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SPACE OF DISTRIBUTIONS INDUCED BY CERTAIN FAMILIES OF LINEAR OPERATORS

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Abstract. We show that some basic concepts and elementary results from the theory of distributions developed by L. Schwartz hold for a finite family $\mathcal{L}$ of linear operators, $L_j : D(L_j) \subseteq X \to X$, when X is a Frechet space equipped with a pairing $[\cdot,\cdot] : X \times Y \to \mathbf{K}$ ($\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$) and the family of adjoints, L_j^*, satisfies a certain invariance condition.

Introduction. Given a Frechet space X and a finite family $\mathcal{L}$ of linear operators, $L_j : D(L_j) \subseteq X \to X$, $j = 1, 2, \dots, n$, we can study the problem of "extending adequately" each L_j. In the case $X = L^1_{\text{loc}}(\Omega)$, where $\Omega \subseteq \mathbf{R}^n$ is an open set, L. Schwartz solved this question for the family of partial differential operators $L_j = \partial/\partial x_j$, by developing the theory of distributions.

It is the purpose of this paper to present some general conditions that X and $\mathcal{L}$ can have in order that Schwartz's concepts apply and some of the elementary results of his theory of distributions be valid.

Our working setting is that of a Frechet space X equipped with a "compatible" pairing $[\cdot,\cdot] : X \times Y \to \mathbf{K}$ ($\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$). Then, the adjoint operator L^* of $L : D(L) \subseteq X \to X$ is defined when $D(L)$ is a "test space". Given a family $\mathcal{L}$ of linear operators on X, $L_1, \dots, L_n$, we show that a space of distributions for $\mathcal{L}$ can be developed if each $D(L_j) \subseteq X$ is a test space and if there is a test space $\Psi \subseteq Y$ that is left invariant by each L_j^*, in a way to be precised in (2.1).

This abstract space of distributions is based on a duality notion defined by means of the pairing $(X, Y; [\cdot,\cdot])$ and, in a "weak" sense, solves the extension problem posed for the family $\mathcal{L}$. As in the approach of Gelfand and Shilov [4], distributions are defined as elements of the dual of the test space Ψ, which is endowed with an inductive limit topology. However, in our work the test space (corresponding to Gelfand and Shilov's *fundamental space*) is not arbitrary but constructed from X and $\mathcal{L}$.

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Finally, we must mention Ehrenpreis [3] extended Schwartz methods to certain families of linear operators acting on functions defined on a countable at infinity locally compact space.

1. Notation and terminology. Troughout all this work X is a Frechet space and X' is its dual space. X' together with its strong topology will be called the *strong dual* of X and denoted by X'_s. We employ the notation

$$\langle x, \phi \rangle = \phi(x), \qquad x \in X \text{ and } \phi \in X'$$

On X we will consider fixed an increasing sequence of seminorms, $\{p_l\}$, which generate its topology. Let X_l be the space X together with the locally convex (not necessarily Hausdorff) topology defined by the seminorm p_l. Then

$$X' = \bigcup_l X'_l. \tag{1.1}$$

Proceeding as in the case of normed spaces, it results that

$$\|\phi\|_l = \sup\{|\langle x, \phi\rangle| : p_l(x) \leq 1\}, \qquad \phi \in X'_l,$$

defines a norm on X'_l. Also, under this norm, X'_l is a Banach space and

$$|\langle x, \phi \rangle| \leq p_l(x)\|\phi\|_l, \qquad x \in X,\ \phi \in X'_l. \tag{1.2}$$

LEMMA 1.1. *The following continuous inclusions hold*:

$$X'_1 \hookrightarrow \ldots \hookrightarrow X'_l \hookrightarrow X'_{l+1} \hookrightarrow \ldots \hookrightarrow X'_s.$$

Proof. From condition $p_l \leq p_{l+1}$ it follows that $X_{l+1} \hookrightarrow X_l$, and hence $X'_l \hookrightarrow X'_{l+1}$. Fixing l, we now show that $X'_l \hookrightarrow X'_s$. Let $\{\phi_k\}$ be a sequence in X'_l such that

$$\phi_k \to 0 \qquad \text{in } X'_l. \tag{1.3}$$

Let B be a bounded subset of X. Then, (1.2) and (1.3) imply $\phi_k \to 0$ uniformly on B. This shows that $\phi_k \to 0$ in X'_s.

Let $n \in \mathbf{N}$ and $T = \{T_1, \ldots, T_n\}$ be a family of linear operators in a vector space Z. Given a nonnegative integer l and a set A we will take $A^{(0)} = \{0\}$ and $A^{(l)} = A \times \ldots \times A$ (l times). To describe any composition between members of T we introduce the following notation.

Let $I_n = \{1, \ldots, n\}$. Then, for $\gamma = (\gamma_1, \ldots, \gamma_l) \in I_n^{(l)}$, $l = 0, 1, \ldots$, we define $[\gamma] = l$, $T_\gamma = T_{\gamma_1} \circ \ldots \circ T_{\gamma_l}$, where T_0 is the identity operator in Z. In this context γ will be called a *subindex* and $[\gamma]$ its *length*.

Suppose now that Z is a Banach space, with norm $\|\cdot\|$, and let $m = 1, 2, \ldots, +\infty$. Then, we define $V^m(Z, T)$ to be the locally convex space obtained by considering in $\bigcap_{[\gamma]\leq m} D(T_\gamma)$ the topology determined by the family of seminorms

$$|z|_\gamma = \|T_\gamma(z)\|, \qquad [\gamma] \leq m. \tag{1.4}$$

Notice that $V^m(Z;T)$ is a normed space for $m \in \mathbb{N}$ and $V^{+\infty}(Z;T)$ is metrizable. The following fundamental properities of $V^m(Z;T)$ are clear.

LEMMA 1.2. *Let Z be a Banach space and $T_j : D(T_j) \subseteq Z \to Z$ a linear operator, $j = 1, 2, \dots, n$. Then:*

(i) *a sequence $\{z_k\}$ converges to z in $V^m(Z;T)$ if and only if*

$$T_\gamma z_k \to T_\gamma z \quad in\ Z, \qquad [\gamma] \leq m;$$

(ii) *for any subindex γ, $T_\gamma : V^{+\infty}(Z;T) \to V^{+\infty}(Z;T)$ is continuous;*

(iii) $V^{+\infty}(Z;T) \hookrightarrow V^{m+1}(Z;T) \hookrightarrow V^m(Z;T) \hookrightarrow Z$, $m \in \mathbb{N}$.

In (iii) of the above lemma we employ the notation $E \hookrightarrow F$ to state that $E \subseteq F$ and that the inclusion of E into F is continuous.

Assume Y is a linear space and $[\cdot\,,\cdot] : X \times Y \to$ is a bilinear mapping having the following properties:

(a) If $y \in Y$ and $[x, y] = 0$ for all $x \in X$, then $y = 0$.

(b) If $x \in X$ and $[x, y] = 0$ for all $y \in Y$, then $x = 0$.

(c) For each $y \in Y$, the linear functional Jy is continuous, where

$$\langle x, Jy \rangle = [x, y]. \tag{1.5}$$

Then, we say that $(X, Y; [\cdot\,,\cdot])$ is a P-space.

From the properties of a P-space $(X, Y; [\cdot\,,\cdot])$ and definition (1.5) it follows that $J : Y \to X'$ is an injective linear operator. Since this map allows us to identify Y with the subspace $J(Y) \subseteq X'$, we call it the *canonical identification* of Y into X'.

We will always consider in $Y_l = J^{-1}(X_l')$ the *negative norm*

$$\|y\|_l^- = \|Jy\|_l = \sup\{|[x, y]| : p_l(x) \leq 1\}, \qquad x \in X,\ y \in Y_l. \tag{1.6}$$

Then $J_l = J|Y_l : Y_l \to X_l'$ is an isometry, and

$$|[x, y]| \leq p_l(x)\|y\|_l^-, \qquad x \in X,\ y \in Y_l. \tag{1.7}$$

Let Φ be a vector subspace of X. If $y \in Y$ and $[x, y] = 0$ for all $x \in \Phi$ imply $y = 0$, then Φ is called a *test space* for Y. In a similar manner we define that a vector subspace $\Psi \subseteq Y$ is a test space for X.

Given a P-space $(X, Y; [\cdot\,,\cdot])$ and a linear operator $L : D(L) \to X$ where $D(L) \subseteq X$ is a test space for Y, then we can consider its *adjoint* operator $L^* : D(L^*) \subseteq Y \to Y$, defined by the usual condition

$$[Lx, y] = [x, L^*y], \qquad x \in D(L),\ y \in D(L^*).$$

It is easy to verify that L^* is a well defined linear operator.

Remark 1.1. In the case of complex scalars, we want to point out that all our development is valid if instead of being linear in the second variable, the map $[\cdot,\cdot] : X \times Y \to \mathbf{C}$ is conjugate linear. This applies specially to the case of a complex Hilbert space.

Let $(X, Y; [\cdot,\cdot])$ be a P-space. If a subspace $\Phi \subseteq X$ is dense, then Φ clearly is a test space for Y. Next we see the reciprocal is not true.

Example 1.1. Let X and Y be the Banach spaces $L^\infty(\mathbf{R}^n)$ and $L^1(\mathbf{R}^n)$, respectively. Defining

$$[u,v] = \int_{\mathbf{R}^n} uv\,dx, \qquad u \in L^\infty(\mathbf{R}^n),\ v \in L^1(\mathbf{R}^n),$$

we see that $(X, Y; [\cdot,\cdot])$ is a P-space. Take $\Phi = C_c^\infty(\mathbf{R}^n)$, the space of those functions on $\mathbf{R}^n$ which are indefinitely differentiable and have compact support. Although $C_c^\infty(\mathbf{R}^n)$ is not dense in $L^\infty(\mathbf{R}^n)$, from the du Bois-Reymond lemma [**1**, p. 59] it follows that $C_c^\infty(\mathbf{R}^n)$ is a test space for $L^1(\mathbf{R}^n)$.

2. Space of distributions for linear operators in the class $\mathcal{C}(\Psi)$. Let $(X, Y; [\cdot,\cdot])$ be a P-space and consider a finite family of linear opertors $\mathcal{L} = \{L_1, \dots, L_n\}$. To construct a space of distributions for $\mathcal{L}$, the extension we make of Schwartz methods requires that the adjoint operators $L_1^*, \dots, L_n^*$ be defined and that they leave invariant a test space Ψ for X. This motivates the next definition.

Le Ψ be a test space for X. Given a linear operator $L : D(L) \subseteq X \to X$, we say that L belongs to the class $\mathcal{C}(\Psi)$, and write $L \in \mathcal{C}(\Psi)$, if $D(L)$ is a test space for Y, $\Psi \subseteq D(L^*)$, and

$$L^*(\Psi \cap Y_l) \subseteq \Psi \cap Y_l, \qquad l = 1, 2, \dots, \tag{2.1}$$

where $Y_l = J^{-1}(X_l')$, J being the canonical identification of Y in X'.

Example 2.1. Let $\Omega \subseteq \mathbf{R}^n$ be a nonempty open set. Fix a sequence $\{K_l\}$ of compact sets such that $K_l \subseteq \operatorname{int} K_{l+1}$ and $\Omega = \bigcup_l K_l$. For $u \in L^p_{\text{loc}}(\Omega)$, $1 \le p < \infty$, consider the seminorms

$$p_l(u) = \left\{ \int_{K_l} |u|^p\,dx \right\}^{1/p}.$$

Then, $X = L^p_{\text{loc}}(\Omega)$ is a Frechet space. Next, take $Y = L_c^{p'}(\Omega)$ the space of all those functions in $L^{p'}(\Omega)$ that have compact support contained in Ω where $1/p + 1/p' = 1$. Defining

$$[u,v] = \int uv\,dx, \qquad u \in L^p_{\text{loc}}(\Omega),\ v \in L_c^{p'}(\Omega)$$

we can verify that $(X, Y; [\cdot,\cdot])$ is a P-space. Moreover, the du Bois-Reymond lemma implies that $\Phi = \Psi = C_c^\infty(\Omega)$ is a test space for $L^p_{\text{loc}}(\Omega)$ and $L_c^{p'}(\Omega)$ respectively.

Let $X_l = (X, p_l)$, $l = 1, 2, \dots$, and $Y_l = \{v \in L^{p'}(\Omega) : \operatorname{supp}(v) \subseteq K_l\}$. Then, Y_l is a Banach subpace of $L^{p'}(\Omega)$. Given $v \in Y_l$, it is clear that $Jv \in X_l'$. Using the usual identification betwen $L^p(K_l)'$ and $L^{p'}(K_l)$, it follows that $J: Y_l \to X_l'$ is an isometric isomorphism.

Hence, for $\Psi = C_c^\infty(\Omega)$, we have $\Psi \cap Y_l = \{\varphi \in C_c^\infty(\Omega) : \operatorname{supp}(\varphi) \subseteq K_l\}$.

Returning to our discussion, let $\mathcal{L} = \{L_1, \dots, L_n\}$ be a family of linear operators in the class $\mathcal{C}(\Psi)$. For each $l \in \mathrm{N}$, let $L_{j,l}^*$ be the restriction to $\Psi \cap Y_l$ of L_j^*, and $\mathcal{L}^*(l)$ the family of these restrictions. We now define the metrizable locally convex space Ψ_l, as $\Psi_l = \Psi \cap Y_l$ with the topology inherited by $V^{+\infty}(Y_l; \mathcal{L}^*(l))$. It follows from Lemma 1.1 that $\Psi_l \hookrightarrow \Psi_{l+1}$, and Lemma 1.2 implies

$$\Psi_l \hookrightarrow Y_l, \qquad l = 1, 2, \dots . \tag{2.2}$$

Also, from (1.1), we have $\Psi = \bigcup_l \Psi_l$.

Let us consider in Ψ the inductive limit topology defined by the increasing sequence of locally convex Hausdorff spaces $\Psi_l \hookrightarrow \Psi_{l+1}$. The next result shows that this topology turns Ψ into a locally convex Hausdorff space, which is simply referred to as the *test space*. This test space will be denoted by $\Psi(X; \mathcal{L})$ or simply by Ψ.

LEMMA 2.1. *If $\mathcal{L}$ is a finite family of operators in $\mathcal{C}(\Psi)$, then:* (i) *the canonical identification $J : \Psi(X; \mathcal{L}) \to X_s'$ is continuous;* (ii) *$\Psi(X; \mathcal{L})$ is a Hausdorff space.*

Proof. (i) From the properties of an inductive limit topology, it is sufficient to show that, for each $l \in \mathrm{N}$, $J : \Psi_l \to X_s'$ is continuous (Schwartz [7, p. 20, Cor. A22]. For this, let us note that J can be expressed as the composition

$$\Psi_l \hookrightarrow Y_l \xrightarrow{J_l} X_l' \hookrightarrow X_s'.$$

From (2.2), the definition of $\|\cdot\|_l$, and Lemma 1.1, it follows that the composition is continuous.

(ii) It is immediate from the fact that J is one-to-one, and X_s' is Hausdorff.

Example 3.1 shows that the test space $\Psi(X; \mathcal{L})$ is not necessarily complete. Moreover, applying the theory of inductive limits, we can see that several properties of the spaces Ψ_l are inherited to the test space $\Psi(X; \mathcal{L})$ [2, pp. 6 and 11]. In particular, since each Ψ_l is a locally convex metrizable space, we have the following

PROPOSITION 2.2. *The test space $\Psi(X; \mathcal{L})$ is bornological.*

Given a nonnegative integer m and $\gamma = (\gamma_1, \dots, \gamma_m) \in I_n^{(m)}$ recall the notation $L_\gamma^* = L_{\gamma_1}^* \dots L_{\gamma_m}^*$.

PROPOSITION 2.3. *For each positive integer m and subindex $\gamma \in I_n^{(m)}$, the linear operator $L_\gamma^* : \Psi(X; \mathcal{L}) \to \Psi(X; \mathcal{L})$ is continuous.*

Proof. From the properties of the inductive limit, it suffices to show that each $L_\gamma^* : \Psi_l \to \Psi(X; \mathcal{L})$ is continuous. Since $\Psi_l \hookrightarrow \Psi(X; \mathcal{L})$, from (2.1) it suffices to prove the continuity of each $L_\gamma^* : \Psi_l \to \Psi_l$. But this is immediate from the definition of Ψ_l and Lemma 1.2.

The dual of the test space $\Psi(X;\mathcal{L})$ is now defined to be the space of distributions corresponding to the family $\mathcal{L}$ and the space X.

PROPOSITION 2.4. *The space of distributions $\Psi(X;\mathcal{L})'_s$ is complete.*

Proof. As the strong dual of a bornological space is always complete (Bourbaki, p. 12), the conclusion is immediate from Proposition 2.3.

In the following we represent by $\mathbf{K}$ ($\mathbf{R}$ or $\mathbf{C}$) the field of scalars.

PROPOSITION 2.5. *Let $u : \Psi \to \mathbf{K}$ be a linear functional. Then, u is a distribution if and only if for every $l = 1,2,\dots,$ there exist C_l and $m_l \in \mathbf{N}$, such that*

$$|\langle\psi,u\rangle| \leq C_l \max\{\|L^*_\gamma\psi\|^- : [\gamma] \leq m_l\}, \qquad \psi \in \Psi_l. \tag{2.3}$$

Proof. From the properties of the inductive limit, u is continuous if and only if each restricition $u : \Psi_l \to \mathbf{K}$ is continuous. Since Ψ_l is a locally convex metrizable space, whose topology can be generated by the increasing sequence of norms $\|\psi\|_{\mathcal{L}^*,m} = \max\{\|L^*_\gamma\psi\|^- : [\gamma] \leq m\}$, the continuity of $u : \Psi_l \to \mathbf{K}$ is equivalent to condition (2.3).

Given $x \in X$, we define the linear functional $Ix : \Psi \to \mathbf{K}$ by $\langle\psi, Ix\rangle = [x,\psi]$, $\psi \in \Psi$.

THEOREM 2.6. *The following holds*: (i) *$Ix \in \Psi(X;\mathcal{L})'$, for each $x \in X$;* (ii) *$I : X \to \Psi(X;\mathcal{L})'_s$ is a continuous linear operator.*

Proof. (i) ss an immediate consequence of (1.7) and Proposition 2.6.

(ii) Let $\{x_k\}$ be sequence such that $x_k \to 0$ in X. We must show that $Ix_k \to 0$ in $\Psi(X;\mathcal{L})'_s$. From the definition of strong toplogy this is equivalent to show that $Ix_k \to 0$ uniformly on each bounded subset of $\Psi(X;\mathcal{L})$. Let then B be a bounded subset of $\Psi(X;\mathcal{L})$. From Lemma 2.1, $J(B)$ is a bounded subset of X'_s. Let us pick a sequence $\{r_k\}$ of positive numbers satisfying $r_k \to +\infty$ and $r_k x_k \to 0$ in X. Then, by definition of the strong topology in X'_s, there is a $C > 0$ such that

$$C \geq |\langle r_k x_k, J\psi\rangle| = r_k|[x_k,\psi]| = r_k|\langle\psi, Ix_k\rangle| \qquad (k = 1,2,\dots),$$

for all $\psi \in B$. This implies that $Ix_k \to 0$ uniformly on B.

Since Ψ is a test space for X, $I : X \to \Psi(X;\mathcal{L})'$ is an injective linear operator. This will allow us to identify X with $I(X) \subseteq \Psi(X;\mathcal{L})'$, and so we call I the canonical identification of X in $\Psi(X;\mathcal{L})'$.

Let $\Psi \subseteq Y$ be a test space and $L \in \mathcal{C}(\Psi)$. Then, the *maximal closed extension* of L, $\boldsymbol{L} : D(\boldsymbol{L}) \subseteq X \to X$, is defined by the condition

$$[\boldsymbol{L}x,\psi] = [x, L^*\psi], \qquad x \in D(\boldsymbol{L}),\ \psi \in \Psi.$$

The basic properties of $\boldsymbol{L}$ are summarized in the following lemma, which can be rapidly verified.

LEMMA 2.7. *If $L \in \dot{C}(\Psi)$, then:* (i) *L is a closed linear operator;* (ii) *L is closable and $\tilde{L} \subseteq \mathbf{L}$, where $\tilde{L}$ is the (usual) minimal closed extension of L.*

THEOREM 2.8. *Let $\mathcal{L} = \{L_1, \dots, L_n\}$ be a family of linear operators in $C(\Psi)$. If the identification $I : X \to \Psi(X;\mathcal{L})'$ is onto, then each of the linear operators $L_1, \dots, L_n$ is continuous.*

Proof. Let $x \in X$. From propositions 2.4 and 2.7 it follows that the linear functional $u : \Psi(X;\mathcal{L}) \to \mathbf{K}$, given by $\langle \psi, u\rangle = \langle L_j^* \psi, Ix\rangle$, $\psi \in \Psi$, is continuous. By our hypothesis, there exists a $z \in X$ such that

$$\langle L_j^* \psi, Ix\rangle = \langle \psi, Iz\rangle, \quad \psi \in \Psi \quad \text{or equivalently,} \quad [x, L_j^* \psi] = [z, \psi], \quad \psi \in \Psi.$$

This shows that $x \in D(\mathbf{L})$ and $\mathbf{L}_j x = z$.

Since the linear operator $\mathbf{L}$ is closed and $D(\mathbf{L}_j) = X$, the conclusion is obtained by applying the closed graph theorem.

From the condition (2.1) it is easy to check that for $\mathcal{L} = \{L_1, \dots, L_n\} \subseteq C(\Psi)$, and $\gamma = (\gamma_1, \dots, \gamma_m) \in I_n^{(m)}$ we have

$$[L_\gamma x, y] = [x, L_{\gamma_{op}}^* y], \qquad x \in D(L_\gamma),\ y \in \Psi. \tag{2.4}$$

where $\gamma_{op} = (\gamma_m, \dots, \gamma_1)$.

If $u \in \Psi'$, (2.4) motivates to define the linear functional $L_\gamma u : \Psi \to \mathbf{K}$, in the *weak sense* (or in the *sense of distributions*), as

$$\langle \psi, L_\gamma u\rangle = \langle L_{\gamma_{op}}^* \psi, u\rangle, \qquad \psi \in \Psi. \tag{2.5}$$

It follows from Proposition 2.4 that $L_\gamma u \in \Psi(X;\mathcal{L})'$.

If $x \in X$, then Theorem 2.7 implies that $Ix \in \Psi(X;\mathcal{L})'$. In this case we write $L_\gamma x$ instead of $L_\gamma Ix$, and we interpret in the weak sense. Thus

$$\langle \psi, L_\gamma x\rangle = [x, L_{\gamma_{op}}^* \psi], \qquad \psi \in \Psi. \tag{2.6}$$

It follows from (ii) of the next proposition that, via the identification $x \to Ix$, L_γ is an extension of $\mathbf{L}_\gamma$, which will be called the *weak extension*.

Proceeding directly from the respective definitions, we obtain

PROPOSITION 2.9. (i) *The weak extension $L_\gamma : \Psi(X;\mathcal{L})' \to \Psi(X;\mathcal{L})'$ is the dual operator of $L_{\gamma_{op}}^* : \Psi(X;\mathcal{L}) \to \Psi(X;\mathcal{L})$.*

(ii) *If $x \in D(\mathbf{L}_\gamma)$, then $L_\gamma Ix = I\mathbf{L}_\gamma x$, where $\mathbf{L}_\gamma$ is the maximal closed extension of L_γ.*

(iii) *If $\gamma = (\gamma_1, \dots, \gamma_m) \in I_n^{(m)}$, then $L_\gamma = L_{\gamma_1} \circ \dots \circ L_{\gamma_m}$ in the weak sense.*

Let $u \in \Psi(X;\mathcal{L})'$ and $\gamma \in I_n(m)$. We write $L_\gamma u \in X$, if there exists a $z \in X$ such that $L_\gamma u = Iz$. This is equivalent to the condition $\langle L_{\gamma_{op}}^* \psi, u\rangle = [z, \psi]$, $\psi \in \Psi$. If this is the case, we write $L_\gamma u = z$.

LEMMA 2.10. *If $x \in X$ and $L_j x \in X$, where L_j is the weak extension, then $x \in D(\mathbf{L}_j)$ and $L_j x = \mathbf{L}_j x$.*

Proof. From (2.6) we have $L_j x \in X$ if and only if there is a $z \in X$ such that $[x, L_j^* \psi] = [z, \psi]$, $\psi \in \Psi$. This means precisely that $x \in D(\mathbf{L}_j)$ and $\mathbf{L}_j x = z = L_j x$.

Example 2.2. Let us continue in the context of Example 2.1 with $p = 1$. There we have shown that $Y_l = J^{-1}(X_l') = \{v \in L^\infty(\Omega) : \operatorname{supp}(v) \subseteq K_l\}$, and the norm in Y_l is the L^∞-norm. Thus, for $\Psi = C_c^\infty(\Omega)$, we have $\Psi \cap Y_l = \{\psi \in C_c^\infty(\Omega) : \operatorname{supp}(\psi) \subseteq K_l\}$. From this it follows immediately that the family of operators

$$\partial = \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\},$$

belongs to the class $\mathcal{C}(C_c^\infty(\Omega))$.

Notice now that the convergence $\psi_k \to \psi$ in Ψ_l is equivalent to $\partial^\alpha \psi_k \to \partial^\alpha \psi$ uniformly on the compact set K_l, for every multi-index α. Therefore

$$\Psi(L^1_{\text{loc}}(\Omega); \partial) = \mathcal{D}(\Omega),$$

the space of test functions on Ω with the Schwartz topology; and

$$\Psi(L^1_{\text{loc}}(\Omega); \partial)' = \mathcal{D}(\Omega)',$$

the space of distributions on Ω.

Going back to our initial context, let $\mathcal{L} = \{L_1, \dots, L_n\}$ be a family of linear operators in the class $\mathcal{C}(\Psi)$. Take $L : D(L) \subseteq X \to X$, of the form

$$L = \sum_{|\gamma| \leq m} a_\gamma L_\gamma, \tag{2.7}$$

where $a_\gamma \in \mathbf{K}$. Note that $L \in \mathcal{C}(\Psi)$.

PROPOSITION 2.11. *Let L be an operator as in* (2.7). *Then:*
(i) $\Psi(X; \mathcal{L}) \hookrightarrow \Psi(X; L)$, (ii) $\Psi(X; L)_s' \hookrightarrow \Psi(X; \mathcal{L})_s'$.

Proof. (i) Being $\Psi(X; \mathcal{L})$ the inductive limit of the increasing sequence of spaces $\{\Psi_l(X; \mathcal{L})\}$, it is enough to show that $\Psi_l(X; \mathcal{L}) \hookrightarrow \Psi(X; L)$. But since $\Psi_l(X; L) \hookrightarrow \Psi(X; L)$, then it is sufficient to estabilish the continuity for each one of the inclusions $\Psi_l(X; \mathcal{L}) \to \Psi_l(X; L)$.

So let $\{y_k\}$ be a sequence such that $y_k \to 0$ in $\Psi_l(X; \mathcal{L})$. Then

$$L_\gamma^* y_k \to 0 \ \text{ in } \Psi_l(X; \mathcal{L}), \text{ for every subindex } \gamma. \tag{2.8}$$

Taking into account that $(L^*)^m$, $m = 0, 1, \dots$, is a linear combination of operators of the form L_γ^*, from (2.8) it follows that

$$(L^*)^m y_k \to 0 \quad \text{in } \Psi_l(X; \mathcal{L}), \quad m = 0, 1, \dots ;$$

hence $y_k \to 0$ in $\Psi_l(X; L)$.

(ii) It is immediate from (i).

3. The normed case. In this section we want to make some remarks concerning the space of distributions $\Psi(X;\mathcal{L})'$, when X is a Banach space.

Let $(X,Y;[\cdot,\cdot])$ be a P-space, where X is a Banach space with norm $\|\cdot\|$. In Y we consider the negative norm $\|\cdot\|^-$ given by (1.6). Let Ψ be a test space for X. Condition (2.1) takes now the simpler form

$$L_j^*(\Psi) \subseteq \Psi, \qquad j = 1,\dots,n. \tag{3.1}$$

Let $\mathcal{L} = \{L_1,\dots,L_n\}$ be a family of operators in the class $\mathcal{C}(\Psi)$. It follows readily that each $L_j^* : D(L_j^*) \subseteq Y \to Y$ is a closed operator. Next, denoting by $\mathcal{L}^*$ the family of these closed operators, we form the family of spaces $V^m(Y;\mathcal{L}^*)$, $m = 0,1,\dots,+\infty$. According to (1.4), we consider on $V^m(Y;\mathcal{L}^*)$, $m < +\infty$, the norm

$$\|y\|_{\mathcal{L}^*,m} = \max\{\|L_\gamma^* y\|^- : [\gamma] \leq m\}.$$

Then $\|y\|_{\mathcal{L}^*,m} \leq \|y\|_{\mathcal{L}^*,m+1}$, $y \in V^{m+1}(Y;\mathcal{L}^*)$.

From the condition (3.1) we have $\Psi \subseteq V^m(Y;\mathcal{L}^*)$, $m = 0,1,\dots,+\infty$; so we can define for $m = 0,1,\dots,+\infty$,

$$V_0^m(Y;\mathcal{L}^*) = \text{closure of } \Psi \text{ in } V^m(Y;\mathcal{L}^*).$$

In this normed case, the test space $\Psi \subseteq V^{+\infty}(Y;\mathcal{L}^*)$ is metrizable, and its topology is generated by the increasing sequence of norms

$$\{\|\cdot\|_{\mathcal{L}^*,m} : m = 0,1,\dots\}.$$

Since Ψ is dense in $V_0^{+\infty}(Y;\mathcal{L}^*)$, from the Hahn-Banach theorem we have

$$\Psi(X;\mathcal{L})' = V_0^{+\infty}(Y;\mathcal{L}^*)'. \tag{3.2}$$

Remark 3.1. If the locally convex metrizable space $V_0^{+\infty}(Y;\mathcal{L}^*)$ is separable (this will happen if Y is separable), then

$$\Psi(X;\mathcal{L})'_s = V_0^{+\infty}(Y;\mathcal{L}^*)'_s.$$

(Grothendieck, p. 62, Corollary 4).

To end this work we want to show that our approach also yields the space of tempered distributions.

Example 3.1. Consider the Banach space $X = L^1(\mathbf{R}^n)$ and let $Y = L^\infty(\mathbf{R}^n)$. Taking

$$[u,v] = \int_{\mathbf{R}^n} uv\,dx, \qquad u \in L^1(\mathbf{R}^n),\ v \in L^\infty(\mathbf{R}^n),$$

it is clear that $(X,Y;[\cdot,\cdot])$ is a P-space. By the du Bois-Reymond lemma, $\Psi = C_c^\infty(\mathbf{R}^n)$ is a test space for $L^1(\mathbf{R}^n)$. Take now the family

$$\mathcal{L} = \left\{\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}, x_1,\dots,x_n\right\},$$

where x_j represents the multiplication by the monomial x_j and note that $\mathcal{L} \subseteq \mathcal{C}(\Psi)$.

PROPOSITION 3.1. *The topology of the test space* $\Psi(L^1(\mathbf{R}^n);\mathcal{L})$ *is the topology* $\Psi = C_c^\infty(\mathbf{R}^n)$ *as a subspace of the Schwartz space* $\mathcal{S}(\mathbf{R}^n)$.

Proof. Let us denote by $\Psi_{\mathcal{S}}$, the space Ψ together with the topology induced as a subspace of the Schwartz space $\mathcal{S}(\mathbf{R}^n)$. Then, the topology of Ψ is generated by the family of seminorms

$$\|\psi\|_{\alpha,\beta} = \sup\{|x^\alpha \partial^\beta \psi(x)| : x \in \mathbf{R}^n\},$$

where α and β are arbitrary multi-indices. On the other hand, the topology of $\Psi = \Psi(L^1(\mathbf{R}^n);\mathcal{L})$ is generated by the family of norms

$$\|\psi\|_m = \max\{\|L_\gamma^*\psi\|_{L^\infty(\mathbf{R}^n)} : [\gamma] \le m\},$$

where

$$\mathcal{L}^* = \left\{-\frac{\partial}{\partial x_1}, \dots, -\frac{\partial}{\partial x_n}, x_1, \dots, x_n\right\}.$$

It is now clear that $\Psi \hookrightarrow \Psi_{\mathcal{S}}$. The other continuous inclusion, $\Psi_{\mathcal{S}} \hookrightarrow \Psi$, is established just by observing that for any subindex γ, $L_\gamma^*\psi$ can be expressed as a sum of terms of the form $x^\alpha \partial^\beta \psi$.

COROLLARY 3.2. (i) $\Psi(L^1(\mathbf{R}^n);\mathcal{L})'$ *is the space of tempered distributions* $\mathcal{S}(\mathbf{R}^n)'$.

(ii) $\Psi(L^1(\mathbf{R}^n);\mathcal{L})'_s = \mathcal{S}(\mathbf{R}^n)'_s$.

Proof. (i) Immediate from the Hahn-Banach theorem, since $C_c^\infty(\mathbf{R}^n)$ is dense in $\mathcal{S}(\mathbf{R}^n)$.

(ii) Since $\mathcal{S}(\mathbf{R}^n)$ has the Heine-Borel property, it is separable (Gelfand-Shilov, p. 58). The result follows from Remark 3.1.

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