PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 49 (63), 1991, 117–122

$\mathcal{K}$-CONVERGENCE AND THE ORLICZ-PETTIS THEOREM

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Abstract. A sequence $\{x_k\}$ in a topological vector space (X, τ) is said to be τ-$\mathcal{K}$ convergent if every subsequence of $\{x_k\}$ has a subsequence $\{x_{n_k}\}$ such that the subseries $\sum x_{n_k}$ is τ-convergent in X. We show that the notion of $\mathcal{K}$ convergence can be used to give a generalization of the Orlicz-Pettis Theorem in the following sense. Let α and β be vector topologies on a vector space X such that $\alpha \subseteq \beta$ and β has a neighborhood base at 0 of α-closed sets. If every α-$\mathcal{K}$ convergent sequence is β-convergent to 0, then every series $\sum x_k$ in X which is α subseries convergent is β subseries convergent. Thus, any statement of the form "α-$\mathcal{K}$ convergence implies β convergence to 0" along with the appropriate accompanying hypothesis implies an Orlicz-Pettis Theorem for α and β. We estabilish such results in a number of different situations.

The classical Orlicz-Pettis Theorem concerning subseries convergence in the weak and norm topologies of a normed linear space has proven to be a very useful result with applications to many situations in measure and integration theory and the geometric theory of B-spaces [3]. In this paper we show that results concerning the $\mathcal{K}$ convergence of sequences introduced by Antosik and Mikusinski can be used to deduce Orlicz-Pettis type results. We then establish several results concerning $\mathcal{K}$ convergent sequences in various settings.

If (G, β) is a topological group, then a (formal) series $\sum x_k$ in G is said to be β-subseries (β-s.s.) convergent if for every subsequence $\{x_{n_k}\}$, the subseries $\sum x_{n_k}$ is β-convergent in G. A sequence $\{x_k\}$ in G is said to be β-$\mathcal{K}$ convergent (or $\mathcal{K}$ convergent with respect to β) if every subsequence of $\{x_k\}$ has a subsequence $\{x_{n_k}\}$ such that the subseries $\sum x_{n_k}$ is β-convergent in G [1, 3.1]. A sequence $\{x_k\}$ such that the series $\sum x_k$ is β-s.s. convergent is clearly β-$\mathcal{K}$ convergent, but the converse does not hold (consider $x_k = 1/k$ in $\mathbf{R}$). Also, a β-$\mathcal{K}$ convergent sequence is β convergent to 0, but the converse does not hold in general [2, 3.3].

In what follows let G be an abelian group with two Hausdorff group topologies α and β on G with $\alpha \subseteq \beta$. A general Orlicz-Pettis Theorem for G is a result which asserts that any α-s.s. convergent series is β-s.s. convergent; for example, the classical Orlicz-Pettis Theorem for normed spaces asserts that any weak-s.s. convergent series is norm-s.s. convergent ([9], [10], [1], [3]). We first show that a

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AMS Subject Classification (1985): Primary 46 A 05

corresponding result concerning $\mathcal{K}$ convergence of sequences in the two topologies will yield an Orlicz-Pettis result as a corollary, and, therefore, such a result can be considerd to be a strengthening of the Orlicz-Pettis Theorem.

LEMMA 1. *A sequence $\{u_k\}$ is β-Cauchy if and only if for every pair of increasing sequences of positive integers $\{p_j\}$, $\{q_j\}$, with $p_j < q_j < p_{j+1}$, $\beta\text{-}\lim(u_{q_j} - u_{p_j}) = 0$.*

THEOREM 2. *Suppose $\alpha \subseteq \beta$ and (G, β) has a neighborhood basis at 0 consisting of α-closed sets (β is F linked to α in Wilansky's terminology* [**13**, 6.1.9.]). *If every α-$\mathcal{K}$ convergent sequence is β-convergent to 0, then every series $\sum x_k$ in G which is α-s.s. convergent is β-s.s. convergent.*

Proof. Let $\sum x_k$ be α-s.s. convergent and let $\{x_{n_k}\}$ be a subsequence of $\{x_k\}$. Set $s_k = \sum_{j=1}^{k} x_{n_j}$. We claim that $\{s_k\}$ is β-Cauchy. Let $\{p_j\}$ and $\{q_j\}$ be as in Lemma 1. By the lemma it suffices to show that

$$z_j = s_{q_j} - s_{p_j} = \sum_{i=p_j+1}^{q_j} x_{n_i}$$

is β-convergent to 0. But, $\{z_j\}$ is α-$\mathcal{K}$ convergent since $\sum z_j$ is α-s.s. convergent being a subseries of $\sum x_j$. Therefore, by hypothesis, $\{z_j\}$ is β-convergent to 0.

If $\{s_k\}$ is α-convergent to $x \in G$, since β is F linked to α it follows that $\{s_k\}$ is β-convergent to x [**13**, 6.1.11].

It follows from Theorem 2 that any result of the form, "$\{x_k\}$ is α-$\mathcal{K}$ convergent implies that $\{x_k\}$ is β-convergent to 0", immediately implies an Orlicz-Pettis result for the topologies α and β, that is, any series which is α-s.s. convergent is β-s.s. convergent. Thus, any result of this form can be regarded as a strengthening of the Orlicz-Pettis Theorem. We consider establishing this result for several diverse situations. As a first example, it was shown in [**1**, 3.7] that if X is a normed space, then any weak-$\mathcal{K}$ convergent sequence is norm convergent to 0. From this result, we then obtain from Theorem 2 the classical Orlicz-Pettis Theorem for normed spaces ([**9**], [**10**]). We can also obtain the analogus result for locally convex spaces.

THEOREM 3. *Let (E, τ) be a Hausdorff, locally convex tvs. If $\{x_k\}$ is $\sigma(E, E')$-$\mathcal{K}$ convergent, then $\tau\text{-}\lim x_k = 0$.*

Proof. By replacing E by span$\{x_k\}$, we may assume that E is separable. To show $\tau\text{-}\lim x_k = 0$, it suffices to show that $\sup\{|\langle x', x_k\rangle| : x' \in U^\circ\} \to 0$, where U is a τ-neighborhood of 0 in E [**8**, 21.3(2)]. For this it suffices to show that $\langle x'_k, x_k\rangle \to 0$ for every $\{x'_k\} \subseteq U^\circ$. Now U° with the weak* topology, $\sigma(E', E)$, is weak* compact by the Banach-Alaoglu Theorem [**8**, 20.9(4)], and is metrizable since E is separable [**8**, 21.3.(4)]. Therefore, $\{x'_k\}$ has a subsequence $\{x'_{n_k}\}$ which is weak* convergent to some $x' \in U^\circ$.

Now consider the matrix $M = [\langle x'_{n_i}, x_{n_j}\rangle]$. In the terminology of [**1**], M is a $\mathcal{K}$-matrix so by the Basic Matrix Theorem 2.2 of [**1**], it follows that $\langle x'_{n_k}, x_{n_k}\rangle \to 0$.

Since this argument can be applied to any subsequence of $\{\langle x'_k, x_k\rangle\}$, it follows that $\langle x'_k, x_k\rangle \to 0$, and the proof is complete.

As an immediate corollary of Theorems 2 and 3, we obtain the locally convex version of the Orlicz-Pettis Theorem [6].

We next consider the weak* topology on the dual of a locally convex space. We first have the following interesting result which gives an improvement of Theorem 3.6 of [1].

THEOREM 4. *Let E be a locally convex tvs such that $(E', \sigma(E', E))$ is a Banach-Mackey space* [**13**, 10.4.3]. *If $\{x'_k\}$ is $\sigma(E', E)$-$\mathcal{K}$ convergent, then $\{x'_k\}$ is weakly convergent to* 0.

Proof. If $\{x'_k\}$ is not weakly convergent to 0, we may assume that there exist $x'' \in E''$ and $\delta > 0$ such that $\langle x'', x'_k\rangle > \delta$ for all k. Then there exists a subsequence $\{x'_{n_k}\}$ such that the series $\sum x'_{n_k}$ is $\sigma(E', E)$ convergent so the partial sums $\left\{\sum_{j=1}^k x'_{n_j}\right\}$ are $\sigma(E', E)$ bounded and, therefore, $\beta(E', E)$ bounded since $(E', \sigma(E', E))$ is a Banach-Mackey space. But then $\left\{\sum_{j=1}^k x'_{n_j}\right\}$ is also $\sigma(E', E'')$ bounded by Mackey's Theorem [**13**, 8.4.1]. However, $\left\langle x'', \sum_{j=1}^m x'_{n_j}\right\rangle \geq \delta m$ for each m implies that $\left\{\sum_{j=1}^m x'_{n_j}\right\}$ is not weakly bounded. This contradiction establishes the result.

If E is a barrelled space, then $(E', \sigma(E', E))$ is a Banach-Mackey space so Theorem 4 is applicable in this case [**13**, 10.4.14].

Note that we cannot obtain an Orlicz-Pettis Theorem for the weak* topology form Theorem 2 since the weak topology is not F linked to the weak* topology in general. Indeed, let e_k be the unit vector in l^∞ with a 1 in the k-th coordinate and 0 elsewhere. The series $\sum e_k$ is weak*-s.s. convergent to the sequence e which has a 1 in each coordinate, the sequence of partial sums $s_n = \sum_{k=1}^n e_k$ is weak Cauchy but is not weakly convergent to e.

The basic Orlicz-Pettis Theorem for the weak* topology on the dual of a B-space is a result due to Diestel and Faires [4]: if X is a B-space, then every weak*-s.s. convergent series in X' is norm s.s. convergent if and only if X' contains no subspace isomorphic to l^∞. We next establish the analogue of this result for $\mathcal{K}$ convergent sequences.

THEOREM 5. *Let X be a B-space whose dual X' contains no subspace isomorphic to l^∞. If $\{x'_k\} \subseteq X'$ is weak* $\mathcal{K}$-convergent, then $\|x'_k\| \to 0$ [so $\{x'_k\}$ is $\|\ \|$-$\mathcal{K}$ convergent to 0 since X' is complete].*

Proof. Suppose $\{\|x'_k\|\}$ doesn't converge to 0. Then we may assume that $\|x'_k\| \geq \delta > 0$ for each k. By Theorem 4 $\{x'_k\}$ is weakly convergent to 0 so $\{x'_k/\|x'_k\|\}$ also converges to 0 since

$$|\langle x'', x'_k/\|x'_k\|\rangle| \leq |\langle x'', x'_k\rangle|/\delta \to 0$$

for every $x'' \in X''$.

We next claim that X' contains a subspace isomorphic to c_0. Otherwise, $\{x'_k/\|x'_k\|\} = \{y'_k\}$ has no subsequence equivalent to the unit vector base $\{e_k\}$ of c_0. By Elton's Theorem [3, p. 253], $\{y'_k\}$ has a subsequence $\{y'_{n_k}\}$ such that if $\{y'_{n_{k_j}}\}$ is an arbitrary subsequence, then

$$\lim_m \left\| \sum_{j=1}^m x'_{n_{k_j}} \right\| = \lim_m \left\| \sum_{j=1}^m \|x'_{n_{k_j}}\| y'_{n_{k_j}} \right\| = \infty$$

since $\{\|x'_{n_{k_j}}\|\} \notin c_0$. Hence, no subsequence of $\{x'_{n_k}\}$ is weak*-$\mathcal{K}$ convergent. It follows that X' contains a subspace isomorphic to c_0. But, then the theorem of Bessaga and Pelcynski [3, V. 10] implies that X' contains a subspace isomorphic to l^∞, and the result follows.

The proof of Theorem 5 is much more complicated than the proof of the Diestel-Faires result on weak*-s.s. convergent series (see, for example [1, 10.10]), using Elton's Theorem and the Bessage-Pelcynski result. However, the techniques used in the proof of the Diestel-Faires result don't seem to carry over to the $\mathcal{K}$ convergent case. It would be desirable to have a simpler proof of Theorem 5 which does not require the use of so much heavy machinery.

We next consider $\mathcal{K}$-convergence for continuous linear operators between normed spaces. Let X and Y be normed linear spaces and $L(X,Y)$ $(K(X,Y))$ the space of all continuous (compact) linear operators from X into Y. The weak operator topology (strong operator topology) on $L(X,Y)$ is the locally convex topology generated by the semi-norms $T \to |\langle y', Tx\rangle|$, $y' \in Y'$, $x \in X$ $(T \to \|Tx\|$, $x \in X)$. We have the following elementary observation.

PROPOSITION 6. *If $\{T_k\} \subseteq L(X,Y)$ is $\mathcal{K}$ convergent with respect to the weak operator topology, then $\{T_k\}$ is $\mathcal{K}$ convergent with respect to the strong operator topology.*

Proof. If a subseries $\sum T_{n_k}$ is convergent in the weak operator topology, then for each $x \in X$ the series $\sum T_{n_k} x$ is weakly convergent. Thus, for each $x \in X$, $\{T_k x\}$ is weakly $\mathcal{K}$ convergent in Y and, therefore, norm $\mathcal{K}$ convergent in Y [1, 3.8].

Proposition 6 cannot be improved to a statement concerning the norm topology of $L(X,Y)$. For example, define $T_k : c_0 \to c_0$ by $T_k(\{t_j\}) = t_k e_k$. Then for each $x = \{t_j\}$, the series $\sum T_k x$ is s.s. convergent in c_0, but for each finite subset $\sigma \subseteq \mathbf{N}$, $\left\|\sum_{k\in\sigma} T_k\right\| = 1$. Hence, $\{T_k\}$ is $\mathcal{K}$ convergent with respect to the strong operator topology but not norm $\mathcal{K}$ convergent.

However, for the space of compact operators, we have the following result which generalizes a theorem of Kalton for subseries convergence in $K(X,Y)$ ([7], [1, 7.5]).

THEOREM 7. *Let X be a B-space and suppose that X' contains no subspace isomorphic to l^∞. If $\{T_k\}$ is $\mathcal{K}$ convergent in $K(X,Y)$ with respect to the weak operator topology, then $\|T_k\| \to 0$. Consequently, $\{T_k\}$ is norm-$\mathcal{K}$ convergent.*

Proof. First, since each T_k has separable range, we may assume that Y is separable.

Next, observe that if the series $\sum_k T_{n_k}$ is convergent in $K(X,Y)$ with respect to the weak operator topology, then for each $y' \in Y$, $x \in X$,

$$(1) \quad \left\langle y', \left(\sum_k T_{n_k}\right) x\right\rangle = \sum_k \langle y', T_{n_k} x\rangle = \sum_k \langle T'_{n_k} y', x\rangle = \left\langle \left(\sum_k T_{n_k}\right)' y', x\right\rangle$$

and $\sum_k T'_{n_k} y'$ is weak* convergent to $(\sum_k T_{n_k})' y'$ for each $y' \in Y'$. Therefore, if $\{T_k\}$ is $\mathcal{K}$ convergent with respect to the weak operator topology, then for each $y' \in Y'$ $\{T'_k y'\}$ is weak*-$\mathcal{K}$ convergent in X', and since X' contains no subspace isomorphic to l^∞,

$$(2) \qquad \|T'_k y'\| \to 0 \quad \text{for each } y' \in Y' \qquad \text{(Theorem 5).}$$

For each k pick $y'_k \in Y'$, $\|y'_k\| = 1$, such that

$$\|T'_k y'_k\| + 1/k \geq \|T'_k\| = \|T_k\|.$$

To show $\|T_k\| \to 0$, it suffices to show that there is a subsequence such that $\|T'_{n_k} y'_{n_k}\| \to 0$ since we can apply this statement to any arbitrary subsequence of $\{T_k\}$. By the separability of Y and the Banach-Alaoglu Theorem, we may assume, by passing to a subsequence if necessary, that $\{y'_k\}$ is weak* convergent to some $y' \in Y$.

Consider the matrix $M = [T'_j(y'_i - y')]$. By the observation above, $\lim_j \|T'_j(y'_i - y')\| = 0$ for each i. By the compactness of each T_j, $\lim_j \|T'_j(y'_i - y')\| = 0$ [5, VI.5.6]. Therefore, there is an increasing sequence of positive integers $\{m_j\}$ such that $\|T'_{m_j}(y'_{m_i} - y')\| < 2^{-i-j}$ for $i \neq j$; for convenience of notation, we assume $m_i = i$. Since $\{T_j\}$ is $\mathcal{K}$ convergent in $K(X,Y)$ with respect to the weak operator topology, there is a subsequence $\{n_j\}$ such that the subseries $\sum T_{n_j}$ is convergent to a compact operator T with respect to the weak operator topology. Using (1), we have

$$\begin{aligned} \|T'_{n_i} y'_{n_i}\| &\leq \|T'_{n_i}(y'_{n_i} - y')\| + \|T'_{n_i} y'\| \\ &\leq \left\|\sum_{j=1}^{\infty} T'_{n_j}(y'_{n_i} - y')\right\| + \left\|\sum_{j=1, j\neq i}^{\infty} T'_{n_j}(y'_{n_i} - y')\right\| + \|T'_{n_i} y'\| \\ &< \|T'(y'_{n_i} - y')\| + \sum_{j=1}^{\infty} 2^{-i-j} + \|T'_{n_i} y'\|. \end{aligned}$$

The first term on the right side of (3) goes to 0 by the compactness of T [5, VI.5.6], the second term is 2^{-i}, and the third term goes to 0 by (2). Hence, $\|T'_{n_i} y'_{n_i}\| \to 0$, and the proof is complete.

Kalton's Theorem [7] is an immediate corollary of Theorem 7.

We can also establish several $\mathcal{K}$ convergence results for the topology of pointwise convergence in certain function spaces. For these results let G be a metric

Abelian topological group. Let S be a compact Hausdorff space and let $C_G(S)$ be the space of all continuous functions $f : S \to G$. Let $|f|_\infty = \sup\{|f(t)| : t \in S\}$, where $|\ |$ is a quasi-norm generating the topology of G. We have the following results.

THEOREM 8. (i) *If $\{f_k\} \subseteq C_G(S)$ is $\mathcal{K}$ convergent with respect to the topology of pointwise convergence in $C_G(S)$, then $|f_k|_\infty \to 0$.*

(ii) *If S is metrizable, $D \subseteq S$ is a dense subset of S and if $\{f_k\}$ is $\mathcal{K}$ convergent with respect to the topology of pointwise convergence on D, then $|f_k|_\infty \to 0$.*

Theorem 8 generalizes Theorems 7.6 and 7.7 of [1]. Theorem 8 can be proved by the same matrix methods employed in [1] so we do not repeat the proofs.

Let $0 < p < \infty$ and let $l^p(G)$ be all G-valued sequences $\{x_k\} = f$ with $\sum |x_k|^p < \infty$. Define a quasi-norm $|\ |_p$ on $l^p(G)$ by $|f|_p = \sum |x_k|^p$ if $0 < p < 1$ and $|f|_p = \left(\sum |x_k|^p\right)^{1/p}$ if $1 \leq p < \infty$. For $l^p(G)$, we have

TEOREM 9. *If $\{f_k\} \subseteq l^p(G)$ is $\mathcal{K}$ convergent with respect to the topology of pointwise convergence on G, then $|f_k|_p \to 0$.*

This result generalizes Theorem 7.9 of [1], and again since the proof is similar, it is omitted.

Finally, an Orlicz-Pettis Theorem of Stiles for series in an F-space (not necessarily convex) with a Schauder basis ([11]; see also [2]) can be improved to a statement about $\mathcal{K}$ convergence [12].

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(Received 07 04 1989)