THE NEUTRIX CONVOLUTION PRODUCT $x_{-}^{-r} \odot x_{+}^{n}$

Brian Fisher and Emin Özçağ

Abstract. Let $f$ and $g$ be distributions in $D'$ and let $f_n(x) = f(x)\tau_n(x)$, where $\tau_n(x)$ is a certain function which converges to the identity function as $n$ tends to infinity. Then the neutrix convolution product $f \odot g$ is defined as the neutrix limit of the sequence $\{f_n \ast g\}$, provided the limit $h$ exists in the sense that $N$-$\lim_{n \to \infty} (f_n \ast g, \phi) = \langle h, \phi \rangle$ for all $\phi$ in $D$. The neutrix convolution products $\ln x_{-} \odot x_{+}^{n}, x_{-}^{n} \odot \ln x_{+}, x_{-}^{n} \odot x_{+}^{n}$ and $x_{-}^{n} \odot x_{+}^{-r}$ for $\mu \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$ are evaluated, from which other neutrix convolution products are deduced.

In the following we let $D$ be the space of infinitely differentiable functions with compact support and let $D'$ be the space of distributions defined on $D$. The convolution product $f \ast g$ of two distributions $f$ and $g$ in $D'$ is then usually defined by the equation $\langle (f \ast g)(x), \phi \rangle = \langle f(y), (g(x), \phi(x+y)) \rangle$ for arbitrary $\delta$ in $D$, provided $f$ and $g$ satisfy either of the conditions: (a) either $f$ or $g$ has bounded support; (b) the supports of $f$ and $g$ are bounded on the same side, (see Gel'fand and Shilov [8]).

It follows that, if the convolution product $f \ast g$ exists by this definition, then

$$f \ast g = g \ast f, \quad (1)$$

$$\quad \quad \quad \quad \quad (f \ast g)' = f' \ast g. \quad (2)$$

This definition of the convolution product is rather restrictive and so the neutrix convolution product was introduced in [2]. In order to define the neutrix convolution product we first of all let $\tau$ be a function in $D$ satisfying the following properties:

(i) $\tau(x) = \tau(-x)$,
(ii) $0 \leq \tau(x) \leq 1$,
(iii) $\tau(x) = 1$ for $|x| \leq 1/2$,
(iv) $\tau(x) = 0$ for $|x| \geq 1$.

The function $\tau_n$ is now defined by

$$\tau_n(x) = \begin{cases} 
1, & |x| \leq n, \\
\tau(n^n x - n^{n+1}), & x > n, \\
\tau(n^n x + n^{n+1}), & x < -n,
\end{cases} \quad \text{for } n = 1, 2, \ldots$$

**AMS Subject Classification (1980):** Primary 46F10
Definition 1. Let $f$ and $g$ be distributions in $\mathcal{D}'$ and let $f_n = f \tau_n$ for $n = 1, 2, \ldots$. Then the neutrix convolution product $f \odot g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided that the limit $h$ exists in the sense that $N - \lim_{n \to \infty} (f_n * g, \phi) = (h, \phi)$, for all $\phi$ in $\mathcal{D}$, where $N$ is the neutrix, (see van der Corput [1]), having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range $N''$ the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \quad (\lambda > 0, \quad r = 1, 2, \ldots)$$

and all the functions which converge to zero in the usual sense, as $n$ tends to infinity.

Note that in this definition the convolution product $f_n * g$ is defined in Gel'fand and Shilov's sense, the distribution $f_n$, having bounded support.

The following theorem was proved in [2], showing that the neutrix convolution product is a generalization of the convolution product.

**Theorem 1.** Let $f$ and $g$ be distributions in $\mathcal{D}'$ satisfying either condition (a) or condition (b) of Gel'fand and Shilov's definition. Then the neutrix convolution product $f \odot g$ exists and $f \odot g = f * g$.

The next two theorems were also proved in [2].

**Theorem 2.** Let $f$ and $g$ be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution product $f \odot g$ exists. Then the neutrix convolution product $f \odot g'$ exists and $(f \odot g)' = f \odot g'$.

Note however that equation (1) does not necessarily hold for the neutrix convolution product and that $(f \odot g)'$ is not necessarily equal to $f' \odot g$.

**Theorem 3.** The neutrix convolution product $x_+^\lambda \odot x_+^s$ exists and

$$x_+^\lambda \odot x_+^s = (-1)^{s+1} B(\lambda + 1, s + 1) x_+^{\lambda + s + 1}$$

for $\lambda > -1$ and $s = 0, 1, 2, \ldots$, where $B$ denotes the Beta function.

Later, the following two theorems were proved in [3] and [4] respectively:

**Theorem 4.** The neutrix convolution product $x_-^\lambda \odot x_+^s$ exists and satisfies equation (3) for $\lambda < -1, \lambda \neq -2, -3, \ldots$ and $s = 0, 1, 2, \ldots$.

**Theorem 5.** The neutrix convolution product $x_-^s \odot x_+^\lambda$ exists and

$$x_-^s \odot x_+^\lambda = (-1)^{s+1} B(\lambda + 1, s + 1) x_+^{\lambda + s + 1}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = 0, 1, 2, \ldots$.

The next two theorems were proved in [5].

**Theorem 6.** The neutrix convolution product $x_-^\lambda \odot x_+^{s-\lambda}$ exists and

$$x_-^\lambda \odot x_+^{s-\lambda} = (-1)^{s+1} B(-s - 1, s + 1 - \lambda) x_+^{s+1}$$

$$+ \frac{(-1)^{s+1} (\lambda)^{s+1}}{(s + 1)!} \{ \pi \cot(\pi \lambda) x_+^{s+1} - x_+^{s+1} \ln |x| \},$$
The neutrix convolution product $x_+^{-r} \otimes x_+^s$

for $\lambda > -1$, $\lambda \neq 0, 1, 2, \ldots$ and $s = -1, 0, 1, 2, \ldots$, where

$$(\lambda)_s = \begin{cases} 1, & s = 0, \\ \prod_{i=0}^{s-1}(\lambda - i), & s \geq 1. \end{cases}$$

In this theorem, $B$ again denotes the Beta function but is defined as in [7] by

$$B(\lambda, \mu) = \lim_{n \to \infty} \int_{1/n}^{1-1/n} t^{\lambda-1}(1-t)^{\mu-1} \, dt.$$  

This definition is in agreement with the usual definition of $B(\lambda, \mu)$ when $\lambda, \mu \neq 0, -1, -2, \ldots$ but defines $B(\lambda, \mu)$ when $\lambda$ or $\mu$ take the values $0, -1, -2, \ldots$.

**Theorem 7.** The neutrix convolution product $x_+^\lambda \otimes x_+^{-s-\lambda}$ exists and

$$x_+^\lambda \otimes x_+^{-s-\lambda} = \frac{\pi \cot(\pi \lambda)}{(-1-\lambda)_{s-1}} \delta(s-2)(x) - \frac{(-1)^s(s-2)!}{(-1-\lambda)_{s-1}} x_+^{-s+1},$$

for $\lambda > -1$, $\lambda \neq 0, 1, 2, \ldots$ and $s = 2, 3, \ldots$.

The next theorem was proved in [6].

**Theorem 8.** The neutrix convolution product $x_+^\lambda \otimes x_+^\mu$ exists and

$$x_+^\lambda \otimes x_+^\mu = B(-\lambda - \mu - 1, \mu + 1)x_+^{\lambda+\mu+1}$$

$$+ B(-\lambda - \mu - 1, \lambda + 1)x_+^{\lambda+\mu+1},$$

for $\lambda, \mu, \lambda + \mu \neq 0, \pm 1, \pm 2, \ldots$.

In the following we are going to consider the neutrix convolution products $x_+^{-r} \otimes x_+^\mu$ and $x_+^\mu \otimes x_+^{-r}$, where $x_+^{-r}$ is defined by

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!} (\ln x_+)^{(r)}$$

and $x_+^{-r}$ is defined by $x_+^{-r} = (-x_+)^{-r}$, but first of all we prove

**Theorem 9.** The neutrix convolution products $\ln x_- \otimes x_+^\mu$ and $x_+^\mu \otimes \ln x_+$

exist and

$$\ln x_- \otimes x_+^\mu = -\frac{x_+^{\mu+1}}{\mu + 1} \ln x_+ + \frac{\gamma + \psi(-\mu - 1)}{\mu + 1} x_+^{\mu+1}, \quad (4)$$

$$x_- \otimes \ln x_+ = -\frac{x_-^{\mu+1}}{\mu + 1} \ln x_- + \frac{\gamma + \psi(-\mu - 1)}{\mu + 1} x_-^{\mu+1} \quad (5)$$

for $\mu \neq 0, \pm 1, \pm 2, \ldots$, where $\gamma$ denotes Euler's constant, $\psi = \Gamma'/\Gamma$ and $\Gamma$ denotes the Gamma function.
Proof. We will suppose first of all that $\mu > -1$ and $\mu \neq 0, 1, 2, \ldots$ so that $x_+^\mu$ is a locally summable function. Put $(\ln x_-)_n = \ln x_+ \tau_n(x)$. Then
\[
(\langle (\ln x_-)_n * x_+^\mu, \phi(x) \rangle = (\langle (\ln y_-)_n, (x_+^\mu, \phi(x + y) \rangle))
\]
\[
= \int_{-n}^{0} \int_{-n}^{0} \ln(-y)\tau_n(y) \int_{a}^{b} (x-y)^{\mu} \phi(x) \, dx \, dy
\]
\[
= \int_{a}^{b} \phi(x) \int_{-n}^{0} \ln(-y)(x-y)^{\mu} \, dy \, dx
\]
\[
+ \int_{a}^{b} \phi(x) \int_{-n}^{0} \ln(-y)\tau_n(y)(x-y)^{\mu} \, dy \, dx
\]
for $n > -a$ and arbitrary $\phi$ in $D$ with support of $\phi$ contained in the interval $[a, b]$.

When $x < 0$, we have on making the substitution $y = xu^{-1}$
\[
\int_{-n}^{0} \ln(-y)(x-y)^{\mu} \, dy = \int_{-n}^{x} \ln(-y)(x-y)^{\mu} \, dy
\]
\[
= (-x)^{\mu+1} \ln(-x) \int_{-x/n}^{1} u^{-\mu-2}(1 - u)^{\mu} \, du
\]
\[
- (-x)^{\mu+1} \int_{-x/n}^{1} u^{-\mu-2} \ln(1 - u)^{\mu} \, du
\]
\[
= I_{1n} - I_{2n}.
\]
Choosing an integer $r > \mu + 1$, we have
\[
\int_{-x/n}^{1} u^{-\mu-2}(1 - u)^{\mu} \, du = \int_{-x/n}^{1} u^{-\mu-2} \left[(1 - u)^{\mu} - \sum_{i=0}^{r} \frac{(-1)^i(\mu)_i}{i!} u^i \right]
\]
\[
+ \sum_{i=0}^{r} \frac{(-1)^i(\mu)_i}{i!(i - \mu - 1)} \left[1 - (x/n)^{i-\mu-1}\right]
\]
and it follows that
\[
N\lim_{n \to \infty} I_{1n} = B(-\mu - 1, \mu + 1)(-x)^{\mu+1} \ln(-x) = 0,
\]
where $B$ denotes the Beta function, see [7] or [8]. Further,
\[
\int_{-x/n}^{1} u^{-\mu-2} \ln(1 - u)^{\mu} \, du = \int_{-x/n}^{1} u^{-\mu-2} \ln \left[(1 - u)^{\mu} - \sum_{i=0}^{r} \frac{(-1)^i(\mu)_i}{i!} u^i \right] \, du
\]
\[
- \sum_{i=0}^{r} \frac{(-1)^i(\mu)_i}{i!(i - \mu - 1)^2} \left[(i - \mu - 1) \left(-\frac{x}{n}\right)^{i-\mu-1} \ln \left(-\frac{x}{n}\right) + 1 - \left(-\frac{x}{n}\right)^{i-\mu-1}\right]
\]
and it follows that $N\lim_{n \to \infty} I_{2n} = B_{10}(-\mu - 1, \mu + 1)(-x)^{\mu+1}$, where
\[
B_{10}(-\mu - 1, \mu + 1) = \left[\partial B(\lambda, \mu + 1)/\partial \lambda\right]_{\lambda=\mu-1} = 0,
\]
The neutrix convolution product \( x^{-\mu}_- \otimes x^\mu_+ \)

see [7]. Thus \( N-\lim_{n \to \infty} I_{2n} = 0 \) and so

\[
N-\lim_{n \to \infty} \int_{-n}^{0} \ln(-y)(x - y)^\mu_+ dy = 0. \tag{7}
\]

When \( x > 0 \), we have on making the substitution \( y = x(1 - u^{-1}) \)

\[
\int_{-n}^{0} \ln(-y)(x - y)^\mu_+ dy = \int_{-n}^{0} \ln(-y)(x - y)^\mu dy
\]

\[
= x^{\mu+1} \ln x \int_{x/(x+n)}^{1} u^{-\mu-2} du + x^{\mu+1} \int_{x/(x+n)}^{1} u^{-\mu-2} \ln(1 - u) du
\]

\[
- x^{\mu+1} \int_{x/(x+n)}^{1} u^{-\mu-2} \ln u du
\]

\[
= I_{3n} + I_{4n} - I_{5n}.
\]

We have

\[
x^{\mu+1} \ln x \int_{x/(x+n)}^{1} u^{-\mu-2} du = -\frac{x^{\mu+1} \ln x}{\mu + 1} + \frac{n^{\mu+1}}{\mu + 1} \left( 1 + \frac{x}{n} \right)^{\mu+1} \ln x
\]

and it follows that

\[
N-\lim_{n \to \infty} I_{3n} = -\frac{x^{\mu+1} \ln x}{\mu + 1}.
\]

Making the substitution \( u = 1 - v \) we have

\[
\int_{x/(x+n)}^{1} u^{-\mu-2} \ln(1 - u) du = \int_{0}^{n/(x+n)} \ln v(1 - v)^{-\mu-2} dv
\]

\[
= \int_{0}^{n/(x+n)} \ln v \left[ (1 - v)^{-\mu-2} - \sum_{i=0}^{r} \frac{(-1)^i (\mu + 2)_i}{i!} v^i \right] dv
\]

\[
+ \sum_{i=0}^{r} \frac{(-1)^i (\mu + 2)_i (1 + x/n)^{-i-1} \ln(1 + x/n)}{i! (i + 1)^2} + \frac{(1 + x/n)^{-i-1}}{i + 1}
\]

and it follows that

\[
N-\lim_{n \to \infty} \int_{x/(x+n)}^{1} u^{-\mu-2} \ln(1 - u) du
\]

\[
= \int_{0}^{1} \ln v \left[ (1 - v)^{-\mu-2} - \sum_{i=0}^{r} \frac{(-1)^i (\mu + 2)_i v^i}{i!} \right] dv
\]

\[
- \sum_{i=0}^{r} \frac{(-1)^i (\mu + 2)_i}{i! (i + 2)^2} \left( 1 + \frac{x}{n} \right)^{-i-1} = B_{10}(1, -\mu - 1).
\]

Thus \( N-\lim_{n \to \infty} I_{4n} = B_{10}(1, -\mu - 1) x^{\mu+1} \).
Next, we have
\[ \int_{-x/(x+n)}^{1} u^{-\mu-2} \ln u \, du = \frac{(x + n)^{\mu+1} [\ln x + \ln(x + n)]}{(\mu + 1)x^{\mu+1}} - \frac{1}{(\mu + 1)^2} + \frac{(x + n)^{\mu+1}}{(\mu + 1)^2 x^{\mu+1}} \]
and it follows that \( N^{-\text{lim}_{n \to \infty}} I_{5n} = -(\mu + 1)^{-2} x^{\mu+1} \). Now it is easily proved that
\[ B_{10}(1, \mu) = -\frac{\gamma - \psi(1 + \mu)}{\mu}, \quad \mu^{-1} + \psi(u) = \psi(\mu + 1) \]
and so
\[ B_{10}(1, -\mu - 1) + (\mu + 1)^{-2} = \frac{\gamma + \psi(-\mu - 1)}{\mu + 1}. \]
Thus
\[ N^{-\text{lim}} \int_{-n}^{0} \ln(-y)(x - y)^{\mu} \, dy = -\frac{x^{\mu+1} \ln x}{\mu + 1} + \frac{\gamma + \psi(-\mu - 1)}{\mu + 1} x^{\mu+1}. \quad (8) \]
Further, with \( a \leq x \leq b \) and \( n > -a \), we have
\[ \left| \int_{-n}^{-n} \ln(-y) \tau_n(y)(x - y)^{\mu} \, dy \right| = o(n^{\mu-n} \ln n) \]
and so
\[ \lim_{n \to \infty} \int_{-n-n}^{-n} \ln(-y) \tau_n(y)(x - y)^{\mu} \, dy = 0. \quad (9) \]
It now follows from equations (6), (7), (8) and (9) that
\[ N^{-\text{lim}}((\ln x_{-})_n * x_{+}^{\mu}) = (-(\mu + 1)^{-1} x_{+}^{\mu+1} \ln x_{+} + [\gamma + \psi(-\mu - 1)](\mu + 1)^{-1} x_{+}^{\mu+1}, \phi(x)) \]
and equation (4) follows for \( \mu > -1 \) and \( \mu \neq 0, 1, 2, \ldots \).

Now assume that equation (4) holds for \( -k < \mu < -k + 1 \), where \( k \) is some positive integer. This is certainly true when \( k = 1 \). Then by Theorem 1, the neutrix convolution product \( \ln x_{-} \otimes x_{+}^{\mu - 1} \) exists and
\[ \mu \ln x_{-} \otimes x_{+}^{\mu - 1} = -x_{+}^{\mu} \ln x_{+} - (\mu + 1)^{-1} x_{+}^{\mu} + [\gamma + \psi(-\mu - 1)] x_{+}^{\mu} \]
\[ = -x_{+}^{\mu} \ln x_{+} + [\gamma + \psi(-\mu)] x_{+}^{\mu}, \]
since \( \psi(-\mu - 1) - (\mu + 1)^{-1} = \psi(-\mu) \). Equation (4) follows by induction for \( \mu \neq 0, \pm 1, \pm 2, \ldots \).

To prove equation (5) we will again suppose first of all that \( \mu > -1 \) and \( \mu \neq 0, 1, 2, \ldots \), so that \( x_{-}^{\mu} \) is a locally summable function. Put \((x_{-}^{\mu})_n = x_{-} \tau_n(x)\). Then
\[ \langle (x_{-}^{\mu})_n \times \ln x_{+}, \phi(x) \rangle = \langle (y_{+}^{\mu})_n, (\ln x_{+}, \phi(x + y)) \rangle \]
The neutrix convolution product $x_n^{-r} \ast x_n^{n}$

\[
\begin{align*}
&= \int_{-n-n-n}^{0} (-y)^\mu \tau_n(x) \int_{a}^{b} \ln(x - y) + \phi(x) \, dx \, dy \\
&= \int_{a}^{b} \phi(x) \int_{-n-n-n}^{0} (-y)^\mu \ln(x - y) + \, dy \, dx \\
&\quad + \int_{a}^{b} \phi(x) \int_{-n-n-n}^{-n} (-y)^\mu \tau_n(y) \ln(x - y) \, dy \, dx \\
&= \int_{-n}^{0} (-y)^\mu \ln(x - y) + \, dy \\
&= -(-x)^(\mu+1) \int_{-x/n}^{1} u^{-(\mu+2)} du \\
&\quad + (-x)^(\mu+1) \ln(-x) \int_{-x/n}^{1} u^{-(\mu+2)} du \\
&\quad + (-x)^(\mu+1) \int_{-x/n}^{1} u^{-(\mu+2)} (1 - u) \, du \\
&= -J_1n + J_2n + J_3n.
\end{align*}
\]

for $n > -a$ and arbitrary $\phi$ in $D$ with support of $\phi$ contained in the interval $[a, b]$.

When $x < 0$, we have on making the substitution $y = xu^{-1}$

\[
\begin{align*}
&\int_{-n}^{0} (-y)^\mu \ln(x - y) + \, dy = \int_{-x/n}^{1} \frac{(-n/x)^{\mu+1}}{\mu + 1} - \frac{1 - (-n/x)^{\mu+1}}{\mu + 1} \\
&\quad + (-x)^{\mu+1} \ln(-x) \int_{-x/n}^{1} u^{-(\mu+2)} du \\
&\quad + (-x)^{\mu+1} \int_{-x/n}^{1} u^{-(\mu+2)} (1 - u) \, du \\
&= -J_1n + J_2n + J_3n.
\end{align*}
\]

We have

\[
\int_{-x/n}^{1} u^{-(\mu+2)} ln u \, du = \frac{(-n/x)^{\mu+1}}{\mu + 1} - \frac{1 - (-n/x)^{\mu+1}}{(\mu + 1)^2}
\]

and it follows that $N^{-}\lim_{n \to \infty} J_1n = -(\mu + 1)^{-2}(-x)^{\mu+1}$.

Next we have

\[
\int_{-x/n}^{1} u^{-(\mu+2)} du = \frac{1 - (-n/x)^{\mu+1}}{\mu + 1}
\]

and it follows that

\[
N^{-}\lim_{n \to \infty} J_2n = -\frac{(-x)^{\mu+1}}{\mu + 1} \ln(-x)
\]

Making the substitution $u = 1 - v$ we have

\[
\begin{align*}
&\int_{-x/n}^{1} u^{-(\mu+2)} \ln(1 - u) \, du = \int_{0}^{(x+n)/n} \ln v (1 - v)^{-\mu-2} \, dv \\
&= \int_{0}^{(x+n)/n} \ln v \left[ (1 - v)^{-\mu-2} - \sum_{i=0}^{r} \frac{(-1)^i(\mu+2)_{i}}{i!} v^i \right] \, dv \\
&\quad + \sum_{i=0}^{r} \frac{(-1)^i(\mu+2)_{i}}{i!} \left[ \frac{(1 + x/n)^{i+1}}{i + 1} - \frac{(1 + x/n)^{i+1}}{(i + 1)^2} \right]
\end{align*}
\]
and it follows as above that $N\text{-}\lim_{n \to \infty} J_{3n} = B_{10}(1, -\mu - 1)(-x)^{\mu + 1}$. Thus

$$N\text{-}\lim_{n \to \infty} \int_{-n}^{0} (-y)^{\mu} \ln(x - y)_+ dy = -\frac{(-x)^{\mu + 1}\ln(-x)}{\mu + 1} + B_{10}(1, -\mu - 1)(-x)^{\mu + 1} + \frac{(-1)^{\mu + 1}}{(\mu + 1)^2} + \frac{\gamma + \psi(-\mu - 1)}{\mu + 1}(-x)^{\mu + 1}$$

(11)

as above.

When $x > 0$, we have on making the substitution $y = x(1 - u^{-1})$

$$\int_{-n}^{0} (-y)^{\mu} \ln(x - y)_+ dy = \int_{-n}^{0} (-y)^{\mu} \ln(x - y) dy = x^{\mu + 1} \ln x \int_{x/(x+n)}^{1} u^{-\mu-2}(1 - u)^{\mu} du - x^{\mu + 1} \int_{x/(x+n)}^{1} u^{-\mu - 2} \ln u(1 - u)^{\mu} du$$

$$= J_{4n} - J_{5n}.$$ 

It follows similarly to above that $N\text{-}\lim_{n \to \infty} J_{4n} = B(-\mu - 1, \mu + 1)x^{\mu + 1}\ln x = 0$ and $N\text{-}\lim_{n \to \infty} J_{5n} = B_{10}(-\mu - 1, \mu + 1)[x/(x+n)]^{\mu + 1} = 0$. Thus

$$N\text{-}\lim_{n \to \infty} \int_{-n}^{0} (-y)^{\mu} \ln(x - y)_+ dy = 0.$$ (12)

Further, with $a \leq x \leq b$, and $n > -a$, we have

$$\left| \int_{-n-n-n}^{-n} (-y)^{\mu} r_n(y) \ln(x - y) dy \right| = o(n^{\mu-n}\ln n)$$

and so

$$\lim_{n \to \infty} \int_{-n-n-n}^{-n} (-y)^{\mu} r_n(y) \ln(x - y) dy = 0.$$ (13)

It now follows from equations (10), (11), (12) and (13) that

$$N\text{-}\lim_{n \to \infty} \langle (x_+^\mu)_{n} \ast \ln x_+, \phi(x) \rangle = \langle (-\mu - 1)^{-1}x_{-}^{\mu + 1}\ln x_+ + [\gamma + \psi(-\mu - 1)](\mu + 1)^{-1}x_{-}^{\mu + 1}, \phi(x) \rangle$$

and equation (5) follows for $\mu > -1$ and $\mu \neq 0, 1, 2, \ldots$.

Finally assume that equation (5) holds for $-k < \mu < -k + 1$. This is certainly true when $k = 1$. The convolution product $(x_{-}^\mu)_{n} \ast \ln x_+$ exists by Gel’fand and Shilov’s definition and so equations (2) hold. Thus if $\phi$ is an arbitrary function in $D$ with support contained in the interval $[a,b]$

$$\langle [(x_{-}^\mu)_{n} \ast \ln x_+]', \phi(x) \rangle = -\langle (x_{-}^\mu)_{n} \ast \ln x_+, \phi'(x) \rangle = -\mu \langle (x_{-}^{\mu-1})_{n} \ast \ln x_+ \phi(x) + [(x_{-}^\mu r_n)(x)] \ast \ln x_+, \phi(x) \rangle$$
and so
\[ \mu((x_n^{-1})_n \ast \ln x_+, \phi(x)) = \langle (x_n^{-1})_n \ast \ln x_+, \phi'(x) \rangle + ([x_n^{-1} \tau_n'(x)] \ast \ln x_+, \phi(x)) \].

The support of \( x_n^{-1} \tau_n'(x) \) is contained in the interval \([-n - n^{-n}, -n]\) and so with \( n > -a \), it follows as above that
\[ \langle [x_n^{-1} \tau_n'(x)] \ast \ln x_+, \phi(x) \rangle = \int_a^b \phi(x) \int_{n-n^{-n}}^{n} (-y) \tau_n'(y) \ln(x-y) \, dy \, dx, \]
where on the domain of integration, \((-y)\mu\) and \(\ln(x-y)\) are locally summable functions. Integrating by parts, it follows that
\[ \int_{n-n^{-n}}^{n} (-y) \tau_n'(y) \ln(x-y) \, dy = n^\mu \ln(x+n) + \int_{n-n^{-n}}^{n} \left[ \mu(-y)^{\mu-1} \ln(x-y) + (-y)^{\mu}(x-y)^{-1} \right] \tau_n(y) \, dy. \]
Choosing a positive integer \( r \) greater than \( \mu \), we see that
\[ n^\mu \ln(x+n) = n^\mu \ln n + n^\mu \sum_{i=1}^{r} \frac{x^i}{i n^i} + o(1/n) \]
and so since \( \mu \) is not an integer, \( N\lim_{n \to \infty} n^\mu \ln(x+n) = 0 \). Further, it follows as in the proof of equation (9) that
\[ N\lim_{n \to \infty} \int_{n-n^{-n}}^{n} \left[ \mu(-y)^{\mu-1} \ln(x-y) + (-y)^{\mu}(x-y)^{-1} \right] \tau_n(y) \, dy = 0. \]
Thus
\[ N\lim_{n \to \infty} (x_n^{-1})_n \ast \ln x_+, \phi(x)) = N\lim_{n \to \infty} ((x_n^{-1})_n \ast \ln x_+, \phi'(x)) = (x_+^{\mu} \ast \ln x_+, \phi'(x)) \]
by our assumption. This proves that the neutrix convolution product \( x_n^{-1} \ast \ln x_+ \) exists and
\[ \mu x_n^{-1} \ast \ln x_+ = -(x_+^{\mu} \ast \ln x_+)' = -x_+^{\mu} \ln x_+ - (\mu + 1)^{-1} x_+^{\mu} + [\gamma + \psi(-\mu - 1)] x_+^{\mu} = -x_+^{\mu} \ln x_+ + [\gamma + \psi(-\mu)] x_+^{\mu}, \]
as above. Equation (5) now follows by induction for \( \mu \neq 0, \pm 1, \pm 2, \ldots \). This completes the proof of the theorem.

**Corollary 1.** The neutrix convolution products \( \ln x_+ \ast x_+^{\mu} \) and \( x_+^{\mu} \ast \ln x_- \) exist and
\[ \ln x_+ \ast x_+^{\mu} = -\frac{1}{\mu + 1} x_+^{\mu+1} \ln x_+ + \frac{\gamma + \psi(-\mu - 1)}{\mu + 1} x_+^{\mu+1} \]
\[ x^\mu_+ \odot \ln x_- = -\frac{1}{\mu + 1} x^\mu+1_+ + \frac{\gamma + \psi(-\mu - 1)}{\mu + 1} x^\mu+1_+ , \]

for \( \mu \neq 0, \pm 1, \pm 2, \ldots \).

**Proof.** The results of the corollary follow immediately on replacing \( x \) by \(-x\) in equations (4) and (5).

**Corollary 2.** The neutrix convolution products \( \ln |x| \odot x^\mu_+ \), \( x^\mu_+ \odot \ln |x| \), \( \ln |x| \odot x^\mu_- \) and \( x^\mu_- \odot \ln |x| \) exist and

\[
\ln |x| \odot x^\mu_+ = \frac{\pi \cot \pi \mu}{\mu + 1} x^\mu+1_+ , \tag{14}
\]

\[
= x^\mu_+ \odot \ln |x| , \tag{15}
\]

\[
\ln |x| \odot x^\mu_- = \frac{\pi \cot \pi \mu}{\mu + 1} x^\mu+1_- , \tag{16}
\]

\[
= x^\mu_- \odot \ln |x| , \tag{17}
\]

for \( \mu \neq 0, \pm 1, \pm 2, \ldots \).

**Proof.** The convolution product \( \ln x_+ \ast x^\mu_- \) exists by Gel'fand and Shilov's definition and it is easily proved that

\[
\ln x_+ \ast x^\mu_- = (\mu + 1)^{-1} x^\mu+1_+ \ln x_+ + B_{10}(1, \mu + 1) x^\mu+1_+ \\
= (\mu + 1)^{-1} x^\mu+1_+ \ln x_+ - \frac{\gamma + \psi(\mu + 2)}{\mu + 1} x^\mu+1_+ \\
= \ln x_+ \odot x^\mu_- .
\]

Since the neutrix convolution product is clearly distributive with respect to addition, it follows that

\[
\ln x_- \odot x^\mu_+ + \ln x_+ \odot x^\mu_- = \ln |x| \odot x^\mu_+ \\
\quad = \frac{\psi(-\mu - 1) - \psi(\mu + 2)}{\mu + 1} x^\mu+1_+ \\
\quad = \frac{\pi \cot \pi \mu}{\mu + 1} x^\mu+1_+ ,
\]

since it can be easily proved that \( \psi(-\mu - 1) - \psi(\mu + 2) = \pi \cot \pi \mu \). This proves equation (14).

Equation (15) follows on noting that the neutrix convolution products of \( \ln x_- \) and \( \ln x_+ \) with \( x^\mu_- \) are commutative.

Replacing \( x \) by \(-x\) in equations (14) and (15) gives us equations (16) and (17).

**Corollary 3.** The neutrix convolution products \( \ln |x| \odot |x| = x^\mu \) and \( |x|^\mu \odot \ln |x| \) exist and

\[
\ln |x| \odot |x|^\mu = \frac{\pi \cot \pi \mu}{\mu + 1} |x|^\mu = |x|^\mu \odot \ln |x| .
\]
for \( \mu \neq 0, \pm 1, \pm 2, \ldots \).

**Proof.** The equations follow from equations (14), (15), (16), and (17) on noting that \(|x|^\mu = x_+^\mu + x_-^\mu\).

**Theorem 10.** The neutrix convolution products \( x_+^{-r} \otimes x_+^\mu \) and \( x_-^{-r} \otimes x_+^\mu \) exist and

\[
x_+^{-r} \otimes x_+^\mu = \frac{(\mu)_{r-1}}{(r-1)!} \left\{ x_+^{\mu-r+1} \ln x_+ - [\gamma + \psi(-\mu + r - 1)] x_+^{\mu-r+1} \right\}
\]

(18)

\[
x_-^{-r} \otimes x_+^\mu = \frac{(\mu)_{r-1}}{(r-1)!} \left\{ x_-^{\mu-r+1} \ln x_- - [\gamma + \psi(-\mu + r - 1)] x_-^{\mu-r+1} \right\},
\]

(19)

for \( \mu \neq 0, \pm 1, \pm 2, \ldots \) and \( r = 1, 2, \ldots \).

**Proof.** We put \((x_+^{-r})_n = x_+^{-r} \tau_n(x)\) for \( r = 1, 2, \ldots \) so that \((\ln x_-)_n = -(x_-^{-1})_n + \ln x_- \tau_n(x)\). Then, if \( \phi \) is an arbitrary function in \( D \) with support contained in the interval \([a, b]\), we have from above

\[
((\ln x_-)_n * x_+^\mu, \phi(x)) = -((\ln)_n * x_+^\mu, \phi'(x))
\]

\[
= -((x_-^{-1})_n * x_+^\mu, \phi(x)) + \ln x_- \tau_n(x), \phi(x))
\]

and so

\[
((x_-^{-1})_n * x_+^\mu, \phi(x)) = ((\ln x_-)_n * x_+^\mu, \phi'(x)) + ((\ln x_- \tau_n(x)) * x_+^\mu, \phi(x)).
\]

The support of \( \ln x_- \tau_n(x) \) is contained in the interval \([-n - n^{-n}, -n]\) and so with \( n > -a \), it follows that

\[
((\ln x_- \tau_n(x)) * x_+^\mu, \phi(x)) = \int_a^b \left( \phi(x) \int_{-n - n^{-n}}^{-n} \ln(-y) \tau_n'(y)(x - y)^\mu \right) dy dx,
\]

where on the domain of integration \( \ln(-y) \) and \((x - y)^\mu\) are locally summable functions. Integrating by parts, it follows as above that

\[
N \lim_{n \to \infty} \int_a^b \phi(x) \int_{-n - n^{-n}}^{-n} \ln(-y) \tau_n'(y)(x - y)^\mu dy dx = 0.
\]

Thus

\[
N \lim_{n \to \infty} ((x_-^{-1})_n * x_+^\mu, \phi(x)) = N \lim_{n \to \infty} ((\ln x_-)_n * x_+^\mu, \phi'(x))
\]

\[
= (\ln x_- \otimes x_+^\mu, \phi'(x))
\]

by our assumption. This proves that the neutrix convolution product \( x_-^{-1} \otimes x_+^\mu \) exists and

\[
x_-^{-1} \otimes x_+^\mu = (\ln x_- \otimes x_+^\mu)' = x_+^\mu \ln x_+ - [\gamma + \psi(-\mu)] x_+^\mu
\]

as above for \( \mu \neq 0, \pm 1, \pm 2, \ldots \). Equation (18) is therefore proved for the case \( r = 1 \).
Now assume that equation (18) holds for some \( r \). The convolution product \((x_-^{-r})_n \ast x_+^\mu\) exists by Gel'fand and Shilov's definition and so equations (2) hold. Thus if \( \phi \) is an arbitrary function in \( D \)

\[
\langle (x_-^{-r})_n \ast x_+^\mu, \phi(x) \rangle = -\langle (x_-^{-r})_n \ast x_+^\mu, \phi'(x) \rangle
\]

\[
= r\langle (x_-^{-r-1})_n \ast x_+^\mu, \phi(x) \rangle + \langle [x_-^{-r} \tau'_n(x)] \ast x_+^\mu, \phi(x) \rangle
\]

and so

\[
r\langle (x_-^{-r-1})_n \ast x_+^\mu, \phi(x) \rangle = -\langle (x_-^{-r})_n \ast x_+^\mu, \phi'(x) \rangle - \langle [x_-^{-r} \tau'_n(x)] \ast x_+^\mu, \phi(x) \rangle.
\]

In follows as above that

\[
N\lim_{n \to \infty} \langle [x_-^{-r} \tau'_n(x)] \ast x_+^\mu, \phi(x) \rangle = 0
\]

and so

\[
N\lim_{n \to \infty} r\langle (x_-^{-r-1})_n \ast x_+^\mu, \phi(x) \rangle = -N\lim_{n \to \infty} \langle (x_-^{-r})_n \ast x_+^\mu, \phi'(x) \rangle
\]

\[
= -\langle x_-^{-r} \ast x_+^\mu, \phi'(x) \rangle
\]

by our assumption. Thus \( x_-^{-r-1} \ast x_+^\mu \) exists and

\[
x_-^{-r-1} \ast x_+^\mu = r^{-1}(x_-^{-r} \ast x_+^\mu)'
\]

\[
= (\mu)_{r-1} \frac{1}{r!} \left\{ (\mu - r + 1)x_+^\mu \ln x_+ + x_+^{\mu-r} - (\mu - r + 1)[\gamma + \psi(-\mu + r)] x_+^{\mu-r} \right\}
\]

\[
= (\mu)_r \frac{1}{r!} \left\{ x_+^{\mu-r} \ln x_+ - [\gamma + \psi(-\mu + r)] x_+^{\mu-r} \right\}.
\]

Equation (18) now follows by induction for \( \mu \neq 0, \pm 1, \pm 2, \ldots \) and \( r = 1, 2, \ldots \).

Using Theorem 2, it follows from equation (5) that

\[
(x_-^\mu \ast \ln x_+) = x_-^\mu \ast x_+^{-1} = x_-^\mu \ln x_- - [\gamma + \psi(-\mu)] x_-^\mu
\]

and equation (19) follows for the case \( r = 1 \).

Assuming equation (19) holds for some \( r \), it again follows from Theorem 2 that

\[
(x_-^\mu \ast x_+^{-r}) = -rx_-^\mu \ast x_+^{-r-1}
\]

\[
= (\mu)_{r-1} \frac{1}{(r-1)!} \left\{ -(\mu - r + 1)x_+^{\mu-r} \ln x_- - x_+^{\mu-r} + (\mu - r + 1)[\gamma + \psi(-\mu + r - 1)] x_+^{\mu-r} \right\}
\]

\[
= -\frac{(\mu)_r}{(r-1)!} \left\{ x_-^{\mu-r} \ln x_- - [\gamma + \psi(-\mu + r)] x_-^{\mu-r} \right\}.
\]

Equation (19) now follows by induction for \( \mu \neq 0, \pm 1, \pm 2, \ldots \) and \( r = 1, 2, \ldots \).
Corollary 1. The neutrix convolution products $x_+^{-r} \otimes x_-^\mu$ and $x_+^\mu \otimes x_-^{-r}$ exist and
\[
x_+^{-r} \otimes x_-^\mu = \left(\frac{\mu}{(r-1)!}\right) \{x_-^{\mu-r+1} \ln x_- - [\gamma + \psi(-\mu + r - 1)]x_-^{\mu-r+1}\},
\]
\[
x_+^\mu \otimes x_-^{-r} = \left(\frac{\mu}{(r-1)!}\right) \{x_+^{\mu-r+1} \ln x_+ - [\gamma + \psi(-\mu + r - 1)]x_+^{\mu-r+1}\},
\]
for $\mu \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$.

Proof. The results of the corollary follow immediately on replacing $x$ by $-x$ in equations (18) and (19).

Corollary 2. The neutrix convolution products $x_-^{-r} \otimes x_+^\mu$, $x_+^\mu \otimes x_-^{-r}$, $x_-^{-r} \otimes x_-^\mu$, and $x_-^\mu \otimes x_-^{-r}$ exist and
\[
x_-^{-r} \otimes x_+^\mu = \frac{(-1)^{r-1}(\mu - 1)\pi \cot \pi \mu}{(r-1)!} x_+^{\mu-r+1}
= x_+^\mu \otimes x_-^{-r}, \tag{20}
\]
\[
x_-^{-r} \otimes x_-^\mu = \frac{(-\mu - r - 1)\pi \cot \pi \mu}{(r-1)!} x_-^{\mu-r+1}
= x_-^\mu \otimes x_-^{-r} \tag{23}
\]
for $\mu \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$.

Proof. The convolution product $x_+^{-r} \ast x_+^\mu$ exists by Gel’fand and Shilov’s definition and it is easily proved that
\[
x_+^{-r} \ast x_+^\mu = \frac{(-1)^{r-1}(\mu - 1)}{(r-1)!} \{x_+^{\mu-r+1} \ln x_+ - [\gamma + \psi(\mu - r + 2)]x_+^{\mu-r+1}\}
= x_+^{-r} \otimes x_+^\mu.
\]
Since $x_- = x_-^{-r} + (-1)^rx_-^{-r}$ we have
\[
x_+^{-r} \otimes x_-^\mu + (-1)^r x_-^{-r} \otimes x_+^\mu = x_-^{-r} \otimes x_-^\mu
\]
\[
= \frac{(-1)^{r}(\mu - 1)[\psi(\mu - r + 2) - \psi(-\mu + r - 1)]x_+^{\mu-r+1}}{(r-1)!}
\]
\[
= \frac{(-1)^{r+1}(\mu - 1)\cot \pi \mu}{(r-1)!} x_+^{\mu-r+1}
\]
since $\psi(\mu - r + 2) - \psi(-\mu + r - 1) = - \cot \pi (\mu - r) = - \cot \pi \mu$. This proves equation (20).

Equation (21) follows on noting that the neutrix convolution products of $x_-^{-r}$ and $x_+^{-r}$ whith $x_+^\mu$ are commutative.

Replacing $x$ by $-x$ in equations (20) and (21) gives us equations (22) and (23).
Corollary 3. The neutrix convolution products \( x^{-r} \odot |x|^\mu \) and \(|x|^\mu \odot x^{-r}\) exist and
\[
x^{-r} \odot |x|^\mu = \begin{cases} 
-\frac{(\mu)_{r-1} \pi \cot \pi \mu}{(r-1)!} |x|^{|\mu-r|+1}, & \text{even } r \\
\frac{(\mu)_{r-1} \pi \cot \pi \mu}{(r-1)!} \text{sgn } x \cdot |x|^{|\mu-r|+1}, & \text{odd } r
\end{cases}
\]
for \( \mu \neq 0, \pm 1, \pm 2, \ldots \) and \( r = 1, 2, \ldots \).

Proof. The results follow from equations (20), (21), (22) and (23) on noting that \(|x|^\mu = x^\mu_+ + x^\mu_-\), \( \text{sgn } x \cdot |x|^\mu = x^\mu_+ - x^\mu_- \).

References

[4] B. Fisher, On the neutrix convolution product \( x^\mu \odot x_\nu^\lambda \), submitted for publication.
[6] B. Fisher, The neutrix convolution product \( x^\lambda \odot x_\mu^\nu \), submitted for publication.

Department of Mathematics
The University
Leicester, LE1 7RH, England

(Received 08 04 1990)