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SOME REMARKS ON THE FIXED POINT STABILITY FOR NONEXPANSIVE MAPPINGS

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Abstract. Fixed point stability results are presented for nonexpansive mappings of nonempty weakly closed subsets of a Banach space X into the family of nonempty closed subsets of X.

Introduction. The study of the existence of fixed points for nonexpansive mappings in Hilbert spaces in [3] was later extended in [1] to Banach spaces satisfying Opial's condition.

The technics, used by Lami Dozo in [1], are based on the approximation of a nonexpansive mapping by a sequence of contractive mappings. In [2] and [4] some stability theorems are proved. The aim of this paper is to prove some fixed point stability results under weaker hypotheses with respect to those considered by Lim in [2].

Let X be a Banach space and $C(X)$ (respect. $K(X)$) the family of nonempty closed (respect. compact) subsets of X. Let $H(\cdot, \cdot)$ be the Hausdorff distance, i.e.:

$$H(A,B) = \max\Big\{\sup_{x\in B} d(x,A), \sup_{x\in A} d(x,B)\Big\},$$

$$\text{where} \quad d(x,C) = \inf_{x\in C} \|x-c\| \quad \text{and} \quad A,B,C \in C(X).$$

A Banach space X satisfies Opial's condition if for any element $x_0 \in X$ and for any sequence (x_n) weakly convergent to x_0 ($x_n \rightharpoonup x_0$) we have:

$$\liminf_n \|x_n - x_0\| < \liminf_n \|x_n - y\| \qquad \text{for any } y \neq x_0.$$

Let $T : X \to C(X)$ be a mapping. T is said to be nonexpansive (respec. t-contractive) if for any x, y in X we have:

$$H(Tx,Ty) \le \|x-y\| \quad (\text{respec. } H(Tx,Ty) \le t\|x-y\|) \quad \text{with } t \in [0,1).$$

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Let x be an element of X; then x is called a fixed point of T if $x \in Tx$.

In [2] Teck Choung Lim proves the following result.

THEOREM 1. *Let X be a Banach space satisfying Opial's condition. Let B be a nonempty weakly closed subset of X and let $T : B \to C(X)$ and $T_n : B \to K(X)$, with $n \in \mathbf{N}$, be mappings such that $T_n x \to Tx$ for any x in B in the sense of the Hausdorff metric. If for any n in $\mathbf{N}$ x_n is a fixed point of T_n and $x_n \rightharpoonup x_0$, then x_0 is a fixed point of T.*

Remark 2. Under the conditions of Theorem 1, T is a nonexpansive mapping. In fact let x, y be in B and let ε be a positive number, then there exists $n_0 = n(x, y, \varepsilon) > 0$ such that for any $n > n_0$ we have: $H(T_n x, Tx) < \varepsilon$ and $H(T_n y, Ty) < \varepsilon$, and so

$$H(Tx, Ty) \leq H(Tx, T_n x) + H(T_n x, T_n y) + H(T_n y, Ty) \leq 2\varepsilon + \|x - y\|$$

and, for $\varepsilon \to 0$, we have $H(Tx, Ty) \leq \|x - y\|$.

Remark 3. Under the conditions of Theorem 1, the mapping T is such that $Tx \in K(X)$, for any $x \in B$, because Tx is the limit, in the Hausdorff metric, of the compact subsets $T_n x$.

Theorem 1 can be improved in the following way:

THEOREM 4. *Let X be a Banach space satisfying Opial's condition, let B be a nonempty weakly closed subset of X and let $T, T_n : B \to C(X)$ $(n \in \mathbf{N})$ be nonexpansive mappings such that $T_n x \to Tx$ for any x in B in the sense of the Hausdorff metric. If for any n in $\mathbf{N}$, x_n is a fixed point of T_n, $x_n \rightharpoonup x_0$ and Tx_0 is compact, then x_0 is a fixed point of T.*

Proof. For any n in $\mathbf{N}$, let h_n and d_n be such that:

$$h_n = H(T_n x_n, T_n x_0) \leq \|x_n - x_0\| \quad \text{and} \quad d_n = H(T_n x_0, Tx_0);$$

by hypothesis we have $\lim_n d_n = 0$. By the fact that x_n is in $T_n x_n$, there exist, for any m in $\mathbf{N}$, elements $y_{n,m}$ in $T_n x_0$ such that: $\|x_n - y_{m,n}\| \leq d_n + 1/m$. By the compactness of Tx_0, there exist a subsequence z_{n,m_k} such that $z_{n,m_k} \xrightarrow[k]{} z_n \in Tx_0$ and a sequence $z_{n_h} \xrightarrow[h]{} z_0 \in Tx_0$. Of course, for any n in $\mathbf{N}$, we have: $x_n - z_{n,m_k} \xrightarrow[k]{} x_n - z_n$ and so:

$$\begin{aligned}\|x_{n_h} - z_{n_h}\| = \lim_k \|x_{n_h} - z_{n_h,m_k}\| &\leq \liminf_k (\|x_{n_h} - y_{n_h,m_k}\| + \|y_{n_h,m_k} - z_{n_h,m_k}\|)\\ &\leq \liminf_k (h_{n_h} + d_{n_h} + 2/m_k) = h_{n_h} + d_{n_h} \|x_{n_h} - z_0\|\\ &\leq \|x_{n_h} - z_{n_h}\| + \|z_{n_h} - z_0\| \leq h_{n_h} + d_{n_h} + \|z_{n_h} - z_0\|\end{aligned}$$

and also

$$\liminf_h \|x_{n_h} - z_0\| \leq \liminf_h (\|x_{n_h} - x_0\| + d_{n_h} + \|z_{n_h} - z_0\|) = \liminf_h \|x_{n_h} - x_0\|.$$

By Opial's condition we have $z_0 = x_0$, and by the fact that $z_0 \in Tx_0$, we have that x_0 is a fixed point of T.

Remark 5. In Theorem 4 the assumption of the compactness of Tx_0 cannot be omitted, as the following example proves. Let $X = B$ be the usual space l^2. Of course, X satisfies Opial's condition. Let $T, T_n : X \to C(X)$ be the following mappings:

$$Tx = \left\{ y \in X : \frac{1}{2} \le \|y\| \le \frac{1}{2} + \frac{\|x\|}{\|x\|+1} \right\}$$

$$T_n x = \left\{ y \in X : \frac{1}{2} \le \|y\| \le \frac{1}{2} + \left(1 - \frac{1}{n}\right) \frac{\|x\|}{\|x\|+1} \right\}$$

$$H(T_n x, Tx) \le \frac{1}{n} \frac{\|x\|}{\|x\|+1} \to 0, \quad \text{for any } x \in X$$

$$H(T_n x, T_n y) \le \left(1 - \frac{1}{n}\right) \left\| \frac{\|x\|}{\|x\|+1} - \frac{\|y\|}{\|y\|+1} \right\| \le \|x - y\|, \quad \text{for any } x, y \in X.$$

For any $n \in \mathbf{N}$, let x_n be the following element of X: $x_n = (1 - 1/n)e_n$, where $e_n = (0, \dots, 0, 1, 0, \dots)$ (the n-th coordinate entry is 1). It is easy to prove:

1) $x_n \in T_n x_n$, for any $n \in \mathbf{N}$,

2) $x_n \to 0$,

3) $T0 = \{y \in X : \|y\| = 1/2\}$ is not compact,

4) 0 is not a fixed point of T.

Remark 6. In Theorem 4 the assumption that Tx_0 is compact cannot be omitted even if we suppose that: $\lim_n \sup_{x \in X} H(T_n x, Tx) = 0$ (see the previous example).

Now we prove another fixed point stability result with weaker assumptions about X.

THEOREM 7. *Let X be a Banach space, let B be a nonempty weakly closed subset of X and let $T, T_n : B \to C(X)$ be nonexpansive mappings such that $T_n x \to Tx$ for any x in B in the sense of the Hausdorff metric. If, for any n in* $\mathbf{N}$, *x_n is a fixed point of T_n, $x_n \to x$ (in the strong sense) and Tx_0 is weakly compact, then x_0 is a fixed point of T.*

Proof. For any n in $\mathbf{N}$, let h_n and d_n be such that:

$$h_n = H(T_n x_n, T_n x_0) \le \|x_n - x_0\| \quad \text{and} \quad d_n = H(T_n x_0, Tx_0).$$

By hypothesis, we have: $\lim_n h_n = \lim_n d_n = 0$. Since x_n is in $T_n x_n$, there exist, for any m in $\mathbf{N}$, elements $y_{n,m}$ in $T_n x_0$ such that $\|x_n - y_{n,m}\| \le h_n + 1/m$, and elements $z_{n,m}$ in Tx_0 such that $\|y_{n,m} - z_{n,m}\| \le d_n + 1/m$.

By the weak compactness of Tx_0, there exist a subsequence $z_{n_k,m} \rightharpoonup_k z_m \in Tx_0$ and a sequence $z_{m_k} \rightharpoonup_k z_0 \in Tx_0$. For any m in $\mathbf{N}$ we have: $x_{n_k} - z_{n_k,m} \rightharpoonup_k x_0 - z_m$, and so

$$\|x_0 - z_m\| \le \liminf_h \|x_{n_k} - z_{n_k,m}\| \le \liminf_h (\|x_{n_k} - y_{n_k,m}\| + \|y_{n_k,m} - z_{n_k,m}\|)$$

$$\leq \liminf_h (h_{n_h} + d_{n_h} + 2/m) = 2/m.$$

Hence $z_m \to x_0$ and by the fact that $z_{n_k} \rightharpoonup z_0$ we have $z_0 = x_0 \in Tx_0$, and so x_0 is a fixed point of T.

In Theorem 7 the assumption that Tx_0 is weakly compact can be omitted if the sequence (T_n) is such that $T_n x \subset Tx$ for any x in B and for any n in $\mathbf{N}$; more precisely

THEOREM 8. *Let X be a Banach space, let B be a nonempy weakly closed subset of X. Let $T, T_n : B \to C(X)$ be nonexpansive mappings such that: (1) $T_n x \to Tx$ for any x in B in the sense of the Hausdorff metric; (2) $T_n x \subset Tx$ for any x in B and any n in $\mathbf{N}$. If, for any n in $\mathbf{N}$, x_n is a fixed point of T_n, and $x_n \to x_0$, then x_0 is a fixed point of T.*

Proof. We have

$$d(x_0, Tx_0) \leq \|x_0 - x_n\| + d(x_n, Tx_0) \leq \|x_0 - x_n\| + H(Tx_n, Tx_0)$$

(in fact, for any C, D in $C(X)$ and c in C we have $d(c, D) \leq H(C, D)$). By the fact that T is nonexpansive, we obtain: $d(x_0, Tx_0) = 0$ and so x_0 is a fixed point of T. Such a result is not true if we replace the assumption "$x_n \to x_0$" by "$x_n \rightharpoonup x_0$" (see the example of Remark 5).

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