \qquad \alpha = 0,1,\dots,m+1.
\end{aligned} \tag{2.31}$$

and t_{m+1} the unit vector in the direction of $'DT_{m+1}^i/dt$, i.e.

$$\text{(a) } 'DT_{m+1}^i/dt = k_{m+1}t_{m+1}^i, \qquad \text{(b) } g_{ij}t_{m+1}^i t_{m+1}^j = 1. \tag{2.32}$$

Differentiating the relation (2.31b) for $\alpha = 0, \dots, m-1$ we obtain $K_{m+1}^0 = 0$, $K_{m+1}^1 = 0$, ..., $K_{m+1}^{m-1} = 0$, so (2.31a) reduces to the form

$$'DT_{m+1}^i/dt = K_{m+1}^m T_m^i + K_{m+1}^{m+1}T_{m+1}^i + K_{m+1}^{m+2}T_{m+2}^i. \tag{2.33}$$

Differentiating (2.31b) for $\alpha = m$ and $\alpha = m+1$, we get

$$\text{(a) } K_{m+1}^m = -K_m^{m+1}, \qquad \text{(b) } K_m^m = -K. \tag{2.34}$$

From (2.31a) and (2.29b) we have

$$k_m^2 = (K_{m+1}^m)^2 + (K_{m+1}^{m+1})^2 + (K_{m+1}^{m+2})^2. \tag{2.35}$$

Substituting (2.34a), (2.34b) and (2.30f) into (2.35) we get

(2.36) $$(K_{m+1}^{m+2})^2 = k_{m+1}^2 - k_m^2 + k_{m-1}^2 - \cdots + (-1)^{m+1}k_0^2 + \delta_{m+1}K^2.$$

From (2.33)–(2.36) it follows that (2.29) and (2.30) are true for $i = n-2$. For $i = n-1$ we have

(2.37) $$\text{(a) } {}'DT_{n-1}^i/dt = K_{n-1}^0 T_0^i + K_{n-1}^1 T_1^i + \cdots + K_{n-1}^{n-2}T_{n-2}^i + K_{n-1}^{n-1}T_{n-1}^i,$$
$$\text{(b) } g_{ij}T_\alpha^i T_{n-1}^j = \delta_{\alpha,n-1}, \qquad \alpha = 0,1,\dots,n-1,$$

because ${}'DT_{n-1}/dt$ is in $T_H(E)$ spanned by $T_0, T_1, \dots, T_{n-1}$. On the other hand, we have as usual

(2.38) $$\text{(a) } {}'DT_{n-1}^i/dt = k_{n-1}t_{n-1}^i, \qquad \text{(b) } g_{ij}t_{n-1}^i t_{n-1}^j = 1.$$

Using the same method as before, from (2.37) and (2.38), we obtain (2.39), where:
(2.39)
$$\text{(a) } K_{n-1}^0 = 0,\ K_{n-1}^1 = 0,\ \dots,\ K_{n-1}^{n-3} = 0,\ K_{n-1}^{n-2} = -K_{n-2}^{n-1},\ K_{n-1}^{n-1} = -K,$$
$$\text{(b) } k_{n-1}^2 = (K_{n-1}^{n-2})^2 + (K_{n-1}^{n-1})^2.$$

Substituting (2.39) into (2.37) we obtain

(2.40) $${}'DT_{n-1}^i/dt = K_{n-1}^{n-2}T_{n-2}^i + K_{n-1}^{n-1}T_{n-1}^i,$$

where

$$K_{n-1}^{n-2} = K_{n-2}^{n-1} \pm \sqrt{k_{n-2}^2 - k_{n-3}^2 + \cdots + (-1)^{n-2}k_0^2 + \delta_{n-2}K^2}, \quad K_{n-1}^{n-1} = -K.$$

So we proved

THEOREM 2.1. *The complete list of Frenet formulae in RLd for the horizontal curve is given by*

$${}'DT_\alpha^i/dt = K_\alpha^{\alpha-1}T_{\alpha-1}^i + K_\alpha^\alpha T_\alpha^i + K_\alpha^{\alpha+1}T_{\alpha+1}^i,$$

where

$$g_{ij}T_\alpha^i T_\beta^j = \delta_{\alpha\beta}, \quad K_0^{-1} = 0, \quad K_{n-1}^n = 0, \quad K_i^i = -K, \quad K_\alpha^{\alpha+1} = -K_{\alpha+1}^\alpha,$$
$$(K_\alpha^{\alpha+1})^2 = k_\alpha^2 - k_{\alpha-1}^2 + k_{\alpha-2}^2 - \cdots + (-1)^\alpha k_0^2 + \delta_\alpha K^2,$$
$$k_\alpha^2 = g_{ij}\frac{{}'DT_\alpha^i}{dt}\frac{{}'DT_\alpha^j}{dt}, \qquad \frac{{}'DT_\alpha^i}{dt} = k_\alpha t_\alpha^i.$$

THEOREM 2.2. *The curvatures $k_0, k_1, \dots, k_{n-1}$ of the horizontal curve (2.4) and the vector of recurrency λ_i of the space RLd are connected by the formula*

(2.41) $$k_{n-1}^2 - k_{n-2}^2 + k_{n-3}^2 - \cdots + (-1)^{n-1}k_0^2 + \delta_{n-1}K^2 = 0, \qquad 2K = \lambda_k\, dx^k/dt.$$

Proof. Substituting (2.36) for $m = n-2$ into (2.39b) we obtain (2.41). For $K = 0$, i.e. when $\lambda_k = 0$ or $\lambda_k\, dx^k = 0$, (2.41) reduces to the form

(2.42) $$k_{n-1}^2 - k_{n-2}^2 + \cdots + (-1)^{n-1}k_0^2 = 0.$$

Remark. If t, n, b denote the unit tangent vector, normal vector and binormal vector in the Euclidian space, then (2.42) reduces to

$k_2^2 - k_1^2 + k_0^2 = (b')^2 - (n')^2 + (t')^2 = 0$, where $b' = db/ds$, $n' = dn/ds$, $t' = dt/ds$.

3. Frenet formulae in RLd for the vertical curve. Let us consider the vertical curve $x^i = x^i(t)$, $y^i = y^i(t)$ for which the tangent vector has the form

$$(3.1)\qquad \begin{aligned} &T = \bar{T} = \bar{T}^i \dot{\partial}_i, \quad \frac{dx^i}{dt}\delta_i = 0 \implies \frac{dx^i}{dt} = 0 \implies x^i(x+dt) = x^i(t) \\ &\bar{T}^i = \frac{dy^i}{dt} + N^i_j \frac{dx^j}{dt} = \frac{dy^i}{dt}. \end{aligned}$$

Let us introduce the notation

$$(3.2)\qquad {}''DQ = \nabla_{\bar{T}} Q \quad (Q \text{ is any tensor and } T \text{ is determined by (3.1).})$$

If we denote by $\bar{T}_0$ the unit tangent vector in the direction of $\bar{T}$ then we have

$$(3.3)\qquad \text{(a) } \bar{T}^i = \bar{k}(t)\bar{T}^i_0, \quad \text{(b) } \bar{g}_{ij}(x(t), y(t))\bar{T}^i_0\bar{T}^j_0 = 1 \implies \bar{g}_{ij}\bar{T}^i\bar{T}^j = \bar{k}^2.$$

From (1.27) and (3.2) we have

$$\frac{{}''D\bar{T}_0}{dt} = \frac{{}''D\bar{T}^i_0}{dt}\dot{\partial}_i = \bar{T}^i_0|_j \frac{dy^j}{dt}\dot{\partial}_i$$

As the vector ${}''DT_0/dt$ is in $T_V(E)$, it can be decomposed in the direction of $\bar{T}_0$ and $\bar{T}_1$, where $\bar{T}_0$ and $\bar{T}_1$ are unit mutualy normal vectors in $T_V(E)$, $\bar{T}_0 = \bar{T}^i_0\dot{\partial}_i$, $\bar{T}_1 = \bar{T}^i_1\dot{\partial}_i$. We may write

$$(3.4)\qquad \text{(a) } \bar{g}_{ij}\bar{T}^i_0\bar{T}^j_1 = 0, \qquad \text{(b) } \bar{g}_{ij}\bar{T}^i_1\bar{T}^j_1 = 1$$

$$(3.5)\qquad {}''D\bar{T}^i_0/dt = \bar{K}^0_0\bar{T}^i_0 + \bar{K}^1_0\bar{T}^i_1.$$

On the other hand, if we denote by $\bar{t}_0$, $\bar{t}_0 = \bar{t}^i_0\dot{\partial}_i$, the unit vector in the direction of ${}''D\bar{T}_0/dt$, we have

$$(3.6)\qquad \text{(a) } {}''D\bar{T}^i_0/dt = \bar{k}_0\bar{t}^i_0 \qquad \text{(b) } \bar{g}_{ij}\bar{t}^i_0\bar{t}^j_0 = 1.$$

From $dx^k = 0$ and (1.29) we obtain ${}''D\bar{g}_{ij} = \bar{g}_{ij}|_k\,\delta y^k$. From this relation and (1.30) we get

$$(3.7)\qquad \begin{aligned} &\text{(a)} \quad {}''D\bar{g}_{ij}/dt = \bar{\mu}_k\bar{g}_{ij}\,dy^k/dt = 2\bar{K}\bar{g}_{ij}, \\ &\text{(b)} \quad 2\bar{K} = \bar{\mu}_k\,dy^k/dt = \bar{\mu}_k\bar{T}^k = \bar{\mu}_k\bar{T}^i_0\bar{k}. \end{aligned}$$

From (3.3b) we have

$$(3.8)\qquad \frac{{}''D\bar{g}_{ij}}{dt}\bar{T}^i_0\bar{T}^j_0 + 2\bar{g}_{ij}\frac{{}''D\bar{T}^i_0}{dt}\bar{T}^j_0 = 0.$$

Substituting (3.7a), (3.7b), (3.5), (3.4a) into (3.8) we obtain

$$\bar{K}_0^0 = -\bar{K}. \tag{3.9}$$

From (3.6a) and (3.6b) we have

$$\bar{g}_{ij}\frac{''D\bar{T}_0^i}{dt}\frac{''D\bar{T}_0^j}{dt} = \bar{k}_0^2. \tag{3.10}$$

On the other hand, using (3.4) and (3.5) we have

$$\bar{g}_{ij}\frac{''D\bar{T}_0^i}{dt}\frac{''D\bar{T}_0^j}{dt} = (\bar{K}_0^0)^2 + (\bar{K}_0^1)^2. \tag{3.11}$$

From (3.10) and (3.11) we get

$$\bar{k}_0^2 = (\bar{K}_0^0)^2 + (\bar{K}_0^1)^2. \tag{3.12}$$

Using (3.9) from (3.12) we have

$$(\bar{K}_0^1)^2 = \bar{k}_0^2 - \bar{K}^2. \tag{3.13}$$

Substituting (3.9) and (3.13) into (3.5) we obtain the first Frenet formula in RLd for the vertical curve:

$$''D\bar{T}_0^i/dt = -\bar{k}T_0^i \pm \sqrt{\bar{k}_0^2 - \bar{K}^2}\bar{T}_1^i.$$

The other formulae may be obtained in the same way as those for the horizontal curve. We have

THEOREM 3.1. *The complete list of Frenet formulae for the vertical curve in RLd is*

$$''D\bar{T}_\alpha^i = \bar{K}_\alpha^{\alpha-1}\bar{T}_{\alpha-1}^i + \bar{K}_\alpha^\alpha\bar{T}_\alpha^i + \bar{K}_\alpha^{\alpha+1}\bar{T}_{\alpha+1}^i,$$

where

$$\bar{g}_{ij}\bar{T}_\alpha^i T_\beta^j = \delta_{\alpha\beta}, \quad \bar{K}_0^1 = 0, \quad \bar{K}_{n-1}^n = 0, \quad K_i^i = -\bar{K}, \quad \bar{K}_\alpha^{\alpha+1} = -\bar{K}_{\alpha+1}^\alpha,$$
$$(\bar{K}_\alpha^{\alpha+1})^2 = \bar{k}_\alpha^2 - \bar{k}_{\alpha-1}^2 + \bar{k}_{\alpha-2}^2 - \dots + (-1)^\alpha\bar{k}_0^2 + \delta_\alpha\bar{K}^2,$$
$$\bar{k}_\alpha^2 = \bar{g}_{ij}\frac{''D\bar{T}_\alpha^i}{dt}\frac{''D\bar{T}_\alpha^j}{dt}, \quad \frac{''DT_\alpha^i}{dt} = \bar{k}_\alpha\bar{t}_\alpha^i, \quad \bar{g}_{ij}\bar{t}_\alpha^i\bar{t}_\alpha^j = 1, \qquad \alpha, \beta = 1, 2, \dots, n-1.$$

THEOREM 3.2. *The curvature $\bar{k}_0, \bar{k}_1, \dots, \bar{k}_{n-1}$ of the vertical curve* (3.1) *and the vector of recurrency $\bar{\mu}_i$ of the space RLd are connected by*

$$\bar{k}_{n-1}^2 - \bar{k}_{n-2}^2 + \bar{k}_{n-3}^2 - \dots + (-1)^{n-1}k_0^2 + \delta_{n-1}\bar{K}^2 = 0, \quad \bar{K} = \bar{\mu}_k\bar{T}^k = \bar{k}\bar{\mu}_i\bar{T}_0^i.$$

The proofs of Theorems 3.1 and 3.2 are similar to the proofs of Theorems 2.1 and 2.2 respectively.

REFERENCES

[1] G. S. Asanov, *Finsler Geometry, Relativity and Gauge Theories*, D. Reidler, 1987.

[2] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler spaces*, Kaseish Press, 1986, Japan.

[3] R. Miron, *Vector bundles in Finsler Geometry*, The proceedings of the national seminar on Finsler Geometry, Braşov, 1983, 147–186.

[4] A. Moór, *Über eine Übertragungstheorie der metrischen Linienelementräume mit rekurrentem Grundtensor*, Tensor (N.S) 29 (1975), 47–63.

[5] P. K. Rashevsky, *Rimanova geometrija i tenzornij analiz*, Nauka, Moskva, 1967.

[6] H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, 1959.

[7] J. A. Schouten, *Ricci Calculus*, Springer-Verlag, 1954.

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