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THE NEUTRIX CONVOLUTION PRODUCT $x_-^{-r} \circledast x_+^\mu$

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Abstract. Let f and g be distributions in $\mathcal{D}'$ and let $f_n(x) = f(x)\tau_n(x)$, where $\tau_n(x)$ is a certain function which converges to the identity function as n tends to infinity. Then the neutrix convolution product $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided the limit h exists in the sense that $N\text{-}\lim_{n\to\infty}\langle f_n * g, \phi\rangle = \langle h, \phi\rangle$ for all ϕ in $\mathcal{D}$. The neutrix convolution products $\ln x_- \circledast x_+^\mu$, $x_-^\mu \circledast \ln x_+$, $x_-^{-r} \circledast x_+^\mu$ and $x_-^\mu \circledast x_+^{-r}$ for $\mu \neq 0, \pm 1, \pm 2, \dots$ and $r = 1, 2, \dots$ are evaluated, from which other neutrix convolution products are deduced.

In the following we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The convolution product $f * g$ of two distributions f and g in $\mathcal{D}'$ is then usually defined by the equation $\langle (f*g)(x), \phi\rangle = \langle f(y), \langle g(x), \phi(x+y)\rangle\rangle$ for arbitary δ in $\mathcal{D}$, provided f and g satisfy either of the conditions: (a) either f or g has bounded support; (b) the supports of f and g are bounded on the same side, (see Gel'fand and Shilov [8]).

It follows that, if the convolution product $f * g$ exists by this definition, then

$$f * g = g * f, \tag{1}$$

$$(f * g)' = f * g' = f' * g. \tag{2}$$

This definition of the convolution product is rather restrictive and so the neutrix convolution product was introduced in [2]. In order to define the neutrix convolution product we first of all let τ be a function in $\mathcal{D}$ satisfying the following properties:

(i) $\tau(x) = \tau(-x)$, (iii) $\tau(x) = 1$ for $|x| \leq 1/2$,

(ii) $0 \leq \tau(x) \leq 1$, (iv) $\tau(x) = 0$ for $|x| \geq 1$.

The function τ_n is now defined by

$$\tau_n(x) = \begin{cases} 1, & |x| \leq n, \\ \tau(n^n x - n^{n+1}), & x > n, \\ \tau(n^n x + n^{n+1}), & x < -n, \end{cases} \qquad \text{for } n = 1, 2, \dots.$$

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Definition 1. Let f and g be distributions in $\mathcal{D}'$ and let $f_n = f\tau_n$ for $n = 1, 2, \ldots$. Then the neutrix convolution product $f \circledast g$ is defined as the neutrix limit of the sequence $\{f_n * g\}$, provided that the limit h exists in the sense that $N\text{-}\lim_{n\to\infty}\langle f_n * g, \phi\rangle = \langle h, \phi\rangle$, for all ϕ in $\mathcal{D}$, where N is the neutrix, (see van der Corput [1]), having domain $N' = \{1, 2, \ldots, n, \ldots\}$ and range N'' the real numbers, with negligible functions finite linear sums of the functions

$$n^\lambda \ln^{r-1} n, \quad \ln^r n \qquad (\lambda > 0,\ r = 1, 2, \ldots)$$

and all the functions which converge to zero in the usual sense, as n tends to infinity.

Note that in this definition the convolution product $f_n * g$ is defined in Gel'fand and Shilov's sense, the distribution f_n having bounded support.

The following theorem was proved in [2], showing that the neutrix convolution product is a generalization of the convolution product.

THEOREM 1. *Let f and g be distributions in $\mathcal{D}'$ satisfying either condition* (a) *or condition* (b) *of Gel'fand and Shilov's definition. Then the neutrix convolution product $f \circledast g$ exists and $f \circledast g = f * g$.*

The next two theorems were also proved in [2].

THEOREM 2. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the neutrix convolution product $f \circledast g$ exists. Then the neutrix convolution product $f \circledast g'$ exists and $(f \circledast g)' = f \circledast g'$.*

Note however that equation (1) does not necessarily hold for the neutrix convolution product and that $(f \circledast g)'$ is not necessarily equal to $f' \circledast g$.

THEOREM 3. *The neutrix convolution product $x_-^\lambda \circledast x_+^s$ exists and*

$$x_-^\lambda \circledast x_+^s = (-1)^{s+1} B(\lambda + 1, s + 1) x_-^{\lambda+s+1} \tag{3}$$

for $\lambda > -1$ and $s = 0, 1, 2, \ldots$, where B denotes the Beta function.

Later, the following two theorems were proved in [3] and [4] respectively:

THEOREM 4. *The neutrix convolution product $x_-^\lambda \circledast x_+^s$ exists and satisfies equation* (3) *for $\lambda < -1$, $\lambda \neq -2, -3, \ldots$ and $s = 0, 1, 2, \ldots$.*

THEOREM 5. *The neutrix convolution product $x_-^s \circledast x_+^\lambda$ exists and*

$$x_-^s \circledast x_+^\lambda = (-1)^{s+1} B(\lambda + 1, s + 1) x_+^{\lambda+s+1}$$

for $\lambda \neq 0, \pm 1, \pm 2, \ldots$ and $s = 0, 1, 2, \ldots$.

The next two theorems were proved in [5].

THEOREM 6. *The neutrix convolution product $x_-^\lambda \circledast x_+^{s-\lambda}$ exists and*

$$x_-^\lambda \circledast x_+^{s-\lambda} = (-1)^{s+1} B(-s-1, s+1-\lambda) x^{s+1} + \frac{(-1)^{s+1}(\lambda)_{s+1}}{(s+1)!}\{\pi \cot(\pi\lambda) x_+^{s+1} - x^{s+1} \ln|x|\},$$

for $\lambda > -1$, $\lambda \neq 0, 1, 2, \ldots$ *and* $s = -1, 0, 1, 2, \ldots$, *where*

$$(\lambda)_s = \begin{cases} 1, & s = 0, \\ \prod_{i=0}^{s-1}(\lambda - i), & s \geq 1. \end{cases}$$

In this theorem, B again denotes the Beta function but is defined as in [7] by

$$B(\lambda, \mu) = \underset{n\to\infty}{N\text{-}\lim} \int_{1/n}^{1-1/n} t^{\lambda-1}(1-t)^{\mu-1}\, dt.$$

This definition is in agreement with the usual definition of $B(\lambda, \mu)$ when $\lambda, \mu \neq 0, -1, -2, \ldots$ but defines $B(\lambda, \mu)$ when λ or μ take the values $0, -1, -2, \ldots$.

THEOREM 7. *The neutrix convolution product* $x_-^\lambda \circledast x_+^{-s-\lambda}$ *exists and*

$$x_-^\lambda \circledast x_+^{-s-\lambda} = \frac{\pi \cot(\pi\lambda)}{(-1-\lambda)_{s-1}} \delta^{(s-2)}(x) - \frac{(-1)^s (s-2)!}{(-1-\lambda)_{s-1}} x^{-s+1},$$

for $\lambda > -1$, $\lambda \neq 0, 1, 2, \ldots$ *and* $s = 2, 3, \ldots$.

The next theorem was proved in [6].

THEOREM 8. *The neutrix convolution product* $x_-^\lambda \circledast x_+^\mu$ *exists and*

$$\begin{aligned} x_-^\lambda \circledast x_+^\mu = {} & B(-\lambda-\mu-1, \mu+1) x_-^{\lambda+\mu+1} \\ & + B(-\lambda-\mu-1, \lambda+1) x_+^{\lambda+\mu+1}, \end{aligned}$$

for $\lambda, \mu, \lambda + \mu \neq 0, \pm 1, \pm 2, \ldots$.

In the following we are going to consider the neutrix convolution products $x_-^{-r} \circledast x_+^\mu$ and $x_-^\mu \circledast x_+^{-r}$, where x_+^{-r} is defined by

$$x_+^{-r} = \frac{(-1)^{r-1}}{(r-1)!} (\ln x_+)^{(r)}$$

and x_-^{-r} is defined by $x_-^{-r} = (-x)_+^{-r}$, but first of all we prove

THEOREM 9. *The neutrix convolution products* $\ln x_- \circledast x_+^\mu$ *and* $x_-^\mu \circledast \ln x_+$ *exist and*

$$\ln x_- \circledast x_+^\mu = -\frac{x_+^{\mu+1}}{\mu+1} \ln x_+ + \frac{\gamma + \psi(-\mu-1)}{\mu+1} x_+^{\mu+1}, \tag{4}$$

$$x_- \circledast \ln x_+ = -\frac{x_-^{\mu+1}}{\mu+1} \ln x_- + \frac{\gamma + \psi(-\mu-1)}{\mu+1} x_-^{\mu+1} \tag{5}$$

for $\mu \neq 0, \pm 1, \pm 2, \ldots$, *where* γ *denotes Euler's constant,* $\psi = \Gamma'/\Gamma$ *and* Γ *denotes the Gamma function.*

Proof. We will suppose first of all that $\mu > -1$ and $\mu \neq 0, 1, 2, \ldots$ so that x_+^μ is a locally summable function. Put $(\ln x_-)_n = \ln x_- \tau_n(x)$. Then

$$\begin{aligned}
\langle (\ln x_-)_n * x_+^\mu, \phi(x)\rangle &= \langle (\ln y_-)_n, \langle x_+^\mu, \phi(x+y)\rangle\rangle \\
&= \int_{-n-n^{-n}}^0 \ln(-y)\tau_n(y) \int_a^b (x-y)_+^\mu \phi(x)\, dx dy \\
&= \int_a^b \phi(x) \int_{-n}^0 \ln(-y)(x-y)_+^\mu \, dy dx \\
&\quad + \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n(y)(x-y)^\mu \, dy dx \qquad (6)
\end{aligned}$$

for $n > -a$ and arbitary ϕ in $\mathcal{D}$ with support of ϕ contained in the interval $[a, b]$.

When $x < 0$, we have on making the substitution $y = xu^{-1}$

$$\begin{aligned}
\int_{-n}^0 \ln(-y)(x-y)_+^\mu \, dy &= \int_{-n}^x \ln(-y)(x-y)^\mu \, dy \\
&= (-x)^{\mu+1} \ln(-x) \int_{-x/n}^1 u^{-\mu-2}(1-u)^\mu \, du \\
&\quad - (-x)^{\mu+1} \int_{-x/n}^1 u^{-\mu-2} \ln u (1-u)^\mu \, du \\
&= I_{1n} - I_{2n}.
\end{aligned}$$

Choosing an integer $r > \mu + 1$, we have

$$\begin{aligned}
\int_{-x/n}^1 u^{-\mu-2}(1-u)^\mu \, du &= \int_{-x/n}^1 u^{-\mu-2}\Big[(1-u)^\mu - \sum_{i=0}^r \frac{(-1)^i(\mu)_i}{i!} u^i\Big] \\
&\quad + \sum_{i=0}^r \frac{(-1)^i(\mu)_i}{i!(i-\mu-1)}[1 - (x/n)^{i-\mu-1}]
\end{aligned}$$

and it follows that

$$N\text{-}\lim_{n\to\infty} I_{1n} = B(-\mu-1, \mu+1)(-x)^{\mu+1}\ln(-x) = 0,$$

where B denotes the Beta function, see [7] or [8]. Further,

$$\begin{aligned}
\int_{-x/n}^1 u^{-\mu-2} \ln u (1-u)^\mu \, du &= \int_{-x/n}^1 u^{-\mu-2} \ln u \Big[(1-u)^\mu - \sum_{i=0}^r \frac{(-1)^i(\mu)_i}{i!} u^i\Big] \, du \\
- \sum_{i=0}^r \frac{(-1)^i(\mu)_i}{i!(i-\mu-1)^2}&\left[(i-\mu-1)\left(-\frac{x}{n}\right)^{i-\mu-1} \ln\left(-\frac{x}{n}\right) + 1 - \left(-\frac{x}{n}\right)^{i-\mu-1}\right]
\end{aligned}$$

and it follows that $N\text{-}\lim_{n\to\infty} I_{2n} = B_{10}(-\mu-1, \mu+1)(-x)^{\mu+1}$, where

$$B_{10}(-\mu-1, \mu+1) = [\partial B(\lambda, \mu+1)/\partial\lambda]_{\lambda=\mu-1} = 0,$$

see [7]. Thus $N\text{-}\lim_{n\to\infty} I_{2n} = 0$ and so

$$N\text{-}\lim_{n\to\infty} \int_{-n}^{0} \ln(-y)(x-y)_+^\mu \, dy = 0. \tag{7}$$

When $x > 0$, we have on making the substitution $y = x(1-u^{-1})$

$$\begin{aligned}\int_{-n}^{0} \ln(-y)(x-y)_+^\mu \, dy &= \int_{-n}^{0} \ln(-y)(x-y)^\mu \, dy \\ &= x^{\mu+1}\ln x \int_{x/(x+n)}^{1} u^{-\mu-2}\, du + x^{\mu+1}\int_{x/(x+n)}^{1} u^{-\mu-2}\ln(1-u)\, du \\ &\quad - x^{\mu+1}\int_{x/(x+n)}^{1} u^{-\mu-2}\ln u\, du \\ &= I_{3n} + I_{4n} - I_{5n}.\end{aligned}$$

We have

$$x^{\mu+1}\ln x \int_{x/(x+n)}^{1} u^{-\mu-2}\, du = -\frac{x^{\mu+1}\ln x}{\mu+1} + \frac{n^{\mu+1}}{\mu+1}\left(1+\frac{x}{n}\right)^{\mu+1}\ln x$$

and it follows that

$$N\text{-}\lim_{n\to\infty} I_{3n} = -\frac{x^{\mu+1}\ln x}{\mu+1}.$$

Making the substitution $u = 1 - v$ we have

$$\begin{aligned}\int_{x/(x+n)}^{1} u^{-\mu-2}\ln(1-u)\, du &= \int_{0}^{n/(x+n)} \ln v(1-v)^{-\mu-2}\, dv \\ &= \int_{0}^{n/(x+n)} \ln v\left[(1-v)^{-\mu-2} - \sum_{i=0}^{r}\frac{(-1)^i(\mu+2)_i}{i!}v^i\right] dv \\ &\quad + \sum_{i=0}^{r}\frac{(-1)^i(\mu+2)_i}{i!}\left[\frac{(1+x/n)^{-i-1}\ln(1+x/n)}{i+1} + \frac{(1+x/n)^{-i-1}}{(i+1)^2}\right]\end{aligned}$$

and it follows that

$$\begin{aligned}N\text{-}\lim_{n\to\infty} \int_{x/(x+n)}^{1} & u^{-\mu-2}\ln(1-u)\, du \\ &= \int_{0}^{1} \ln v\left[(1-v)^{-\mu-2} - \sum_{i=0}^{r}\frac{(-1)^i(\mu+2)_i}{i!}v^i\right] dv \\ &\quad - \sum_{i=0}^{r}\frac{(-1)^i(\mu+2)_i}{i!(i+2)^2}\left(1+\frac{x}{n}\right)^{-i-1} = B_{10}(1,-\mu-1).\end{aligned}$$

Thus $N\text{-}\lim_{n\to\infty} I_{4n} = B_{10}(1,-\mu-1)x^{\mu+1}$.

Next, we have

$$\int_{x/(x+n)}^{1} u^{-\mu-2}\ln u\,du = \frac{(x+n)^{\mu+1}[\ln x + \ln(x+n)]}{(\mu+1)x^{\mu+1}} - \frac{1}{(\mu+1)^2} + \frac{(x+n)^{\mu+1}}{(\mu+1)^2 x^{\mu+1}}$$

and it follows that $N\text{-}\lim_{n\to\infty} I_{5n} = -(\mu+1)^{-2}x^{\mu+1}$. Now it is easily proved that

$$B_{10}(1,\mu) = \frac{-\gamma-\psi(1+\mu)}{\mu}, \qquad \mu^{-1}+\psi(u) = \psi(\mu+1)$$

and so

$$B_{10}(1,-\mu-1) + (\mu+1)^{-2} = \frac{\gamma+\psi(-\mu-1)}{\mu+1}.$$

Thus

$$\underset{n\to\infty}{N\text{-}\lim} \int_{-n}^{0} \ln(-y)(x-y)_+^{\mu}\,dy = -\frac{x^{\mu+1}\ln x}{\mu+1} + \frac{\gamma+\psi(-\mu-1)}{\mu+1}x^{\mu+1}. \tag{8}$$

Further, with $a \le x \le b$ and $n > -a$, we have

$$\left|\int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n(y)(x-y)^{\mu}\,dy\right| = o(n^{\mu-n}\ln n)$$

and so

$$\lim_{n\to\infty} \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n(y)(x-y)^{\mu}\,dy = 0. \tag{9}$$

It now follows from equations (6), (7), (8) and (9) that

$$\begin{aligned}&\underset{n\to\infty}{N\text{-}\lim}\langle(\ln x_-)_n * x_+^{\mu},\phi(x)\rangle \\ &\quad = \langle -(\mu+1)^{-1}x_+^{\mu+1}\ln x_+ + [\gamma+\psi(-\mu-1)](\mu+1)^{-1}x_+^{\mu+1},\phi(x)\rangle\end{aligned}$$

and equation (4) follows for $\mu > -1$ and $\mu \ne 0,1,2,\ldots$.

Now assume that equation (4) holds for $-k < \mu < -k+1$, where k is some positive integer. This is certainly true when $k=1$. Then by Theorem 1, the neutrix convolution product $\ln x_- \circledast x_+^{\mu-1}$ exists and

$$\begin{aligned}\mu \ln x_- \circledast x_+^{\mu-1} &= -x_+^{\mu}\ln x_+ - (\mu+1)^{-1}x_+^{\mu} + [\gamma+\psi(-\mu-1)]x_+^{\mu} \\ &= -x_+^{\mu}\ln x_+ + [\gamma+\psi(-\mu)]x_+^{\mu},\end{aligned}$$

since $\psi(-\mu-1) - (\mu+1)^{-1} = \psi(-\mu)$. Equation (4) follows by induction for $\mu \ne 0,\pm1,\pm2,\ldots$.

To prove equation (5) we will again suppose first of all that $\mu > -1$ and $\mu \ne 0,1,2,\ldots$, so that x_-^{μ} is a locally summable function. Put $(x_-^{\mu})_n = x_-\tau_n(x)$. Then

$$\langle (x_-^{\mu})_n * \ln x_+,\phi(x)\rangle = \langle (y_+^{\mu})_n, \langle \ln x_+,\phi(x+y)\rangle\rangle$$

$$= \int_{-n-n^{-n}}^{0} (-y)^{\mu} \tau_n(x) \int_a^b \ln(x-y)_+ \phi(x)\, dx dy$$
$$= \int_a^b \phi(x) \int_{-n}^{0} (-y)^{\mu} \ln(x-y)_+ \, dy dx$$
$$+ \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^{\mu} \tau_n(y) \ln(x-y)\, dy dx \tag{10}$$

for $n > -a$ and arbitrary ϕ in $\mathcal{D}$ with support of ϕ contained in the interval $[a, b]$.

When $x < 0$, we have on making the substitution $y = xu^{-1}$

$$\int_{-n}^{0} (-y)^{\mu} \ln(x-y)_+ \, dy = \int_{-n}^{x} (-y)^{\mu} \ln(x-y)\, dy$$
$$= -(-x)^{\mu+1} \int_{-x/n}^{1} u^{-\mu-2} \ln u \, du$$
$$+ (-x)^{\mu+1} \ln(-x) \int_{-x/n}^{1} u^{-\mu-2}\, du$$
$$+ (-x)^{\mu+1} \int_{-x/n}^{1} u^{-\mu-2} \ln(1-u)\, du$$
$$= -J_{1n} + J_{2n} + J_{3n}.$$

We have

$$\int_{-x/n}^{1} u^{-\mu-2} \ln u \, du = \frac{(-n/x)^{\mu+1}}{\mu+1} - \frac{1-(-n/x)^{\mu+1}}{(\mu+1)^2}$$

and it follows that $N\text{-}\lim_{n\to\infty} J_{1n} = -(\mu+1)^{-2}(-x)^{\mu+1}$.

Next we have

$$\int_{-x/n}^{1} u^{-\mu-2}\, du = -\frac{1-(-n/x)^{\mu+1}}{\mu+1}$$

and it follows that

$$N\text{-}\lim_{n\to\infty} J_{2n} = -\frac{(-x)^{\mu+1}\ln(-x)}{\mu+1}.$$

Making the substitution $u = 1 - v$ we have

$$\int_{-x/n}^{1} u^{-\mu-2} \ln(1-u)\, du = \int_0^{(x+n)/n} \ln v (1-v)^{-\mu-2}\, dv$$
$$= \int_0^{(x+n)/n} \ln v \left[(1-v)^{-\mu-2} - \sum_{i=0}^{r} \frac{(-1)^i(\mu+2)_i}{i!} v_i \right] dv$$
$$+ \sum_{i=0}^{r} \frac{(-1)^i(\mu+2)_i}{i!} \left[\frac{(1+x/n)^{i+1}\ln(1+x/n)}{i+1} - \frac{(1+x/n)^{i+1}}{(i+1)^2} \right]$$

and it follows as above that $N\text{-}\lim_{n\to\infty} J_{3n} = B_{10}(1, -\mu - 1)(-x)^{\mu+1}$. Thus

$$\begin{aligned} &\underset{n\to\infty}{N\text{-}\lim} \int_{-n}^{0} (-y)^\mu \ln(x-y)_+ \, dy \\ &= -\frac{(-x)^{\mu+1}\ln(-x)}{\mu+1} + B_{10}(1,-\mu-1)(-x)^{\mu+1} + \frac{(-1)^{\mu+1}}{(\mu+1)^2} \\ &= -\frac{(-x)^{\mu+1}\ln(-x)}{\mu+1} + \frac{\gamma+\psi(-\mu-1)}{\mu+1}(-x)^{\mu+1} \end{aligned} \tag{11}$$

as above.

When $x > 0$, we have on making the substitution $y = x(1-u^{-1})$

$$\begin{aligned} &\int_{-n}^{0} (-y)^\mu \ln(x-y)_+ \, dy = \int_{-n}^{0} (-y)^\mu \ln(x-y)\, dy \\ &= x^{\mu+1}\ln x \int_{x/(x+n)}^{1} u^{-\mu-2}(1-u)^\mu \, du - x^{\mu+1}\int_{x/(x+n)}^{1} u^{-\mu-2}\ln u(1-u)^\mu \, du \\ &= J_{4n} - J_{5n}. \end{aligned}$$

It follows similarly to above that $N\text{-}\lim_{n\to\infty} J_{4n} = B(-\mu-1, \mu+1)x^{\mu+1}\ln x = 0$ and $N\text{-}\lim_{n\to\infty} J_{5n} = B_{10}(-\mu-1, \mu+1)[x/(x+n)]^{\mu+1} = 0$. Thus

$$\underset{n\to\infty}{N\text{-}\lim} \int_{-n}^{0} (-y)^\mu \ln(x-y)_+ \, dy = 0. \tag{12}$$

Further, with $a \le x \le b$, and $n > -a$, we have

$$\left| \int_{-n-n^{-n}}^{-n} (-y)^\mu \tau_n(y) \ln(x-y)\, dy \right| = o(n^{\mu-n}\ln n)$$

and so

$$\lim_{n\to\infty} \int_{-n-n^{-n}}^{-n} (-y)^\mu \tau_n(y)\ln(x-y)\, dy = 0. \tag{13}$$

It now follows from equations (10), (11), (12) and (13) that

$$\begin{aligned} &\underset{n\to\infty}{N\text{-}\lim} \langle (x_-^\mu)_n * \ln x_+, \phi(x)\rangle \\ &= \langle -(\mu+1)^{-1}x_-^{\mu+1}\ln x_- + [\gamma+\psi(-\mu-1)](\mu+1)^{-1}x_-^{\mu+1}, \phi(x)\rangle \end{aligned}$$

and equation (5) follows for $\mu > -1$ and $\mu \ne 0, 1, 2, \ldots$.

Finally assume that equation (5) holds for $-k < \mu < -k+1$. This is certainly true when $k = 1$. The convolution product $(x_-^\mu)_n * \ln x_+$ exists by Gel'fand and Shilov's definition and so equations (2) hold. Thus if ϕ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[a, b]$

$$\begin{aligned} \langle [(x_-^\mu)_n * \ln x_+]', \phi(x)\rangle &= -\langle (x_-^\mu)_n * \ln x_+, \phi'(x)\rangle \\ &= -\mu\langle (x_-^{\mu-1})_n * \ln x_+ \phi(x)\rangle + \langle [x_-^\mu \tau_n'(x)] * \ln x_+, \phi(x)\rangle \end{aligned}$$

and so

$$\mu\langle (x_-^{\mu-1})_n * \ln x_+, \phi(x)\rangle = \langle (x_-^\mu)_n * \ln x_+, \phi'(x)\rangle + \langle [x_-^\mu \tau_n'(x)] * \ln x_+, \phi(x)\rangle.$$

The support of $x_-^\mu \tau_n'(x)$ is contained in the interval $[-n - n^{-n}, -n]$ and so with $n > -a$, it follows as above that

$$\langle [x_-^\mu \tau_n'(x)] * \ln x_+, \phi(x)\rangle = \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} (-y)^\mu \tau_n'(y) \ln(x-y)\, dydx,$$

where on the domain of integration, $(-y)^\mu$ and $\ln(x-y)$ are locally summable functions. Integrating by parts, it follows that

$$\begin{aligned}&\int_{-n-n^{-n}}^{-n} (-y)^\mu \tau_n'(y) \ln(x-y)\, dy \\ &\quad = n^\mu \ln(x+n) + \int_{-n-n^{-n}}^{-n} [\mu(-y)^{\mu-1}\ln(x-y) + (-y)^\mu (x-y)^{-1}]\tau_n(y)\, dy.\end{aligned}$$

Choosing a positive integer r greater than μ, we see that

$$n^\mu \ln(x+n) = n^\mu \ln n + n^\mu \sum_{i=1}^{r} \frac{x^i}{in^i} + o(1/n)$$

and so since μ is not an integer, $N\text{–}\lim_{n\to\infty} n^\mu \ln(x+n) = 0$. Further, it follows as in the proof of equation (9) that

$$\lim_{n\to\infty} \int_{-n-n^{-n}}^{-n} [\mu(-y)^{\mu-1}\ln(x-y) + (-y)^\mu (x-y)^{-1}]\tau_n(y)\, dy = 0.$$

Thus

$$\begin{aligned}\underset{n\to\infty}{N\text{–}\lim}\, \mu\langle (x_-^{\mu-1})_n * \ln x_+, \phi(x)\rangle &= \underset{n\to\infty}{N\text{–}\lim}\langle (x_-^\mu)_n * \ln x_+, \phi'(x)\rangle \\ &= \langle x_-^\mu \circledast \ln x_+, \phi'(x)\rangle\end{aligned}$$

by our assumption. This proves that the neutrix convolution product $x_-^{\mu-1} \circledast \ln x_+$ exists and

$$\begin{aligned}\mu x_-^{\mu-1} \circledast \ln x_+ &= -(x_-^\mu \circledast \ln x_+)' \\ &= -x_-^\mu \ln x_- - (\mu+1)^{-1} x_-^\mu + [\gamma + \psi(-\mu-1)]x_-^\mu \\ &= -x_-^\mu \ln x_- + [\gamma + \psi(-\mu)]x_-^\mu,\end{aligned}$$

as above. Equation (5) now follows by induction for $\mu \neq 0, \pm 1, \pm 2, \ldots$. This completes the proof of the theorem.

COROLLARY 1. *The neutrix convolution products* $\ln x_+ \circledast x_-^\mu$ *and* $x_+^\mu \circledast \ln x_-$ *exist and*

$$\ln x_+ \circledast x_-^\mu = -\frac{1}{\mu+1} x_-^{\mu+1} \ln x_- + \frac{\gamma + \psi(-\mu-1)}{\mu+1} x_-^{\mu+1}$$

$$x_+^\mu \circledast \ln x_- = -\frac{1}{\mu+1}x_+^{\mu+1} + \frac{\gamma+\psi(-\mu-1)}{\mu+1}x_+^{\mu+1},$$

for $\mu \neq 0, \pm 1, \pm 2, \ldots$.

Proof. The results of the corollary follow immediately on replacing x by $-x$ in equations (4) and (5).

COROLLARY 2. *The neutrix convolution products* $\ln|x| \circledast x_+^\mu$, $x_+^\mu \circledast \ln|x|$, $\ln|x| \circledast x_-^\mu$ *and* $x_-^\mu \circledast \ln|x|$ *exist and*

$$\ln|x| \circledast x_+^\mu = \frac{\pi\cot\pi\mu}{\mu+1}x_+^{\mu+1}, \tag{14}$$

$$= x_+^\mu \circledast \ln|x|, \tag{15}$$

$$\ln|x| \circledast x_-^\mu = \frac{\pi\cot\pi\mu}{\mu+1}x_-^{\mu+1}, \tag{16}$$

$$= x_-^\mu \circledast \ln|x|, \tag{17}$$

for $\mu \neq 0, \pm 1, \pm 2, \ldots$.

Proof. The convolution product $\ln x_+ * x_+^\mu$ exists by Gel'fand and Shilov's definition and it is easily proved that

$$\begin{aligned}\ln x_+ * x_+^\mu &= (\mu+1)^{-1}x_+^{\mu+1}\ln x_+ + B_{10}(1,\mu+1)x_+^{\mu+1}\\ &= (\mu+1)^{-1}x_+^{\mu+1}\ln x_+ - \frac{\gamma+\psi(\mu+2)}{\mu+1}x_+^{\mu+1}\\ &= \ln x_+ \circledast x_+^\mu.\end{aligned}$$

Since the neutrix convolution product is clearly distributive with respect to addition, it follows that

$$\begin{aligned}\ln x_- \circledast x_+^\mu + \ln x_+ \circledast x_+^\mu &= \ln|x| \circledast x_+^\mu\\ &= \frac{\psi(-\mu-1)-\psi(\mu+2)}{\mu+1}x_+^{\mu+1}\\ &= \frac{\pi\cot\pi\mu}{\mu+1}x_+^{\mu+1},\end{aligned}$$

since it can be easily proved that $\psi(-\mu-1)-\psi(\mu+2) = \pi\cot\pi\mu$. This proves equation (14).

Equation (15) follows on noting that the neutrix convolution products of $\ln x_-$ and $\ln x_+$ with x_+^μ are commutative.

Replacing x by $-x$ in equations (14) and (15) gives us equations (16) and (17).

COROLLARY 3. *The neutrix convolution products* $\ln|x| \circledast |x|^\mu$ *and* $|x|^\mu \circledast \ln|x|$ *exist and*

$$\ln|x| \circledast |x|^\mu = \frac{\pi\cot\pi\mu}{\mu+1}|x|^\mu = |x|^\mu \circledast \ln|x|$$

for $\mu \neq 0, \pm 1, \pm 2, \ldots$.

Proof. The equations follow from equations (14), (15), (16), and (17) on noting that $|x|^\mu = x_+^\mu + x_-^\mu$.

THEOREM 10. *The neutrix convolution products* $x_-^{-r} \circledast x_+^\mu$ *and* $x_-^\mu \circledast x_+^{-r}$ *exist and*

$$x_-^{-r} \circledast x_+^\mu = \frac{(\mu)_{r-1}}{(r-1)!}\{x_+^{\mu-r+1}\ln x_+ - [\gamma + \psi(-\mu + r - 1)]x_+^{\mu-r+1}\} \tag{18}$$

$$x_-^\mu \circledast x_+^{-r} = \frac{(\mu)_{r-1}}{(r-1)!}\{x_-^{\mu-r+1}\ln x_- - [\gamma + \psi(-\mu + r - 1)]x_-^{\mu-r+1}\}, \tag{19}$$

for $\mu \neq 0, \pm 1, \pm 2, \ldots$ *and* $r = 1, 2, \ldots$.

Proof. We put $(x_-^{-r})_n = x_-^{-r}\tau_n(x)$ for $r = 1, 2, \ldots$ so that $(\ln x_-)_n' = -(x_-^{-1})_n + \ln x_-\tau_n'(x)$. Then, if ϕ is an arbitrary function in $\mathcal{D}$ with support contained in the interval $[a, b]$, we have from above

$$\begin{aligned}\langle[(\ln x_-)_n * x_+^\mu]', \phi(x)\rangle &= -\langle(\ln)_n * x_+^\mu, \phi'(x)\rangle \\ &= -\langle(x_-^{-1})_n * x_+^\mu, \phi(x)\rangle + \langle\ln x_-\tau_n'(x), \phi(x)\rangle\end{aligned}$$

and so

$$\langle(x_-^{-1})_n * x_+^\mu, \phi(x)\rangle = \langle(\ln x_-)_n * x_+^\mu, \phi'(x)\rangle + \langle[\ln x_-\tau_n'(x)] * x_+^\mu, \phi(x)\rangle.$$

The support of $\ln x_-\tau_n'(x)$ is contained in the interval $[-n - n^{-n}, -n]$ and so with $n > -a$, it follows that

$$\langle[\ln x_-\tau_n'(x)] * x_+^\mu, \phi(x)\rangle = \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^\mu \, dy dx,$$

where on the domain of integration $\ln(-y)$ and $(x - y)^\mu$ are locally summable functions. Integrating by parts, it follows as above that

$$\underset{n\to\infty}{N\text{-}\lim} \int_a^b \phi(x) \int_{-n-n^{-n}}^{-n} \ln(-y)\tau_n'(y)(x-y)^\mu \, dy dx = 0.$$

Thus

$$\begin{aligned}\underset{n\to\infty}{N\text{-}\lim}\langle(x_-^{-1})_n * x_+^\mu, \phi(x)\rangle &= \underset{n\to\infty}{N\text{-}\lim}\langle(\ln x_-)_n * x_+^\mu, \phi'(x)\rangle \\ &= \langle\ln x_- \circledast x_+^\mu, \phi'(x)\rangle\end{aligned}$$

by our assumption. This proves that the neutrix convolution product $x_-^{-1} \circledast x_+^\mu$ exists and

$$x_-^{-1} \circledast x_+^\mu = (\ln x_- \circledast x_+^\mu)' = x_+^\mu \ln x_+ - [\gamma + \psi(-\mu)]x_+^\mu$$

as above for $\mu \neq 0, \pm 1, \pm 2, \ldots$. Equation (18) is therefore proved for the case $r = 1$.

Now assume that equation (18) holds for some r. The convolution product $(x_-^{-r})_n * x_+^\mu$ exists by Gel'fand and Shilov's definition and so equations (2) hold. Thus if ϕ is an arbitrary function in $\mathcal{D}$

$$\begin{aligned}\langle[(x_-^{-r})_n * x_+^\mu]', \phi(x)\rangle &= -\langle(x_-^{-r})_n * x_+^\mu, \phi'(x)\rangle \\ &= r\langle(x_-^{-r-1})_n * x_+^\mu, \phi(x)\rangle + \langle[x_-^{-r}\tau_n'(x)] * x_+^\mu, \phi(x)\rangle\end{aligned}$$

and so

$$r\langle(x_-^{-r-1})_n * x_+^\mu, \phi(x)\rangle = -\langle(x_-^{-r})_n * x_+^\mu, \phi'(x)\rangle - \langle[x_-^{-r}\tau_n'(x)] * x_+^\mu, \phi(x)\rangle.$$

In follows as above that

$$\operatorname*{N-lim}_{n\to\infty}\langle[x_-^{-r}\tau_n'(x)] * x_+^\mu, \phi(x)\rangle = 0$$

and so

$$\begin{aligned}\operatorname*{N-lim}_{n\to\infty} r\langle(x_-^{-r-1})_n * x_+^\mu, \phi(x)\rangle &= -\operatorname*{N-lim}_{n\to\infty}\langle(x_-^{-r})_n * x_+^\mu, \phi'(x)\rangle \\ &= -\langle x_-^{-r} * x_+^\mu, \phi'(x)\rangle\end{aligned}$$

by our assumption. Thus $x_-^{-r-1} \circledast x_+^\mu$ exists and

$$\begin{aligned}x_-^{-r-1} \circledast x_+^\mu &= r^{-1}(x_-^{-r} * x_+^\mu)' \\ &= \frac{(\mu)_{r-1}}{r!}\{(\mu - r + 1)x_+^{\mu-r}\ln x_+ + x_+^{\mu-r} \\ &\quad -(\mu - r + 1)[\gamma + \psi(-\mu + r - 1)x_+^{\mu-r}]\} \\ &= \frac{(\mu)_r}{r!}\{x_+^{\mu-r}\ln x_+ - [\gamma + \psi(-\mu + r)]x_+^{\mu-r}\}.\end{aligned}$$

Equation (18) now follows by induction for $\mu \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$.

Using Theorem 2, it follows from equation (5) that

$$(x_-^\mu \circledast \ln x_+)' = x_-^\mu \circledast x_+^{-1} = x_-^\mu \ln x_- - [\gamma + \psi(-\mu)]x_-^\mu$$

and equation (19) follows for the case $r = 1$.

Assuming equation (19) holds for some r, it again follows from Theorem 2 that

$$\begin{aligned}(x_-^\mu \circledast x_+^{-r})' &= -rx_-^\mu \circledast x_+^{-r-1} \\ &= \frac{(\mu)_{r-1}}{(r-1)!}\{-(\mu - r + 1)x_-^{\mu-r}\ln x_- - x_-^{\mu-r} \\ &\quad +(\mu - r + 1)[\gamma + \psi(-\mu + r - 1)]x_-^{\mu-r}\} \\ &= -\frac{(\mu)_r}{(r-1)!}\{x_-^{\mu-r}\ln x_- - [\gamma + \psi(-\mu + r)]x_-^{\mu-r}\}.\end{aligned}$$

Equation (19) now follows by induction for $\mu \neq 0, \pm 1, \pm 2, \ldots$ and $r = 1, 2, \ldots$.

COROLLARY 1. *The neutrix convolution products* $x_+^{-r} \circledast x_-^{\mu}$ *and* $x_+^{\mu} \circledast x_-^{-r}$ *exist and*

$$x_+^{-r} \circledast x_-^{\mu} = \frac{(\mu)_{r-1}}{(r-1)!}\{x_-^{\mu-r+1}\ln x_- - [\gamma + \psi(-\mu + r - 1)]x_-^{\mu-r+1}\},$$

$$x_+^{\mu} \circledast x_-^{-r} = \frac{(\mu)_{r-1}}{(r-1)!}\{x_+^{\mu-r+1}\ln x_+ - [\gamma + \psi(-\mu + r - 1)]x_+^{\mu-r+1}\},$$

for $\mu \neq 0, \pm1, \pm2, \ldots$ *and* $r = 1, 2, \ldots$.

Proof. The results of the corollary follow immediately on replacing x by $-x$ in equations (18) and (19).

COROLLARY 2. *The neutrix convolution products* $x^{-r} \circledast x_+^{\mu}$, $x_+^{\mu} \circledast x^{-r}$, $x^{-r} \circledast x_-^{\mu}$ *and* $x_-^{\mu} \circledast x^{-r}$ *exist and*

$$x^{-r} \circledast x_+^{\mu} = \frac{(-1)^{r-1}(\mu)_{r-1}\pi\cot\pi\mu}{(r-1)!}x_+^{\mu-r+1} \tag{20}$$

$$= x_+^{\mu} \circledast x^{-r}, \tag{21}$$

$$x^{-r} \circledast x_-^{\mu} = \frac{-(\mu)_{r-1}\pi\cot\pi\mu}{(r-1)!}x_-^{\mu-r+1} \tag{22}$$

$$= x_-^{\mu} \circledast x^{-r} \tag{23}$$

for $\mu \neq 0, \pm1, \pm2, \ldots$ *and* $r = 1, 2, \ldots$.

Proof. The convolution product $x_+^{-r} * x_+^{\mu}$ exists by Gel'fand and Shilov's definition and it is easily proved that

$$x_+^{-r} * x_+^{\mu} = \frac{(-1)^{r-1}(\mu)_{r-1}}{(r-1)!}\{x_+^{\mu-r+1}\ln x_+ - [\gamma + \psi(\mu - r + 2)]x_+^{\mu-r+1}\}$$
$$= x_+^{-r} \circledast x_+^{\mu}.$$

Since $x^{-r} = x_+^{-r} + (-1)^r x_-^{-r}$ we have

$$x_+^{-r} \circledast x_+^{\mu} + (-1)^r x_-^{-r} \circledast x_+^{\mu} = x^{-r} \circledast x_+^{\mu}$$
$$= \frac{(-1)^r(\mu)_{r-1}}{(r-1)!}[\psi(\mu - r + 2) - \psi(-\mu + r - 1)]x_+^{\mu-r+1}$$
$$= \frac{(-1)^{r-1}(\mu)_{r-1}\cot\pi\mu}{(r-1)!}x_+^{\mu-r+1}$$

since $\psi(\mu-r+2)-\psi(-\mu+r-1) = -\cot\pi(\mu-r) = -\cot\pi\mu$. This proves equation (20).

Equation (21) folows on noting that the neutrix convolution products of x_-^{-r} and x_+^{-r} whith x_+^{μ} are commutative.

Replacing x by $-x$ in equations (20) and (21) gives us equations (22) and (23).

COROLLARY 3. *The neutrix convolution products $x^{-r} \circledast |x|^\mu$ and $|x|^\mu \circledast x^{-r}$ exist and*

$$x^{-r} \circledast |x|^\mu = \begin{cases} -\dfrac{(\mu)_{r-1}\pi\cot\pi\mu}{(r-1)!}|x|^{\mu-r+1}, & \text{even } r \\ \dfrac{(\mu)_{r-1}\pi\cot\pi\mu}{(r-1)!}\operatorname{sgn} x\cdot|x|^{\mu-r+1}, & \text{odd } r \end{cases}$$

for $\mu \neq 0, \pm1, \pm2, \ldots$ and $r = 1, 2, \ldots$.

Proof. The results follow from equations (20), (21), (22) and (23) on noting that $|x|^\mu = x_+^\mu + x_-^\mu$, $\operatorname{sgn} x \cdot |x|^\mu = x_+^\mu - x_-^\mu$.

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