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SOME FIXED POINT THEOREMS AND THEIR APPLICATIONS TO BEST SIMULTANEOUS APPROXIMATIONS

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Abstract. Rao and Mariadoss obtained certain results for T-invariant points in a set of K-approximants for a point x in X, a normed linear space, where K is a subset of X. In this paper we extend such results in the context of operators acting on a set of simultaneous K-approximants for a pair of arbitrary but fixed elements x_1 and x_2 in X. By applying a result on common fixed points for a pair of mappings T_1 and T_2 we derive another result on T-invariant point for a set of simultaneous K-approximants for a pair of elements x_1 and x_2 in X.

Introduction. Let X be a real normed linear space and x an element of X, not in the closure of K, where K is a subset of X. The set of best-K-approximants to x consists of those $g_0 \in K$ satisfying

$$|x - g_0| = \inf\{|x - g| : g \in K\}$$

and it is denoted by $P_K(x)$. Let x_1 and x_2 be two elements of X; then the set of best simultaneous K-approximants to x_1, x_2 consists of those $g_0 \in K$ satisfying

$$|x_1 - g_0| + |x_2 - g_0| = \inf\{|x_1 - g| + |x_2 - g| : g \in K\}$$

and it is denoted by $R_K(x_1, x_2)$. For some of important and basic results for best K-approximants to an element $x \in X$, where X is a normed linear space, the reader is referred to Singer [9]. We shall quote some results from [9] to elaborate this concept slightly more than what is being required in the present work. For the notion of best simultaneous K-approximants for a pair $x_1, x_2 \in X$, where X is a normed linear space and some of the results concerned with this notion one can refer to Muthukumar [5]. Narang and Khanna [7] extend the results of [5] to the setting of metric linear spaces. An earlier work of Narang and Ahuja [6] also deals with best simultaneous approximations.

A set K in a normed linear space X is said to be "proximinal" if $P_K(x) \neq \varnothing$ for all $x \in X \setminus K$. It is said to be "semi-Chebyshev" if $P_K(x)$ happens to be utmost

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AMS Subject Classification (1985): Primary 41 A 65 a singleton set for all $x \in X \setminus K$. In case $P_K(x)$ is a singleton set, for all $x \in X \setminus K$ then set K is said to be "Chebysev". In Singer [9] a more interesting case for the situation when K is a linear subspace of X is characterized as follows:

All linear subspaces of X are semi-Chebyshev subspaces if and only if X is strictly convex.

As a corollary we get the result:

All closed linear subspaces of X are Chebyshev subspaces iff X is reflexive and strictly convex.

The characterization as quoted in the latter part in the preceeding paragraph establishes the existence and uniqueness of best K-approximant of an element $x \in X \setminus K$ where K is a subspace of a normed linear space X. On the score of best-simultaneous approximation the work of Muthukumar established the relationship between best approximation and best-simultaneous approximation in a normed linear space in [5]. By suitably defining $M^{\perp}$, the set of elements orthogonal to the elements of M, a subspace of a metric linear space X, the following was proved by Narang and Khanna [7]:

Every pair $x_1, x_2 \in M$ has a best-simultaneous approximation in M which is also a best approximation of the arithmetic means of x_1 and x_2 if x_1, x_2 are linearly dependent and the orthogonality in M is homogenous.

In the most ideal case, i.e. when X is a Hilbert space, it can be easily obtained that the above proposition of [7] reduces to the following: Suppose M is a closed subspace of Hilbert space X. Let $x_1, x_2 \in X \setminus M$. Then the pair x_1, x_2 has a best approximation of the arithmetic mean of x_1 and x_2 (one can observe that the parallelogram identity is the main key property which is used in the proof of such a thing in a Hilbert space).

Next we give definitions of contraction mapping and contractive mapping. Subsequently we give the definition of star-shaped set in a Banach space.

Let T be a self map on X. T is called a contraction if $|Ty - Tz| \leq \alpha|y - z|$, $0 < \alpha < 1$, $y, z \in X$. Banach's contraction principle states that in a complete metric space a contraction map has a unique fixed point. T is called contractive if $|Ty - Tz| < |y - z|$ for $y, z \in X$, $y \neq z$. A subset S of X is called star-shaped if there exists a point p, called star centre, in S such that $\lambda p + (1 - \lambda)z \in S$ for all $z \in S$ and $0 < \lambda < 1$. It is clear that every convex subset is star-shaped but a star-shaped set need not be convex. A more general class of sets containing the star-shaped sets is called "contractive". A set S in X is contractive if there exists a sequence $\{f_n\}$ of contraction mappings of S into itself such that $f_n y \to y$ for each $y \in S$.

Brosowski [1] proved the following:

THEOREM 1.1. *Let T be a contractive linear operator on a normed linear space X. Let K be a T-invariant subset of X and x a T-invariant point. If the set* *of best K-approximants to x is nonempty, convex and compact, then it contains a T-invariant point.*

A generalization of Theorem 1.1 was obtained by Singh [10] in the following manner.

THEOREM 1.2. *Let T be a contractive operator on a normed linear space X. Let C be a T-invariant subset of X and x a T-invariant point. If the set of best C-approximants to x is nonempty compact and star-shaped then it contains a T-invariant point.*

2. In the present section we give some generalizations of Theorems 1.1 and 1.2 in the context of simultaneous best approximations. Let T be a self map on x such that for $y, z \in X$,

$$|Ty - Tz| \leq \alpha|y - z| + \beta\{|y - Ty| + |z - Tz|\} + \gamma\{|y - Tz| + |z - Ty|\} \tag{2.1}$$

where, α, β, γ are non-negative numbers satisfying $\alpha + 2\beta + 2\gamma \leq 1$.

Let M be a subset of X. A sequence $\{y_n\}$ in M is said to be minimizing for the pair x_1 and x_2, if

$$\lim_{n \to \infty} \{|y_n - x_1| + |y_n - x_2|\} = \inf\{|y - x_1| + |y - x_2| : y \in M\}.$$

M is called approximatively compact, if for every pair $x_1, x_2 \in M$ each minimizing sequence $\{y_n\} \subset M$ has a convergent subsequence converging to an element in M.

The following theorem is a generalization of Theorem 1 of Rao and Mariadoss [8] for a mapping T which maps D a set of best simultaneous C-approximants to $x_1, x_2 \in X$ into itself.

THEOREM 2.1. *Let T be a continuous self map on a Banach space X satisfying (2.1). Let C be an approximatively compact and T-invariant subset of X. Let $Tx_i = x_i$ $(i = 1, 2)$ for some x_1, x_2 not in the norm closure of C. If the set of best simultaneous C-approximants to x_1, x_2 is nonempty and star-shaped then it has a T-invariant point.*

Proof. Let D be the set of best simultaneous C-approximants to x_1 and x_2. Then

$$D = \{z \in C : |z - x_1| + |z - x_2| \leq |y - x_1| + |y - x_2|, \forall y \in C\} \tag{2.2}$$

Let $z \in D$. Then by (2.1) and the hypothesis it is clear that

$$\begin{aligned} |x_1 - Tz| + |x_2 - Tz| &= |Tx_1 - Tz| + |Tx_2 - Tz| \\ &\leq \alpha|x_1 - z| + \beta|z - Tz| + \gamma\{|x_1 - Tz| + |z - Tx_1|\} \\ &\quad + \alpha|x_2 - z| + \beta|z - Tz| + \gamma\{|x_2 - Tz| + |z - Tx_2|\}. \end{aligned}$$

After some simplifications we get,

$$(1 - \beta - \gamma)|x_1 - Tz| + (1 - \beta - \gamma)|x_2 - Tz| \leq (\alpha + \beta + \gamma)\{|x_1 - z| + |x_2 - z|\}.$$

That is

$$|x_1 - Tz| + |x_2 - Tz| \leq |x_1 - z| + |x_2 - z| \tag{2.3}$$

since $\alpha + 2\beta + 2\gamma \leq 1$. Also, using (2.2), we get the r.h.s. of inequality (2.3) as $\leq |x_1 - y| + |x_2 - y|$, for all $y \in C$. Hence, $Tz \in D$. Therefore T is a self map on D. Since D is nonempty and star-shaped, there exists a star centre $p \in D$ such that, $\lambda p + (1 - \lambda)z \in D$, for all $z \in D$, $0 < \lambda < 1$. Now taking a sequence k_n of non-negative real numbers ($0 < k_n < 1$) converging to 1, one can define $T_n : D \to D$ for $n = 1, 2, \ldots$ as follows:

$$T_n z = k_n Tz + (1 - k_n)p, \qquad z \in D \tag{2.4}$$

Since T is a self map on D, so is T_n, for each n. Also for $y, z \in D$

$$\begin{aligned} |T_n y - T_n z| &= k_n |Ty - Tz| \\ &\leq k_n \alpha |y - z| + k_n \beta\{|y - Ty| + |z - Tz|\} + k_n \gamma\{|y - Tz| + |z - Ty|\} \end{aligned}$$

where $\alpha k_n + 2k_n\beta + 2k_n\gamma \leq 1$. Therefore by a theorem of Hardy and Rogers [3] has a unique fixed point in D, for each n. Let $T_n z_n = z_n$.

Now the approximative compactness of C implies that D is compact; therefore, there is a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \to z_0$ in D. Again, $z_{n_i} = T_{n_i} z_{n_i} = k_{n_i} T z_{n_i} + (1 - k_{n_i})p$. Considering the assumption that T is continuous and the fact that $k_{n_i} \to 1$ as $i \to \infty$, it follows that $z_0 = Tz_0$. Thus z_0 is a T-invariant point in D, which completes the proof.

Remark. For the case $\beta = \gamma = 0$, the map in Theorem 2.1, becomes nonexpansive and hence contractive. This case gives an extension of the theorem 1.2 of Singh [10] for best simultaneous C-approximant. If T is linear this result would give an extension of the Theorem 1.1 of Brosowski for best simultaneous C-approximant.

The following results given in the setting of a metric space generalize the results of Rao and Mariadoss [8] and can be obtained from Lemma 1 of E. Chandler and Faukner [2], which exploited the property of a contractive set.

LEMMA 1 [2]. *Let C be a compact subset of a metric space X, and T a non-expansive self map of C. If there exist contraction maps $\phi_n : TC \to C$ such that $\phi_n x \to x$ for all x in TC, then T has a fixed point in C.*

THEOREM 2.2. *Let E be a metric space with metric d. Let C be an approximatively compact subset of E. Let T be a non-expansive self map on C and $Tx_1 = x_1$ and $Tx_2 = x_2$. If the set of best simultaneous C-approximants of x_1 and x_2 is non-empty and contractive, then it contains another fixed point of T.*

A generalized version of Theorem 2.2 can further be given for a map T satisfying condition (2.1).

3. *Definition* 3.1. For each bounded subset A of a metric space E, the measure of non-compactness of A, $\alpha[A]$ is defined as, $\alpha[A] = \inf\{\varepsilon > 0 : A$ is covered by a finite number of closed balls centered at points of E of radius $< \varepsilon\}$.

Definition 3.2. A mapping $T : X \to X$ is called condensing if for all bounded sets $D \subset X$ with $\alpha[D] > 0$, $\alpha[T(D)] \leq \alpha[D]$, where $\alpha[D]$ is the measure of non-compactness of D.

THEOREM 3.1. *Let X be a complete, contractive metric space with contractions f_n. Let C be a closed bounded subset of E. If T is a non-expansive and condensing self map on E such that $Tx_1 = x_1$ and $Tx_2 = x_2$ for some $x_1, x_2 \in X$, and the set of all best simultaneous approximants to x_1 and x_2 is nonempty, then it has a T-invariant point.*

Proof. Let D be the set of best simultaneous C-approximants of x_1 and x_2. Then D is a closed and bounded subset of C and $T(D) \subset D$. Now a direct application of theorem 1 of Chandler and Faukner [2] quoted below will give a T-invariant point of D.

THEOREM 1 [2]. *Let X be a complete contractive metric space with contractions $\{\phi_n\}$. Let C be a closed bounded subset of X and $\phi_n : C \to C$ is non-expansive and condensing, then T has a fixed point in C.*

The following theorem is an extension of Theorem 5 of Rao and Mariadoss for best simultaneous M-approximants and can be proved by an analogous technique.

THEOREM 3.2. *Let X be a complete metric space, M an approximatively compact subset of X and $x_1, x_2 \in X \setminus M$. Let T be a self map on X with $Tx_1 = x_1$ and $Tx_2 = x_2$ and let for some positive integer m, the condition,*

$$d(T^m y, T^m z) \leq \alpha\{d(y, T^m y) + d(z, T^m z)\}, \qquad 0 < \alpha < 1/2$$

holds for $y, z \in M$. If the set of best simultaneous M-approximants to x_1 and x_2 is nonempty, then it has a unique fixed point.

4. The following result which generalizes Theorem 2.1 of Section 2. We use a common fixed point result for the pair of mappings T_1 and T_2 to prove

THEOREM 4.1. *Let T_1 and T_2 be a pair of continuous self maps on a Banach space X satisfying $|T_1 x - T_2 y| \leq |x - y|$, for $x, y \in X$ $(x \neq y)$. Let C be an approximatively compact and T_i-invariant $(i = 1, 2)$ subset of X. Let x_1 and x_2 be two common fixed points for the pair T_1 and T_2 not belonging to the norm closure of C. If the set of best simultaneous C-approximants to x_1, x_2 is non-empty and star-shaped, then it has a point which is both T_1- and T_2-invariant.*

Proof. Since x_1 and x_2 are common fixed points of T_1 and T_2, the method of proof of Theorem 2.1 yields $T_1(D) \subset D$ and $T_2(D) \subset D$ easily. Later on we proceed to show that there is a point $z_0 \in D$ such that $T_i z_0 = z_0$ $(i = 1, 2)$. In the course of showing this we use the fact that T_{1n} and T_{2n} defined analogously as in (2.4) have a common fixed point $z_n \in D$ (see theorem 4.5.9, p. 144 in [4]). Also we use the continuity of both T_1 and T_2.

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