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REMARKS ON ABSOLUTELY REGULAR AND REGULAR TEMPERED DISTRIBUTIONS

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Abstract. Several characterizations of the space of absolutely regular tempered distributions are given. Also, some relations between this space and the space of regular tempered distributions are considered.

1. Spaces of absolutely regular tempered distributions $\mathcal{S}'_{\rm ar}$ and regular tempered distrubtions $\mathcal{S}'_{\rm r}$ are defined as follows:

$$\mathcal{S}'_{\rm ar} = \left\{ f \in L^1_{\rm loc}(\mathbf{R}^q) : \text{for every } \varphi \in \mathcal{S}(\mathbf{R}^q),\ f\varphi \in L^1(\mathbf{R}^q) \right\} \quad [5]$$

$$\mathcal{S}'_{\rm r} = \mathcal{S}'(\mathbf{R}^q) \cap L^1_{\rm loc}(\mathbf{R}^q) \quad [2].$$

Szmydt [5, 6], has given several characterizations of absolutely regular tempered distributions. The comprehensive theory of spaces $\mathcal{S}'_{\rm ar}$ and $\mathcal{S}'_{\rm r}$ specially of their topological structures have been given by Dierolf [2].

If $f \in \mathcal{S}'_{\rm ar}$ then by the formulae $\varphi \to f\varphi$ a linear transformation from $\mathcal{S}$ to L^1 is defined. By the closed graph theorem we obtain that this transformation is continuous. This implies the following lemma.

LEMMA. *We have* $\mathcal{S}'_{\rm ar} \subset \mathcal{S}'_{\rm r}$.

2. Let us recall that $f \in L^1_{\rm loc}$ is a slowly increasing function if for some $k \in \mathbf{N}^q$, we have $f(x)(1+x^2)^{-k} \in L^\infty$, where $x^2 = (x,x) = x_1^2 + \cdots + x_q^2$.

THEOREM. *Let* $T \in L^1_{\rm loc}$. *Then the following conditions are equivalent*:

(a) T *is an absolutely regular distribution.*

(b) *There exists* $k \in \mathbf{N}^q$ *such that* $(1+x^2)^{-k}T(x) \in L^1$.

(c) *The function* $F(x) = \int_0^x |T(t)|\,dt$, $x \in \mathbf{R}^q$, *is a slowly increasing function.*

(d) $|T|$ *is a tempered distribution.*

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Proof. (a) $\Longrightarrow$ (b). This is proved by Szmydt [6].

(b) $\Longrightarrow$ (c). This follows from the inequality

$$(1+x^2)^{-k}\left|\int_0^x |T(t)|\,dt\right| \le \left|\int_0^x (1+t^2)^{-k}|T(t)|\,dt\right| \le \int_{\mathbf{R}^q}(1+t^2)^{-k}|T(t)|\,dt, \qquad x\in\mathbf{R}^q.$$

(c) $\Longrightarrow$ (d). Since $F\in\mathcal{S}'$ and $\dfrac{\partial^q}{\partial x_1\ldots\partial x_q}F=|T|$, we have $|T|\in\mathcal{S}'$, as well.

(d) $\Longrightarrow$ (a). Note that the mapping $\varphi(x)\to\check{\varphi}(x)=\varphi(-x)$ is a surjection of $\mathcal{S}$ onto $\mathcal{S}$. Take $\varphi\in\mathcal{S}$ which is a non-negative function. There exists a sequence φ_n from $\mathcal{D}$ such that $\varphi_n\ge 0$, $n\in\mathbf{N}$, and $\varphi_n\to\varphi$, in $\mathcal{S}$. Since $|T|\in\mathcal{S}'$,

$$\lim_{n\to\infty}\int_{\mathbf{R}^q}|T(t)|\check{\varphi}_n(t)\,dt=\langle|T(t)|,\check{\varphi}(t)\rangle<\infty.$$

For almost all $t\in\mathbf{R}^q$, $|T(t)|\check{\varphi}_n(t)\to|T(t)|\check{\varphi}(t)$. By Fatou's lemma we have

$$\int_{\mathbf{R}^q}|T(t)|\check{\varphi}(t)\,dt\le\lim_{n\to\infty}\int_{\mathbf{R}^q}|T(t)|\check{\varphi}_n(t)\,dt<\infty.$$

This implies that $|T(t)|\check{\varphi}(t)\in L^1$.

Let φ be an arbitrary element from $\mathcal{S}$. There exists $\psi\in\mathcal{S}$ such that $\psi>0$ and $|\varphi|\le\psi$ [2, p. 401]. Since we have proved that $|T|\psi\in L^1$, it follows that $|T|\varphi\in L^1$.

Let us remark that from [4, Theorem VI, p. 239] directly follows the equivalence of (d) and

(e) *For every* $\varphi\in\mathcal{D}$, $|T|*\varphi$ *is a slowly increasing function.*

To formulate the next corollary we need the following notation:

$$J(x_k)T(x_1,\ldots,x_k,\ldots,x_q)=J(x_k)T=\int_0^{x_k}T(x_1,\ldots,t,\ldots,x_q)\,dt,$$
$$k=1,\ldots,q,\quad (T\in L^1_{\text{loc}}).$$

Corollary. (i) *If* $T\in\mathcal{S}'_{\text{ar}}$ *then*

$$J(x_{k_1})\ldots J(x_{k_r})T\in\mathcal{S}'_{\text{ar}},\qquad k_1,\ldots,k_r\in\{1,\ldots,q\};$$

(ii) *If* $T\in\mathcal{S}'_{\text{r}}\setminus(\mathcal{S}'_{\text{ar}})$ *and* $T^{(k)}$, $k\in\mathbf{N}^q$, *is from* L^1_{loc} *then* $T^{(k)}\in\mathcal{S}'_{\text{r}}\setminus(\mathcal{S}'_{\text{ar}})$.

Proof. Since (i) directly follows form (c), we will prove only (ii). If $T\in\mathcal{S}_r$ and $T^{(k)}\in L^1_{\text{loc}}$ then $T^{(k)}\in\mathcal{S}'_{\text{r}}$ ([2, (2.2)(b)]). Thus (i) implies (ii).

Let $T\in L^1_{\text{loc}}$. Clearly $H(x)=\exp\left(i\left(\int_0^x|T(t)|\,dt\right)\right)$, $x\in\mathbf{R}^q$, is from $\mathcal{S}'_{\text{r}}$ and thus, $\partial^qH(x)/\partial x_1\ldots\partial x_q\in\mathcal{S}'_{\text{r}}$. Since $|\partial^qH(x)/\partial x_1\ldots\partial x_q|=|T(x)|$, it follows that every non-negative function from L^1_{loc} is the absolute value of some regular tempered distribution.

It was proved in [2, (2.4) Proposition] that $T \in \mathcal{S}'_{\mathrm{r}}$ does not imply $|T| \in \mathcal{S}'_{\mathrm{r}}$. This easily follows from the remark given above, if we take for example, $T(t) = \exp t$, $t \in \mathbf{R}^q$.

3. Following Vladimirov [7], we define the Fourier transformation and the inverse Fourier transformation of an $f \in L^1$ by

$$(\mathcal{F}f(x))(\xi) = \int_{\mathbf{R}^q} f(x)\exp(i(x,\xi))\,dx \qquad \text{and}$$
$$(\widetilde{\mathcal{F}}f(x))(\xi) = (2\pi)^{-q}\int_{\mathbf{R}^q} f(x)\exp(-i(x,\xi))\,dx, \quad \xi \in \mathbf{R}^q.$$

By the same symbols we denote the distributional Fourier and inverse Fourier transformations.

Up to the end of the paper we suppose that $f \in \mathcal{S}'_{\mathrm{ar}}$.

It was noted by Dierolf that $f \in \mathcal{S}'_{\mathrm{ar}}$ iff for some non-negative integer k $(\widetilde{\mathcal{F}}f) \in \mathcal{B}_{1,k}$, where $\mathcal{B}_{1,k}$ is the space if all tempered distributions T for which

$$(\widetilde{\mathcal{F}}T)(\xi)(1+\xi^2)^{-k/2} \in L^1 \qquad [3, \text{p. } 36].$$

Let Λ be the set of all q-th variations of elements -1 and 1 and let $\Pi_{(\sigma_1,\dots,\sigma_q)} = \{\xi \mid \xi_i\sigma_i \geq 0,\ i = 1,\dots,q\}$, $(\sigma_1,\dots,\sigma_q) \in \Lambda$. We put, for $(\sigma_1,\dots,\sigma_q) \in \Lambda$,

$$H_{(\sigma_1,\dots,\sigma_q)}(\xi_1,\dots,\xi_q) = \begin{cases} 1 & \xi \in \Pi_{(\sigma_1,\dots,\sigma_q)} \\ 0 & \text{elsewhere.} \end{cases}$$

We have

$$(\mathcal{F}f)(\xi) = \sum_{(\sigma_1,\dots,\sigma_q)\in\Lambda} \hat{f}(\xi)H_{(\sigma_1,\dots,\sigma_q)}(\xi_1,\dots,\xi_q) \quad \text{(a.e.)}.$$

Let

$$F_{(\sigma_1,\dots,\sigma_q)}(-x+iy) = \int (\mathcal{F}f)(\xi)H_{(\sigma_1,\dots,\sigma_q)}(\xi)\cdot\exp(i(-x+iy,\xi))\,d\xi,$$
$$x \in \mathbf{R}^q,\ y \in \Pi_{(\sigma_1,\dots,\sigma_q)}.$$

Since $(\mathcal{F}f)(\xi)H_{(\sigma_1,\dots,\sigma_q)}(\xi) \in \mathcal{S}'(\Pi_{(\sigma_1,\dots,\sigma_q)})$ (see the footnote in [7, p. 77]) and in the sense of convergence in $\mathcal{S}'(\mathbf{R}^q)$ there holds

$$F_{(\sigma_1,\dots,\sigma_q)}(-x+iy) \to \mathcal{F}((\mathcal{F}f)(\xi)H_{(\sigma_1,\dots,\sigma_q)}(\xi))(-x) \quad \text{as } y \to 0,$$
$$y \in \Pi_{(\sigma_1,\dots,\sigma_q)}, \quad (\sigma_1,\dots,\sigma_q) \in \Lambda$$

[7, Section 12.2], we obtain

$$f(x) = (2\pi)^n \sum_{(\sigma_1,\dots,\sigma_q)\in\Lambda} \lim_{\substack{y\to 0\\ y\in\Pi_{(\sigma_1,\dots,\sigma_q)}}} F_{(\sigma_1,\dots,\sigma_q)}(-x+iy),$$

where the limit is taken in the sense of convergence in $\mathcal{S}'$.

The characterization of analytic functions $F_{(\sigma_1,\dots,\sigma_q)}$, $(\sigma_1,\dots,\sigma_q)\in\Lambda$ is given in [7, Ch. II, 12.1 and 12.2].

In the one-dimensional case the most precise boundary value representation of absolutely regular tempered distributions follows from Carleman [1]. First we recall some definitions which were given by him.

Let $\alpha,\beta\in\mathbf{R}$ such that $\alpha>0$, $\beta>-1$. As in [1, p. 42], we say that a function $\varphi(z)$ regular in the half-plane $D=\operatorname{Im} z>0$ (or $D=\operatorname{Im} z<0$), is of the class (α,β) if the function

$$\frac{1}{(z-z_0')^{\beta}}\,\frac{(z-z_0)^{-\alpha}}{\Gamma(\alpha)}\int_{z_0}^{z}(z-\zeta)^{\alpha-1}\varphi(\zeta)\,d\zeta$$

is bounded in D; z_0 is a point from D and z_0' is a point in the z-plane symmetric to z_0 according to the $\operatorname{Im} z=0$. If $\alpha=0$, we replace the last expression by $(z-z_0)^{-\beta}\varphi(z)$.

If for some (α,β), $\alpha\ge 0$, $\beta\ge -1$, the function $\varphi(z)$, defined on D is regular and of the class (α,β), then φ is called regular and of a finite class.

Using the part (c) of the Theorem and [1, Ch. II Theorems I and II, p. 42 and p. 47] we obtain the following representation of an element from $\mathcal{S}'_{\mathrm{ar}}(\mathbf{R})$:

THEOREM 2. *Let $f\in\mathcal{S}'_{\mathrm{ar}}(\mathbf{R})$. Then there exist analytic functions $f_1(z)$, $z\in\{z:\operatorname{Im} z>0\}$ and $f_2(z)$, $z\in\{z:\operatorname{Im} z<0\}$, regular in $\operatorname{Im} z>0$, respectively $\operatorname{Im} z<0$, and of finite classes such that $f(x)=\lim_{y\to 0^+}(f_1(x+iy)-f_2(x-iy))$ almost everywhere on $\mathbf{R}$. Functions $f_1(z)$ and $f_2(z)$ are determined up to some polynomials.*

Remark. Since $f_1(x+iy)$ and $f_2(x-iy)$ are regular in the half-plane $y>0$, it follows that these functions are bounded on any compact set of the form $[a,b]\times[0,\delta]$ in the half plane $y\ge 0$. Thus, by the Lebesque's theorem we obtain that (4) holds in the sense of weak convergence in $\mathcal{S}'(\mathbf{R})$.

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