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ASYMPTOTIC BEHAVIOR OF BETA TRANSFORM OF A SINGULAR FUNCTION

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Abstract. Let $B(u,v)$ denote the Beta function and

$$\beta_n(f,x) = \frac{\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1} f(t)\,dt}{B(nx, n(1-x))}.$$

We prove that

$$\beta_n(|t-x|,x) = \left(\frac{2x(1-x)}{n\pi}\right)^{1/2} + O\left(\frac{1}{n^{2/3}}\right)$$

for $x \in (0,1)$. Consequently, using a result of Bojanic and Khan [1], it follows that for functions with the first derivative of bounded variation on $[0,1]$ we have

$$\beta_n(f,x) = f(x) + \left(\frac{x(1-x)}{2n\pi}\right)^{1/2} (f'_R(x) - f'_L(x)) + o\left(\frac{1}{\sqrt{n}}\right)$$

for any $x \in (0,1)$.

Introduction. Beta operators were introduced by Lupas [4] and further modified and studied by Khan [3], Upreti [5] and others. Khan in particular studied the approximation properties of beta operators for functions of bounded variation on $[0,1]$. With Khan, we define the beta operator as follows

$$\beta_n(f,x) = \int_0^1 \frac{t^{nx-1}(1-t)^{n(1-x)-1}}{B(nx, n(1-x))} f(t)\,dt$$

where $B(u,v)$ is the Beta function

$$B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}\,dt; \qquad u,v > 0.$$

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The aim of this paper is to study the asymptotic behavior of $\beta_n(f,x)$ for functions whose derivative is of bounded variation. In [1], Bojanic and Khan have proved that for such functions we have

$$\beta_n(f,x) = f(x) + \frac{1}{2}(f'(x+0) - f'(x-0))\beta_n(|t-x|,x) + o\Big(\frac{1}{\sqrt{n}}\Big)$$

where x is a fixed number in $(0,1)$. Consequently, in order to study the asymptotic behavior of this operator it is essential to know the behavior of the special beta transform $\beta_n(|t-x|,x)$.

In this paper we shall give a simple and direct proof of the following

THEOREM. *For a fixed $x \in (0,1)$ we have*

$$\beta_n(|t-x|,x) = \Big(\frac{2x(1-x)}{\pi n}\Big)^{1/2} + O\Big(\frac{1}{n^{2/3}}\Big).$$

COROLLARY. *For a function f whose derivative is of bounded variation in $[0,1]$ we have for any $x \in (0,1)$ fixed,*

$$\beta_n(f,x) = f(x) + \Big(\frac{x(1-x)}{2n\pi}\Big)^{1/2}(f'(x+0) - f'(x-0)) + o\Big(\frac{1}{\sqrt{n}}\Big).$$

Preliminaries. Note that the identities

$$\beta_n((t-x)^2,x) = \frac{x(1-x)}{n+1} \tag{1}$$

$$x(1-x)B(nx, n(1-x)) = \frac{n+1}{n}B(nx+1, n(1-x)+1) \tag{2}$$

can easily be derived from basic properties of the Beta function.

In the following let $0 < a < x < b < 1$. In part I of the paper we shall show that in order to derive the desired asymptotic formula it suffices to investigate the integral

$$I_n(x) = \int_a^b \frac{t^{nx}(1-t)^{n(1-x)}}{B(nx+1, n(1-x)+1)}|t-x|\,dt.$$

We shall then use Laplace's method [2] to estimate this integral in part II of the paper.

Part I. LEMMA 1. *We have*

$$\left|\beta_n(|t-x|,x) - \int_a^b \frac{t^{nx-1}(1-t)^{n(1-x)-1}}{B(nx,n(1-x))}|t-x|\,dt\right| \le \frac{b-a}{(b-x)(x-a)}\cdot\frac{x(1-x)}{n+1}.$$

Proof. Writing

$$\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1}|t-x|\,dt = \Big(\int_0^a + \int_a^b + \int_b^1\Big)t^{nx-1}(1-t)^{n(1-x)-1}|t-x|\,dt$$

we observe that for $t \in [0,a]$ we have $x - t \geq x - a$ and for $t \in [b,1]$ we have $t - x \geq b - x$ and so

$$\int_0^a t^{nx-1}(1-t)^{n(1-x)-1}|t-x|\,dt \leq \frac{1}{x-a}\int_0^a t^{nx-1}(1-t)^{n(1-x)-1}(x-t)^2\,dt$$
$$\leq \frac{1}{x-a}\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1}(x-t)^2\,dt.$$

Similarly we obtain

$$\int_b^1 t^{nx-1}(1-t)^{n(1-x)-1}|t-x|\,dt \leq \frac{1}{b-x}\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1}(x-t)^2\,dt.$$

Combining these estimates together with (1) gives the desired approximation.

Denote now

$$\Delta_1 = \int_a^b \frac{t^{nx-1}(1-t)^{n(1-x)-1}}{B(nx,n(1-x))}|t-x|\,dt - \int_a^b \frac{t^{nx}(1-t)^{n(1-x)}}{B(nx+1,n(1-x)+1)}|t-x|\,dt.$$

LEMMA 2. *We have* $|\Delta_1| \leq \dfrac{2}{n+1}$.

Proof. Using (2) we can write

$$\begin{aligned}
\Delta_1 &= \int_a^b \frac{t^{nx-1}(1-t)^{n(1-x)-1}}{B(nx,n(1-x))}|t-x|\,dt \\
&\quad - \frac{n+1}{nx(1-x)}\int_a^b \frac{t^{nx}(1-t)^{n(1-x)}}{B(nx,n(1-x))}|t-x|\,dt \\
&= \frac{1}{B(nx,n(1-x))}\left\{\int_a^b t^{nx-1}(1-t)^{n(1-x)-1}|t-x|\,dt \right.\\
&\qquad\qquad \left. - \frac{1}{x(1-x)}\int_a^b t^{nx}(1-t)^{n(1-x)}|t-x|\,dt\right\} \\
&\quad - \frac{1}{B(nx,n(1-x))}\int_a^b \frac{t^{nx}(1-t)^{n(1-x)}}{nx(1-x)}|t-x|\,dt \\
&\equiv \frac{I+J}{B(nx,n(1-x))}
\end{aligned}$$

where I denotes the bracketed expression. We can rewrite I as follows

$$\begin{aligned}
I &= \int_a^b t^{nx}(1-t)^{n(1-x)}\left(\frac{1}{t(1-t)} - \frac{1}{x(1-x)}\right)|t-x|\,dt \\
&= \int_a^b t^{nx}(1-t)^{n(1-x)}\frac{(x-t)(1-x-t)}{t(1-t)x(1-x)}|t-x|\,dt \\
&= \frac{1}{x(1-x)}\int_a^b t^{nx-1}(1-t)^{n(1-x)-1}(1-x-t)(x-t)|t-x|\,dt.
\end{aligned}$$

Thus,

$$|I| \le \frac{1}{x(1-x)} \int_0^1 t^{nx-1}(1-t)^{n(1-x)-1}(t-x)^2\,dt.$$

Using this inequality and (1) we find that

$$\left|\frac{I}{B(nx, n(1-x))}\right| \le \frac{1}{n+1}.$$

Finally, using (2) it follows that

$$\left|\frac{J}{B(nx, n(1-x))}\right| \le \frac{b-a}{nx(1-x)} \cdot \frac{B(nx+1, n(1-x)+1)}{B(nx, n(1-x))} = \frac{b-a}{n+1} \le \frac{1}{n+1}$$

and the lemma is proved.

Remark. Notice that combining Lemma 1 and Lemma 2 we obtain the estimate

$$\left|\beta_n(|t-x|, x) - \int_a^b \frac{t^{nx}(1-t)^{n(1-x)}}{B(nx+1, n(1-x)+1)}|t-x|\,dt\right| \le \frac{b-a}{(b-x)(x-a)} \cdot \frac{x(1-x)}{n+1} + \frac{2}{n+1}.$$

With a special choice of $a_n = x - x/\sqrt[3]{n}$, $b_n = x + (1-x)/\sqrt[3]{n}$, $n > 1$, this can further be estimated by $3n^{-2/3}$. Hence, with these a_n and b_n we can write

$$\beta_n(|t-x|, x) = \int_{a_n}^{b_n} \frac{t^{nx}(1-t)^{n(1-x)}}{B(nx+1, n(1-x)+1)}|t-x|\,dt + O(n^{-2/3}) \qquad (3)$$

where the constant in the remainder term thus far can be taken independent of x.

We shall conclude part I with two useful identities and estimates following from these. Straightforward calculations lead to

LEMMA 3. *We have*

(a) $$J_1 = \int_0^1 \frac{t^{nx}(1-t)^{n(1-x)}}{B(nx+1, n(1-x)+1)}(t-x)^2\,dt = \frac{(n-6)x(1-x)+2}{(n+2)(n+3)}$$

and from here it follows that for all $n = 1, 2, \ldots$ *and* $x \in [0,1]$

$$|J_1| \le 1/(n+1);$$

(b) $$J_2 = \int_0^1 \frac{t^{nx}(1-t)^{n(1-x)}}{B(nx+1, n(1-x)+1)}(t-x)^4\,dt = \frac{(3n^2 - 86n + 120)x^2(1-x)^2 + (26n - 120)x(1-x) + 24}{(n+2)(n+3)(n+4)(n+5)}$$

and from here it follows that for all $n = 1, 2, \ldots$ *and* $x \in [0, a]$

$$|J_2| \le 1/n^2. \qquad (4)$$

Part II. We shall now turn to the study of the asymptotic behavior of the integrals

$$I_n = \int_{a_n}^{b_n} \frac{(t^x(1-t)^{1-x})^n}{B(nx+1, n(1-x)+1)} |t-x|\, dt$$

with $a_n = x(1-n^{-1/3})$, $b_n = x + (1-x)n^{-1/3}$. Denote $h(x,t) = x\log t + (1-x)\log(1-t)$. With this notation, we can write

$$I_n = \int_{a_n}^{b_n} \frac{e^{nh(x,t)}}{\int_0^1 e^{nh(x,t)}\, dt} |t-x|\, dt.$$

Clearly $h_t(x,t) = x/t - (1-x)/(1-t) = 0$ if and only if $t = x$ so we can write

$$\begin{aligned} h(x,t) &= h(x,x) + \frac{1}{2}(t-x)^2 h_{tt}(x,x) + \frac{1}{6}(t-x)^3 r(x,t) \\ &= h(x,x) - \frac{(t-x)^2}{2x(1-x)} + \frac{1}{6}(t-x)^3 r(x,t) \end{aligned}$$

where for $t \in [a_n, b_n]$ we have

$$|r(x,t)| \leq \max_{v \in [a_n, b_n]} \left| \frac{\partial^3 h}{\partial t^3}(x,v) \right| \equiv \gamma_n(x).$$

However, easy calculation shows e.g. that for $n \geq 8$

$$|\gamma_n(x)| \leq 16 \left(\frac{1}{x^2} + \frac{1}{(1-x^2)} \right)$$

so we shall simply write in what follows that for $t \in [a_n, b_n]$, $n = 2, 3, \ldots$ we have

$$|r(x,t)| \leq \gamma(x). \tag{5}$$

Next we shall investigate how will I_n change when we replace the function $h(x,t)$ in the numerator of I_n by $h(x,x) - (t-x)^2/(2x(1-x))$. For this purpose let us first consider the difference

$$\begin{aligned} \Delta_2(x,t) &\equiv e^{nh(x,t)} - e^{n(h(x,x)-(t-x)^2/(2x(1-x)))} \\ &= e^{nh(x,t)}\left(1 - e^{-n(h(x,t)-h(x,x)+(t-x)^2/(2x(1-x)))}\right) \\ &= e^{nh(x,t)}\left(1 - e^{-n(t-x)^3 r(x,t)/6}\right). \end{aligned}$$

Hence, using an elementary inequality ($|e^x - 1| \leq |x| e^{|x|}$) and (5) we can now estimate for $t \in [a_n, b_n]$

$$|\Delta_2(x,t)| \leq \frac{n}{6}|t-x|^3 \gamma(x) e^{nh(x,t) + n(b_n-a_n)^3\gamma(x)/6}. \tag{6}$$

Since $b_n - a_n = n^{-1/3}$, it follows that

$$\left| I_n - \frac{1}{B(nx+1, n(1-x)+1)} \int_{a_n}^{b_n} e^{n(h(x,x)-(t-x)^2/(2x(1-x)))} |t-x|\, dt \right|$$

$$\le \frac{1}{B(nx+1,n(1-x)+1)} \int_{a_n}^{b_n} |\Delta_2(x,t)|\,|t-x|\,dt \le n\gamma(x)e^{\gamma(x)}J_2 \le \frac{\gamma(x)e^{\gamma(x)}}{n}$$

where the last inequality follows using estimate (4). Hence we have just shown that

$$I_n = \int_{a_n}^{b_n} \frac{e^{n(h(x,x)-(t-x)^2/(2x(1-x)))}}{B(nx+1,n(1-x)+1)}|t-x|\,dt + O\Big(\frac{1}{n}\Big)$$

and combining this result with (3) we arrive at

LEMMA 4. *We have*

$$\beta_n(|t-x|,x) = \int_{a_n}^{b_n} \frac{e^{n(h(x,x)-(t-x)^2/(2x(1-x)))}}{B(nx+1,n(1-x)+1)}|t-x|\,dt + O\Big(\frac{1}{n^{2/3}}\Big). \qquad (7)$$

Note that the constant in the remainder term depends on x.

In the following lemma we shall further simpify the integral to be estimated.

LEMMA 5. *We have*

$$\beta_n(|t-x|,x) = \frac{\int_{a_n}^{b_n} e^{-n(t-x)^2/(2x(1-x))}|t-x|\,dt}{\int_0^1 e^{-n(t-x)^2/(2x(1-x))}\,dt} + O\Big(\frac{1}{n^{2/3}}\Big).$$

Proof. Consider the difference

$$\begin{aligned}\Delta_3 \equiv &\int_{a_n}^{b_n} \frac{e^{n(h(x,x)-(t-x)^2/(2x(1-x)))}}{B(nx+1,n(1-x)+1)}|t-x|\,dt - \frac{\int_{a_n}^{b_n} e^{-n(t-x)^2/(2x(1-x))}|t-x|\,dt}{\int_0^1 e^{-n(t-x)^2/(2x(1-x))}\,dt}\times\\ &\times \int_{a_n}^{b_n} e^{n(h(x,x)-(t-x)^2/(2x(1-x)))}|t-x|\,dt\times\\ &\times\left\{\frac{1}{\int_0^1 e^{nh(x,t)}\,dt} - \frac{1}{\int_0^1 e^{n(h(x,x)-(t-x)^2/(2x(1-x)))}\,dt}\right\}\\ = &-\frac{\int_{a_n}^{b_n} e^{n(h(x,x)-(t-x)^2/(2x(1-x)))}|t-x|\,dt}{\int_0^1 e^{n(h(x,x)-(t-x)^2/(2x(1-x)))}\,dt}\times\\ &\times\frac{\int_0^1\big(e^{nh(x,t)} - e^{n(h(x,x)-(t-x)^2/(2x(1-x)))}\big)\,dt}{\int_0^1 e^{nh(x,t)}\,dt}.\end{aligned}$$

Estimating $|t-x| \le b_n - a_n = n^{-1/3}$ in the first integral, the first fraction in the last product can simply be estimated by $n^{-1/3}$. In the second fraction, we first use inequality (6) so that we obtain

$$|\Delta_3| \le n^{-1/3} n\gamma(x)e^{\gamma(x)} \frac{\int_0^1 e^{nh(x,t)}|t-x|^3\,dt}{\int_0^1 e^{nh(x,t)}\,dt}$$

and then proceed e.g. as follows

$$\int_0^1 e^{nh(x,t)}|t-x|^3\,dt = \Big(\int_{|t-x|\le 1/\sqrt{n}} + \int_{|t-x|>1\sqrt{n}}\Big) e^{nh(x,t)}|t-x|^3\,dt$$
$$\le \frac{1}{n^{3/2}}\int_0^1 e^{nh(x,t)}\,dt + \sqrt{n}\int_0^1 e^{nh(x,t)}(t-x)^4\,dt.$$

Hence, combining these inequalities with (4) we finally obtain

$$|\Delta_3| \le n^{2/3}\gamma(x)e^{\gamma(x)}\cdot\frac{2}{n^{3/2}} \le \frac{C(x)}{n^{2/3}}$$

and the lemma is proved.

Proof of the Theorem. In view of Lemma 5 it suffices to show that

$$\frac{\int_{a_n}^{b_n} e^{-n(t-x)^2/(2x(1-x))}|t-x|\,dt}{\int_0^1 e^{-n(t-x)^2/(2x(1-x))}\,dt} = \sqrt{\frac{2x(1-x)}{\pi n}} + O\Big(\frac{1}{n}\Big).$$

Denote by K_n, L_n the integrals in the numerator and denominator, respectively, of the fraction on the left side. Substitution $t-x=\sqrt{2x(1-x)}v/n$ in both integrals then leads to

$$K_n = \frac{2x(1-x)}{n}\int_{c_n}^{d_n} e^{-v^2}|v|\,dv$$

where $c_n = -\frac{xn^{1/6}}{\sqrt{2x(1-x)}}$ and $d_n = \frac{(1-x)n^{1/6}}{\sqrt{2x(1-x)}}$. Hence,

$$K_n = \frac{2x(1-x)}{n}\Big\{\int_{-\infty}^{+\infty} e^{-v^2}|v|\,dv + \int_{-\infty}^{c_n} e^{-v^2}v\,dv - \int_{d_n}^{+\infty} e^{-v^2}v\,dv\Big\}$$
$$= \frac{2x(1-x)}{n}\Big\{1 - \frac{1}{2}e^{-xn^{1/3}/(2(1-x))} - \frac{1}{2}e^{-(1-x)n^{1/3}/(2x)}\Big\}$$
$$= \frac{2x(1-x)}{n} + O\Big(\frac{e^{-c(x)n^{1/3}}}{n}\Big), \qquad c(x)>0.$$

Similarly we obtain

$$L_n = \sqrt{\frac{2\pi x(1-x)}{n}} + O\Big(\frac{e^{-c(x)n}}{\sqrt{n}}\Big), \qquad c(x)>0.$$

From these representations of K_n and L_n the theorem follows.

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