PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 49 (63), 1991, 6–16

ON GRAPHS WHOSE ENERGY DOES NOT EXCEED 4

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Abstract. In a recent paper [5] Torgašev described all finite connected graphs whose energy (i.e. the sum of all positive eigenvalues including their multiplicities), does not exceed 3. In this paper, we describe all connected graphs whose energy does not exceed 4. The method applied here differs of the corresponding method in [5].

In this paper we consider only finite connected graphs having no loops or multiple edges. The vertex set of a graph G is denoted by $V(G)$, and its order by $|G|$. The spectrum of such a graph is the set $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ of eigenvalues of its 0-1 adjacency matrix.

The sum of all positive eigenvalues (including their multiplicities) is denoted by $S(G)$ and called the *energy* of G. For any real $a \geq 1$, we can consider the class of graphs $P(a) = \{G \mid S(G) \leq a\}$ and, in this paper we describe completely the class $P(4)$. Briefly, any graph $G \in P(4)$ is called *admissible*, and any other graph — *impossible* (or forbidden) for this class.

We note that in [5] A. Torgašev described completely the class $P(3)$. Hence, in the investigation of the class $P(4)$ we exclude the graphs whose energy is ≤ 3. Therefore, we describe in fact the class $Q(4) = P(4) \setminus P(3)$.

In [5] it is also proved that the class $P(a)$ is finite for any real $a \geq 1$. Our method differs slightly from the corresponding method in [5]. In fact, we describe first the complete set of the so-called canonical graphs in the class $P(4)$, then we generate all this class.

Let H be any connected induced subgraph of a graph G; this is denoted by $H \subseteq G$. Since, by the known interlacing theorem [1; p. 19], we have that any connected subgraph of an admissible graph is also admissible. It implies that the method of forbidden subgraphs can be consistently applied.

We say that two vertices $x, y \in V(G)$ are equivalent in G and denote it by $x \sim y$ if x is nonadjacent to y, and x and y have exactly the same neighbours in G. The relation $\sim$ is obviously an equivalence relation on the vertex set $V(G)$. The

AMS Subject Classification (1980): Primary 05 C 50

corresponding quotient graph is denoted by g, and called the *canonical* graph of G. The last graph is also connected. Also obviously have $g \subseteq G$. For instance, if $G = K_{m_1 m_2 \dots m_p}$ $(p \geq 2)$ is the complete p-partite graph, then its canonical graph is the complete graph K_p. The canonical graph of the complete graph K_n is the same graph K_n.

We say that G is canonical if $|G| = |g|$, thus if G has no two equivalent vertices.

if g is the canonical graph of G, $|g| = k$, and $N_1, \dots, N_k$ are the corresponding sets of equivalent vertices in G, we write $G = g(N_1, \dots, N_k)$, or simply $G = g(n_1, \dots, n_k)$, where $|N_i| = n_i$ $(i = 1, \dots, k)$, understanding that g is a labelled graph. We call $N_1, \dots, N_k$, the characteristic sets of G. Obviously, each set $N_i \subseteq V(G)$ consists only of isolated vertices, and if at least one edge between the sets N_i, N_j $(i \neq j)$ is present, then all possible edges between these sets are also present. Therefore, it is very convenient to display the sets N_i $(i = 1, \dots, k)$ by white (i.e. empty) circles, and all possible edges between the sets N_i and N_j by only one edge between the corresponding circles. If, for example, G is the complete bipartite graph with characteristic subsets N_1, N_2 we can simply denote

$G =$ 

if $|N_i| = n_i$ $(i = 1, 2)$.

Now it is clear that the graph $G = g(n_1, \dots, n_k)$, whose canonical graph is g, is obtained by varying, in an arbitrary way, the values of parameters $n_1, \dots, n_k \in \mathbb{N}$. It should be also noted that $G_1 \subseteq G_2$ holds for two graphs $G_1 = g(n_1, \dots, n_k)$ and $G_2 = g(m_1, \dots, m_k)$, having the same canonical graph g with $n_i \leq m_i$ $(i = 1, \dots, k)$.

If g is the canonical graph of a graph G, we have that $g \subseteq G$, whence we obtain $G \in P(4) \implies g \in P(4)$. Taking into account that $P(4)$ is finite [5], we have that the class $P_0(4)$ of all canonical graphs from the class $P(4)$ is also finite. Therefore, in the investigation of the class $Q(4)$, it is reasonable to describe first the class $Q(4)$, then generate all the class $P(4)$, and consequently the class $Q(4)$.

We also note that many other hereditary problems in the Spectral theory of graphs can be reduced to finding first the corresponding sets of canonical graphs. In this respect one can consult the papers [3], [4] etc.

The creating of the set $P_0(4)$ in this paper is based on the following general theorem proved in [6], which can be very valuable for other similar problems.

THEOREM A. *In all but a sequence of exceptional cases, each connected canonical graph on n vertices $(n \geq 3)$ contains an induced subgraph on $n-1$ vertices, which is also connected and canonical. The exceptional cases are the graphs from Fig. 1. These graphs satisfy the relations $T_0 \subseteq T_1 \subseteq T_2 \subseteq \dots$* □

By a direct inspection of spectra of all connected graphs with at most 7 vertices, we find that the class $P(4)$ contains exactly 39 canonical graphs with at most 7 vertices. They are displayed in Fig. 2.

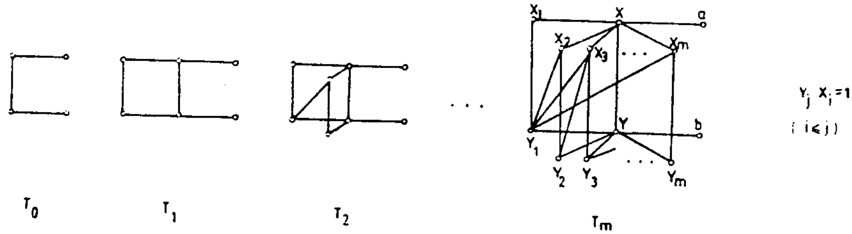

Fig. 1

As it is also known, there are exactly 11.117 connected graphs with 8 vertices. By a direct inspection of their spectra, we find that the class $P(4)$ contains no canonical graphs with 8 vertices.

As a direct consequence, having also in mind Theorem A, we find immediately the following result.

THEOREM 1. *The complete list of all canonical graphs from the class $P(4)$ is given by Fig.* 2.

Hence, the class $P(4)$ contained exactly 39 nonisomorphic canonical graphs.

Besides, we note that some canonical graphs from the Fig. 2 belong to the class $P(3)$, hence they do not belong to the class $Q(4)$. But they can also serve as canonical graphs to some graphs from the class $Q(4)$. These graphs are exactly

$$g_1 = K_2, \quad g_2 = K_3, \quad g_3 = P_4, \quad g_4, \quad g_5 = K_4, \quad g_6 = P_5, \quad g_7,$$

where P_n $(n \geq 2)$ is the path on n vertices.

Next, we say that a canonical graph $g \in Q(4)$ is *simple* if each graph G $(G \neq g)$, whose canonical graph is g, does not belong to this class.

Then we have:

PROPOSITION 1. *The canonical graphs g_{12}, g_{14}, g_{15}, g_{16}, g_{19}, g_{20}, g_{23}, g_{24}, g_{25}, g_{26}, g_{28}, g_{29}, g_{30}, g_{31}, g_{32}, g_{33}, g_{34}, g_{35}, g_{36}, g_{37}, g_{38}, g_{39} from the Fig.* 2 *are simple.*

We note that the proof that a canonical graph $g \in Q(4)$ is simple, is immediate. Since the property $S(G) \leq 4$ is hereditary, it is sufficient only to prove that by adding a new vertex to g, which is equivalent to any already present, always gives an imposible graph. But, it is a matter of routine to check this for any of the 22 graphs mentioned above.

In the sequel, for any of the remaining graphs from the Fig. 2, we give necessary and sufficient conditions under which a corresponding overgraph belongs to the class $Q(4)$.

Let g be any canonical graph from the Fig. 2. Denote $|g| = k$, and let G be any graph whose canonical graph is g. If $N_1, \ldots, N_k$ are the characteristic sets of

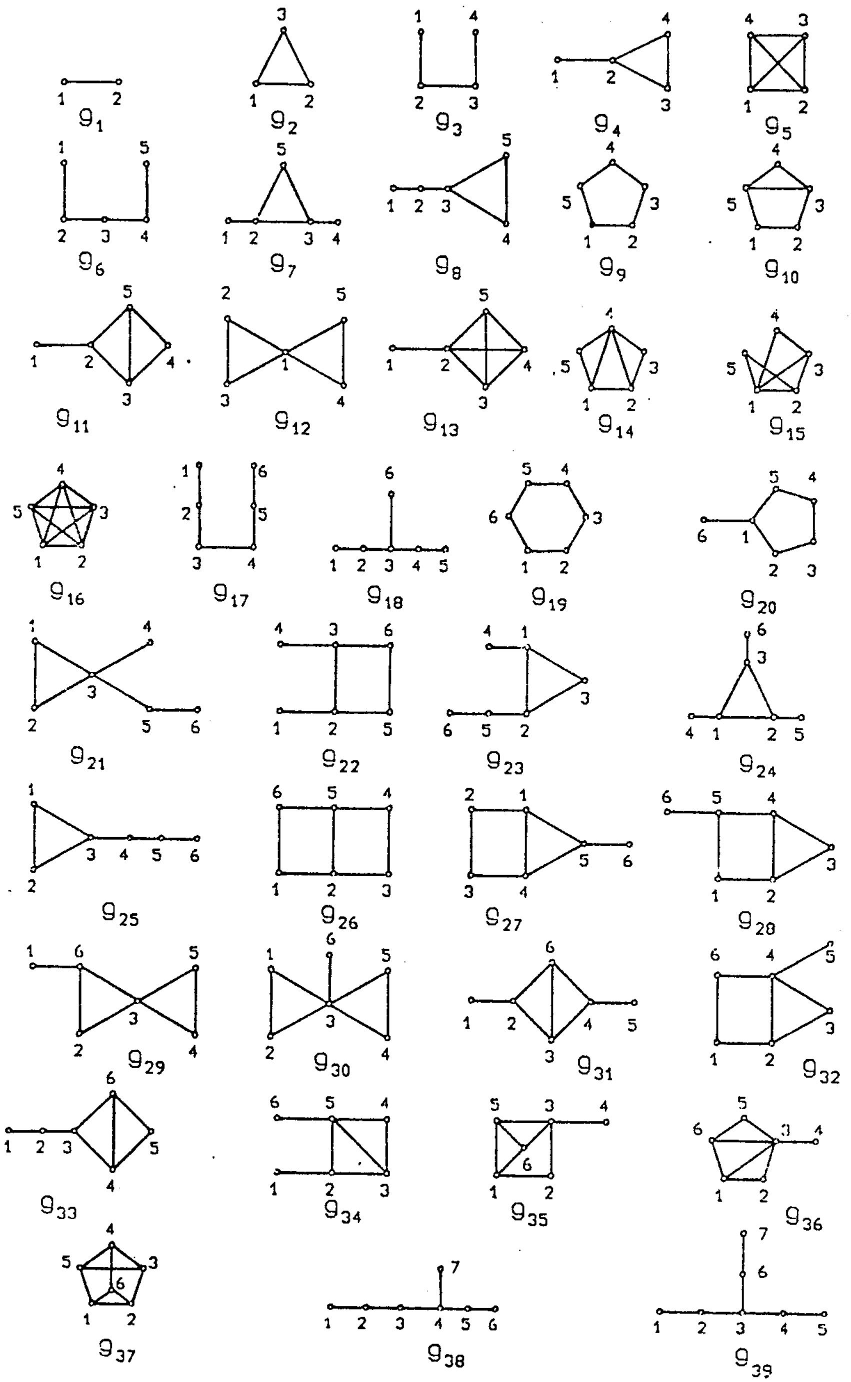
g1
g2
g3
g4
g5
g6
g7
g8
g9
g10
g11
g12
g13
g14
g15
g16
g17
g18
g19
g20
g21
g22
g23
g24
g25
g26
g27
g28
g29
g30
g31
g32
g33
g34
g35
g36
g37
g38
g39

Fig. 2

G and $|N_i| = n_i$ $(i = 1, \ldots, k)$, we denote $G = g(n_1, \ldots, n_k)$ understanding that g is a labelled graph.

PROPOSITION 2. *A graph* $G = g_1(m,n) \in Q(4)$ $(m \leq n)$ *if and only if* $(m,n) = (1,10)$, $(1,11)$, $(1,12)$, $(1,13)$, $(1,14)$, $(1,15)$, $(1,16)$, $(2,5)$, $(2,6)$, $(2,7)$, $(2,8)$, $(3,4)$, $(3,5)$, $(4,4)$.

Proof. Since $g_1 = K_2$ the graph $G = K_{m,n}$ is the complete bipartite graph, hence it will have only one positive eigenvalue $r(G) = \sqrt{mn}$. Therefore $G \in Q(4)$ if and only if $9 < mn \leq 16$, which easily gives the statement.□

PROPOSITION 3. *A graph* $G = g_2(m,n,k)$ $(m \leq n \leq k)$ *belongs to the class* $Q(4)$ *if and only if* $(m,n,k) = (1,1,4)$, $(1,1,5)$, $(1,1,6)$, $(1,2,2)$, $(1,2,3)$, $(2,2,2)$.

Proof. Since $g_2 = K_3$, the graph G is the complete 3-partite graph $K_{m,n,k}$. It has only one positive eigenvalue, which is the maximal root $r(G)$ of the polynomial

$$F(\lambda) = \lambda^3 - (mn + nk + mk)\lambda - 2mnk.$$

Hence $G \in Q(4)$ if and only if $3 < r(G) \leq 4$. Therefore we easily find the statement.□

PROPOSITION 4. *A graph* $G = g_3(m,n,k,l)$, $(m < l$ *or* $m = l,\ n \leq k)$ *belongs to the clas* $Q(4)$ *if and only if* (m,n,k,l) *has one of the following values*:

$$\begin{array}{llll}
(1,1,1,4), & (1,1,1,5), & (1,1,1,6), & (1,1,1,7), \\
(1,1,1,8), & (2,1,1,3), & (2,1,1,4), & (2,1,1,5), \\
(2,1,1,6), & (3,1,1,3), & (3,1,1,4), & (1,1,3,1), \\
(1,1,4,1), & (1,1,5,1), & (1,1,2,2), & (1,1,3,2), \\
(1,1,2,3), & (1,2,1,2), & (1,3,1,2), & (1,4,1,2), \\
(1,2,1,3), & (1,3,1,3), & (1,2,1,4), & (1,2,1,5), \\
(1,2,2,1), & (1,2,3,1), & (2,2,1,2), & (2,2,1,3), \\
(1,2,2,2). & & &
\end{array}$$

Proof. It is easy to check that each of the graphs above belongs to the class $Q(4)$. Besides, it can be easily proved that the graph $g_3(2,2,2,2)$ has the energy > 4, thus it is impossible for the class $P(4)$. Hence, if some $G = g_3(m,n,k,l) \in Q(4)$ $(m \leq l)$ then we necessarily have $m = 1$ or $n = 1$ or $k = 1$.

Next, note that all eigenvalues of such a graph are determined by equation

$$\lambda^4 - (mn + nk + kl)\lambda^2 + mnkl = 0.$$

Whence, these eigenvalues can be explictily found. Therefore, it is easy to prove that $G = g_3(m,n,k,l) \in Q(4)$ if and only if we have

$$9 < mn + nk + kl + 2\sqrt{mnkl} \leq 16.$$

Hence, we immediately get our Proposition. □

PROPOSITION 5. *A graph $G = g_4(m,n,k,l)$ $(k \le l)$ belongs to the class $Q(4)$ if and only if (m,n,k,l) has one of the following values:*

$$\begin{aligned}(m,n,k,l) = &(1,1,1,2), \quad (1,1,1,3), \quad (1,1,1,4), \quad (2,1,1,2),\\ &(2,1,1,3), \quad (3,1,1,1), \quad (3,1,1,2), \quad (4,1,1,1),\\ &(4,1,1,2), \quad (5,1,1,1), \quad (6,1,1,1), \quad (7,1,1,1),\\ &(1,2,1,1), \quad (1,3,1,1), \quad (2,2,1,1), \quad (3,2,1,1),\\ &(1,2,1,2), \quad (1,1,2,2), \quad (2,1,2,2).\end{aligned}$$

We omit the proof in view of the similarities with the previous ones.

PROPOSITION 6. *A graph $G = g_5(m,n,k,l)$ $(m \le n \le k \le l)$ belongs to the class $Q(4)$ if and only if $(m,n,k,l) = (1,1,1,2)$.*

Proof. It is immediate to check that the graph $g_5(1,1,1,2)$ belongs to the class $Q(4)$. Since the graphs $g_5(1,1,2,2)$ and $g_5(1,1,1,3)$ have the energy > 4, we proved the statement. □

PROPOSITION 7. *A graph $G = g_6(m,n,k,l,p)$ $(m < p$ or $m = p,\ n \le l)$ belongs to the class $Q(4)$ if and only if (m,n,k,l,p) has one of the following values:*

$$\begin{aligned}&(1,1,1,1,2), \quad (1,1,1,1,3), \quad (1,1,1,1,4), \quad (1,1,1,1,5),\\ &(2,1,1,1,2), \quad (2,1,1,1,3), \quad (2,1,1,1,4), \quad (3,1,1,1,3),\\ &(1,2,1,1,1), \quad (1,2,1,1,2), \quad (1,2,1,1,3), \quad (1,3,1,1,1),\\ &(1,1,2,1,1), \quad (1,1,3,1,1), \quad (1,1,4,1,1), \quad (1,1,2,1,2),\\ &(1,1,3,1,2), \quad (1,1,2,1,3), \quad (1,2,1,2,1), \quad (1,1,1,2,2),\\ &(1,2,2,1,1), \quad (2,1,2,1,2).\end{aligned}$$

Proof. It is easy to check that all above graphs belong to the class $Q(4)$. Since all the graphs $g_6(m,n,k,l,p)$ $(m < p$ or $m = p,\ n \le l)$, where (m,n,k,l,p) has one of the values

$$\begin{aligned}&(1,1,1,1,6), \quad (1,1,5,1,1), \quad (1,4,1,1,1), \quad (1,1,1,2,3),\\ &(1,1,1,3,2), \quad (1,1,3,1,3), \quad (1,1,4,1,2), \quad (1,1,2,1,4),\\ &(1,2,1,1,4), \quad (1,3,1,1,2), \quad (2,1,1,1,5), \quad (3,1,1,1,4),\\ &(1,2,3,1,1), \quad (1,3,2,1,1), \quad (1,2,1,3,1), \quad (1,1,2,2,2),\\ &(1,2,1,2,2), \quad (2,1,1,2,2), \quad (1,2,2,2,1), \quad (1,2,2,1,2),\\ &(2,1,2,1,3), \quad (2,1,3,1,2),\end{aligned}$$

have the energy > 4, we immediately get the statement. □

In a similar way, we can prove next propositions:

PROPOSITION 8. *A graph* $G = g_7(m,n,k,l,p)$ ($m < l$ *or* $m = l$, $n \leq k$) *belongs to the class* $Q(4)$ *if and only if* (m,n,k,l,p) *has one of the values*

$$\begin{array}{llll} (1,1,1,1,2), & (1,1,1,1,3), & (2,1,1,1,1), & (2,1,1,1,2), \\ (3,1,1,1,1), & (3,1,1,1,2), & (4,1,1,1,1), & (5,1,1,1,1), \\ (1,2,1,1,1), & (1,2,1,2,1), & (1,2,1,3,1), & (2,1,1,2,1), \\ (2,1,1,2,2). & & & \end{array}$$

PROPOSITION 9. *A graph* $G = g_8(m,n,k,l,p)$ ($l \leq p$) *belongs to the class* $Q(4)$ *if and only if* (m,n,k,l,p) *has one of the values*

$$\begin{array}{lll} (1,1,1,1,1), & (1,1,1,1,2), & (2,1,1,1,1), \\ (3,1,1,1,1), & (1,1,2,1,1), & (1,2,1,1,1). \end{array}$$

PROPOSITION 10. *A graph* $G = g_9(m,n,k,l,p)$ ($m < n$ *or* $m = n$, $k \leq p$) *belongs to the clas* $Q(4)$ *if and only if* (m,n,k,l,p) *has one of the values* $(1,1,1,1,1)$, $(1,1,1,1,2)$.

PROPOSITION 11. *A graph* $G = g_{10}(m,n,k,l,p)$ ($m < n$ *or* $m = n$, $k \geq p$) *belongs to the class* $Q(4)$ *if and only if* (m,n,k,l,p) *has one of the values* $(1,1,1,1,1)$, $(1,1,1,2,1)$, $(1,1,2,1,1)$, $(1,2,1,1,1)$.

PROPOSITION 12. *A graph* $G = g_{11}(m,n,k,l,p)$ ($k \leq p$) *belongs to the class* $Q(4)$ *if and only if* (m,n,k,l,p) *has one of the values* $(1,1,1,1,1)$, $(1,1,1,2,1)$, $(2,1,1,1,1)$.

PROPOSITION 13. *A graph* $G = g_{13}(m,n,k,l,p)$ ($k \leq l \leq p$) *belongs to the class* $Q(4)$ *if and only if* (m,n,k,l,p) *has one of the values* $(1,1,1,1,1)$, $(2,1,1,1,1)$.

PROPOSITION 14. *A graph* $G = g_{17}(m,n,k,l,p,q)$ ($m < q$ *or* $m = q$, $n < p$ *or* $m = q$, $n = p$, $k \leq l$) *belongs to the class* $Q(4)$ *if and only if* (m,n,k,l,p,q) *has one of the following values* $(1,1,1,1,1,1)$, $(1,1,1,1,1,2)$.

PROPOSITION 15. *A graph* $G = g_{18}(m,n,k,l,p,q)$ ($m < p$ *or* $m = p$, $n \leq l$) *belongs to the class* $Q(4)$ *if and only if* (m,n,k,l,p,q) *has one of the values* $(1,1,1,1,1,1,)$, $(1,1,1,1,1,2)$, $(1,1,1,1,2,1)$.

PROPOSITION 16. *A graph* $G = g_{21}(m,n,k,l,p,q)$ ($m \leq n$) *belongs to the class* $Q(4)$ *if and only if has one of the values* $(1,1,1,1,1,1)$, $(1,1,1,2,1,1)$.

A graph $G = g_{22}(m,n,k,l,p,q)$ ($m < l$ or $m = l$, $n < k$ or $m = l$, $n = k$, $p \leq q$) belongs to the class $Q(4)$ if and only if (m,n,k,l,p,q) has one of the values $(1,1,1,1,1,1)$, $(1,1,1,2,1,1)$.

A graph $G = g_{27}(m,n,k,l,p,q)$ belongs to the class $Q(4)$ if and only if (m,n,k,l,p,q) has one of the values $(1,1,1,1,1,1)$, $(1,1,1,1,1,2)$.

Propositions 1–16 and Theorem 1 describe completely the class $Q(4)$. As an immediate consequence of Propositions 1–16 and Theorem 1 we have the following result.

THEOREM 2. *There are exactly* 154 *nonisomorphic graphs whose energy does not exceed* 4 *and is greater than* 3. *Their orders run over the set* $\{5, 6 \dots, 17\}$. *All these graphs are represented in the List* 1.

We notice that all graphs in this list are represented in the form

$$n_1\ n_2\ n_3\ a_{12}a_{13}a_{23}\dots a_{1n}a_{2n}\dots a_{n-1,n},$$

where n_1 is the ordering number of a corresponding graph, n_2 is the number of its vertices, n_3 is the number of its edges and $a_{12}a_{13}a_{23}\dots a_{1n}a_{2n}\dots a_{n-1,n}$ is the upper diagonal part of its adjacency matrix.

The list of all nonisomorphic graphs from the class $Q(4)$

```
001 05 05  1 10 001 0101
002 05 05  1 10 001 1100
003 05 06  1 10 001 1110
004 05 06  1 10 110 1001
005 05 06  1 10 001 1101
006 05 06  1 10 110 1010
007 05 07  1 11 111 1000
008 05 07  1 10 011 1101
009 05 07  1 10 111 0011
010 05 08  1 10 111 0111
011 05 08  1 11 111 1001
012 05 09  1 10 111 1111
013 05 10  1 11 111 1111
014 06 05  1 10 001 1000 00010
015 06 05  1 10 001 1000 01000
016 06 05  1 10 001 0100 00001
017 06 06  1 10 001 0101 00100
018 06 06  1 10 110 1000 10000
019 06 06  1 10 001 0100 01001
020 06 06  1 10 001 0100 10010
021 06 06  1 10 001 0100 10100
022 06 06  1 10 001 1000 00101
023 06 06  1 10 100 1000 01001
024 06 06  1 10 110 1000 00001
025 06 06  1 10 001 1000 10100
026 06 06  1 10 011 1000 00010
027 06 06  1 10 001 1100 00100
028 06 06  1 10 001 0100 00011
029 06 06  1 10 001 1010 00001
030 06 07  1 10 001 1101 00100
031 06 07  1 10 110 1000 10001
032 06 07  1 10 110 1000 00011
033 06 07  1 10 001 1101 01000
034 06 07  1 10 110 1000 00101
035 06 07  1 10 110 1010 01000
036 06 07  1 10 001 1000 11100
037 06 07  1 10 001 1000 11001
038 06 07  1 10 001 1000 10110
```

039	06	07	1	10	100	1000	01101	
040	06	07	1	10	001	1010	10100	
041	06	07	1	10	100	1000	11100	
042	06	07	1	10	001	0101	01010	
043	06	07	1	10	001	1110	00010	
044	06	07	1	10	001	1110	01000	
045	06	07	1	10	011	1000	00101	
046	06	07	1	10	011	1001	00001	
047	06	08	1	10	001	1010	10011	
048	06	08	1	10	011	1000	10011	
049	06	08	1	10	100	1000	10111	
050	06	08	1	10	110	1000	11100	
051	06	08	1	10	011	1001	01100	
052	06	08	1	10	011	1000	10110	
053	06	08	1	10	001	1100	01101	
054	06	08	1	10	110	1000	11010	
055	06	08	1	10	001	1000	11101	
056	06	08	1	10	001	1000	11011	
057	06	09	1	10	001	1101	11010	
058	06	09	1	10	001	1101	01110	
059	06	09	1	10	110	1000	01111	
060	06	09	1	10	110	1001	11001	
061	06	09	1	10	100	1111	10001	
062	06	10	1	10	011	1001	11110	
063	06	10	1	10	011	1001	11101	
064	06	11	1	10	100	1111	11110	
065	06	12	1	10	111	0111	11101	
066	07	06	1	10	100	1000	10000	000001
067	07	06	1	10	001	1000	00100	000010
068	07	06	1	10	001	1000	10000	010000
069	07	06	1	10	001	1000	10000	000100
070	07	06	1	10	001	1000	01000	000100
071	07	06	1	10	001	1000	00010	000100
072	07	06	1	10	001	0100	00001	000010
073	07	06	1	10	001	1000	00100	001000
074	07	06	1	10	001	1000	01000	000010
075	07	07	1	10	001	1000	00100	100001
076	07	07	1	10	001	0100	10010	010000
077	07	07	1	10	001	1000	10000	100001
078	07	07	1	10	110	1000	10000	010000
079	07	07	1	10	001	1000	00100	101000
080	07	07	1	10	001	1000	00010	100100
081	07	07	1	10	100	1000	10000	010010
082	07	07	1	10	001	1000	10000	001100
083	07	07	1	10	110	1000	10000	100000
084	07	08	1	10	001	1101	01000	010000
085	07	08	1	10	100	1000	10000	001101
086	07	08	1	10	100	1000	01101	000100
087	07	08	1	10	100	1000	10000	100011
088	07	08	1	10	100	1000	01101	000001
089	07	08	1	10	001	1010	10100	100000
090	07	08	1	10	001	0100	10010	100010
091	07	08	1	10	011	1001	00001	000010
092	07	08	1	10	011	1001	00001	000001
093	07	09	1	10	011	1001	10010	000100
094	07	09	1	10	011	1001	10010	010000
095	07	09	1	10	100	1000	10000	101110
096	07	09	1	10	011	1001	01100	000001
097	07	09	1	10	100	1000	10111	000001

098	07	09	1 10 001 1010 10011 100000
099	07	10	1 10 100 1000 01101 011010
100	07	10	1 10 100 1000 10000 111110
101	07	10	1 10 100 1000 01001 011110
102	07	10	1 10 100 1000 11100 111000
103	07	10	1 10 100 1000 10000 011111
104	07	11	1 10 011 1001 10010 101100
105	07	11	1 10 011 1001 01100 100101
106	07	11	1 10 100 1111 10001 100010
107	07	12	1 10 011 1001 01101 100101
108	08	07	1 10 100 1000 10000 100000 0001000
109	08	07	1 10 001 1000 10000 000100 0001000
110	08	07	1 10 100 1000 10000 000001 0000001
111	08	07	1 10 001 1000 00100 001000 0010000
112	08	07	1 10 001 1000 00100 001000 1000000
113	08	08	1 10 001 1000 00100 001000 0100100
114	08	08	1 10 100 1000 10000 100000 0010100
115	08	08	1 10 001 1000 00010 100100 0001000
116	08	08	1 10 001 1000 00100 101000 0010000
117	08	08	1 10 100 1000 10000 100000 1000010
118	08	08	1 10 001 1000 00010 100100 1000000
119	08	08	1 10 110 1000 10000 010000 1000000
120	08	09	1 10 100 1000 10000 100000 0001110
121	08	09	1 10 100 1000 10000 100000 1010010
122	08	09	1 10 001 1010 10100 100000 0010000
123	08	09	1 10 100 1000 01101 000001 0000010
124	08	09	1 10 001 1010 10100 100000 1000000
125	08	09	1 10 001 1000 00100 001000 0110100
126	08	10	1 10 100 1000 10000 100000 0101011
127	08	10	1 10 011 1001 10010 000100 1000000
128	08	11	1 10 100 1000 10000 100000 0101111
129	08	12	1 10 100 1000 10000 100000 0111111
130	08	13	1 10 100 1111 10001 100010 1000100
131	08	15	1 10 011 1001 01101 100101 1001010
132	08	16	1 10 011 1001 01101 100101 0110101
133	09	08	1 10 100 1000 10000 100000 0001000 10000000
134	09	08	1 10 001 1000 00100 001000 1000000 10000000
135	09	08	1 10 001 1000 10000 000100 0001000 00010000
136	09	08	1 10 100 1000 10000 100000 0001000 00010000
137	09	08	1 10 001 1000 10000 000100 0001000 10000000
138	09	08	1 10 100 1000 10000 000001 0000001 10000000
139	09	09	1 10 100 1000 10000 100000 1000000 00100100
140	09	09	1 10 100 1000 10000 100000 1000000 10000010
141	09	09	1 10 110 1000 10000 010000 1000000 10000000
142	09	14	1 10 100 1000 10000 100000 1000000 01111111
143	10	09	1 10 100 1000 10000 100000 1000000 10000000 000000001
144	10	09	1 10 100 1000 10000 100000 0001000 10000000 000100000
145	10	10	1 10 100 1000 10000 100000 1000000 10000000 100000100
146	10	16	1 10 100 1000 10000 100000 1000000 10000000 011111111
147	11	10	1 10 100 1000 10000 100000 1000000 10000000 100000000 1000000000
148	11	10	1 10 100 1000 10000 100000 1000000 10000000 000000001 1000000000
149	12	11	1 10 100 1000 10000 100000 1000000 10000000 100000000 1000000000 10000000000
150	13	12	1 10 100 1000 10000 100000 1000000 10000000 100000000 1000000000 10000000000 100000000000
151	14	13	1 10 100 1000 10000 100000 1000000 10000000 100000000 1000000000 10000000000 100000000000 1000000000000

152	15	14	1 10 100 1000 10000 100000 1000000 10000000 100000000 1000000000 10000000000 100000000000 1000000000000 10000000000000
153	16	15	1 10 100 1000 10000 100000 1000000 10000000 100000000 1000000000 10000000000 100000000000 1000000000000 10000000000000 100000000000000
154	17	16	1 10 100 1000 10000 100000 1000000 10000000 100000000 1000000000 10000000000 100000000000 1000000000000 10000000000000 100000000000000 1000000000000000

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(Received 26 10 1989; in revised form 30 06 1990)