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AN ESTIMATE OF THE RATE OF CONVERGENCE FOR MODIFIED SZÁSZ-MIRAKYAN OPERATORS OF FUNCTION OF BOUNDED VARIATION

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Abstract. We give an estimate of the rate of convergence for an integral modification of the Szász-Mirakyan operators introduced by Kasana et al. [Int. Conf. Math. Anal. and Appl. Kuwait (1985), Pergamon Press, Oxford, 29–41] of functions of bounded variation by probabilistic approach.

1. Introduction. Let f be a function defined on $[0,\infty)$. The Szász-Mirakyan operator L_n applied to f is

$$L_n(f,x)=\sum_{k=0}^{\infty}e^{-nx}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right).$$

Kasana et. al. [3] proposed modified Szász-Mirakyan operators to approximate functions integrable on $[0,\infty)$ as

$$S_n(f(t);x)=n\sum_{k=0}^{\infty}p_k(nx)\int_0^{\infty}p_k(nt)f(t)\,dt \quad \text{where} \quad p_k(nx)=e^{-nx}\frac{(nx)^k}{k!}.$$

In [1] Bojanic estimated the rate of convergence of Fourier series of functions of bounded variation. In this paper we prove an analogous result for modified Szász-Mirakyan operator by using some results of probability theory considering the functions bounded on $[0,\infty)$. In the end it is shown that our result is essentially the best possible.

2. Auxiliary results. To prove the main result, we shall need the following lemmas:

LEMMA 1. *For every $x\in(0,\infty)$, we have*

$$p_k(nx)\le 2/\left(5\sqrt{nx}\right). \tag{2.1}$$

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Proof. Let $\{\xi_k\}$ be a sequence of independent random variables all having the same Poisson (x) distribution. Define $\eta_n = \sum_{i=1}^{n} \xi_i$; then $P(\eta_n = k) = e^{-nx}(nx)^k/k! = p_k(nx)$. Then $a_1 = E(\eta_n) = nx$, $b_1^2 = (D\eta_n)^{1/2} = nx$ and $\beta_3 = E(\eta_n - E\eta_n)^3 = nx$. Hence,

$$\frac{C\beta_3}{\sqrt{n}b_1^3} = \frac{C \cdot nx}{\sqrt{n}(nx)^{3/2}} \leq \frac{1}{\sqrt{nx}}.$$

We have

$$p_k(nx) = P(k-1 < \eta_n \leq k) = P\left(\frac{k-1-nx}{\sqrt{nx}} < \frac{\eta_n - nx}{\sqrt{nx}} \leq \frac{k-nx}{\sqrt{nx}}\right).$$

By using [2, p. 300], we obtain

$$\left| p_k(nx) - \frac{1}{\sqrt{2\pi}} \int_{(k-1-nx)/\sqrt{nx}}^{(k-nx)/\sqrt{nx}} e^{-t^2/2}\, dt \right| < \frac{2C\beta_3}{\sqrt{n}b_1^3} \leq \frac{2}{\sqrt{nx}}.$$

Since

$$\frac{1}{\sqrt{2\pi}} \int_{(k-1-nx)/\sqrt{nx}}^{(k-nx)/\sqrt{nx}} e^{-t^2/2}\, dt \leq \frac{1}{\sqrt{2\pi nx}},$$

we get the required result (2.1).

LEMMA 2. *For n sufficiently large, we have*

$$x/n \leq S_n((t-x)^2; x) \leq 3x/n. \tag{2.2}$$

Proof. We have by easy calculations

$$S_n(1;x) = 1, \qquad S_n(x;x) = x,$$
$$S_n(t;x) = (1+nx)/n, \qquad S_n(t^2;x) = (x^2n^2 + 4nx + 2)/n^2.$$

Thus $S_n((t-x)^2;x) = 2n^{-1}(x+n^{-1})$. Hence (2.2) follows for sufficiently large n.

LEMMA 3. *Let* $K_n(x,t) = n\sum_{k=0}^{\infty} p_k(nx)p_k(nt)$. *If n is sufficiently large, then*

(i) *For* $0 \leq y < x$ *we have*

$$\int_0^y K_n(x,t)\, dt \leq \frac{3x}{n(x-y)^2}. \tag{2.3}$$

(ii) *For* $x < z < \infty$, *we have*

$$\int_z^{\infty} K_n(x,t)\, dt \leq \frac{3x}{n(z-y)^2}. \tag{2.4}$$

Proof. The results (2.3) and (2.4) can easily be proved by using Lemma 2.

LEMMA 4. *We have*

$$n\int_x^\infty p_k(nt)\,dt=\sum_{j=0}^k p_j(nx).$$

Proof. On the left hand side, we have

$$n\int_x^\infty p_k(nt)\,dt=\int_{nx}^\infty p_k(u)\,du=\frac{1}{k!}\int_{nx}^\infty e^{-u}u^k\,du.$$

Integrating by parts, we get

$$\begin{aligned}n\int_x^\infty p_k(nt)\,dt&=\frac{1}{k!}\left[\{u^k(-e^{-u})\}_{nx}^\infty+\int_{nx}^\infty ke^{-u}u^{k-1}\,du\right]\\&=\frac{1}{k!}(nx)^ke^{-nx}+\frac{1}{(k-1)!}\left[(nx)^{k-1}e^{-nx}+\int_{nx}^\infty(k-1)u^{k-2}e^{-u}\,du\right]\\&=p_k(nx)+p_{k-1}(nx)+\dots+p_0(nx)=\sum_{j=0}^k p_j(nx).\end{aligned}$$

Similarly

$$n\int_0^x p_k(nt)\,dt=1-\sum_{j=0}^k p_j(nx).$$

3. Main result. THEOREM. *Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$ and let $\mathrm{V}_a^b(g_x)$ be the total variation of g_x. Then, for n sufficiently large, we have*

$$\left|S_n(f;x)-\frac{1}{2}(f(x+)+f(x-))\right|\\ \le\frac{1}{5\sqrt{nx}}|f(x+)-f(x-)|+\frac{7}{nx}\sum_{k=1}^n\mathrm{V}_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)\quad(3.1)$$

where for any fixed $x\in(0,\infty)$, define g_x as

$$g_x(t)=\begin{cases}f(t)-f(x+), & x<t<\infty\\ 0, & t=x\\ f(t)-f(x-), & 0\le t<x.\end{cases}$$

Proof. We have

$$\left|S_n(f;x)-\frac{1}{2}(f(x+)+f(x-))\right|\\ \le|S_n(g_x;x)|+\frac{1}{2}|f(x+)-f(x-)|\cdot|S_n(\mathrm{Sign}(t-x);x)|.\quad(3.2)$$

In order to obtain the result we need the estimate for $S_n(g_x;x)$ and $S_n(\mathrm{Sign}(t-x);x)$. Now,

$$\begin{aligned}S_n(\mathrm{Sign}(t-x);x) &= \int_0^\infty \mathrm{Sign}(t-x)K_n(x,t)\,dt\\ &= \int_x^\infty K_n(x,t)\,dt - \int_0^x K_n(x,t)\,dt\\ &= A_n(x) - B_n(x), \qquad \text{say.}\end{aligned}$$

Using Lemma 4, we have

$$\begin{aligned}A_n(x) = \int_x^\infty K_n(x,t)\,dt &= n\sum_{k=0}^\infty p_k(nx)\int_x^\infty p_k(nt)\,dt\\ &= \sum_{k=0}^\infty\Big(p_k(nx)\sum_{j=0}^k p_j(nx)\Big), \quad \text{i.e.}\end{aligned}$$

$$\begin{aligned}A_n(x) &= p_0^2 + p_1(p_0+p_1) + p_2(p_0+p_1+p_2) + \dots\\ &= p_0^2 + p_1^2 + p_2^2 + \dots + p_0(p_1+p_2+p_3+\dots)\\ &\quad + p_1(p_2+p_3+p_4+\dots) + \dots .\end{aligned}$$

Now, $I = (p_0+p_1+p_2+\dots)(p_0+p_1+p_2+\dots)$. Hence $2A_n(x) - I = p_0^2+p_1^2+p_2^2+\dots$. By using (2.1) of Lemma 1, we get

$$|2A_n(x) - I| \le \frac{2}{5\sqrt{nx}}\sum_{k=0}^\infty p_k(nx) = \frac{2}{5\sqrt{nx}}.$$

But $A_n(x) + B_n(x) = 1$, therefore $|A_n(x) - B_n(x)| = |2A_n(x) - 1|$, i.e.

$$|A_n(x) - B_n(x)| \le 2/(5\sqrt{nx}). \tag{3.3}$$

To estimate $S_n(g_x;x)$, we decompose $[0,\infty)$ interval into three parts as follows:

$$\begin{aligned}S_n(g_x;x) = \int_0^\infty g_x(t)K_n(x,t)\,dt &= \int_0^{x-(x/\sqrt{n})} g_x(t)K_n(x,t)\,dt\\ &+ \int_{x-(x/\sqrt{n})}^{x+(x/\sqrt{n})} g_x(t)K_n(x,t)\,dt + \int_{x+(x/\sqrt{n})}^\infty g_x(t)K_n(x,t)\,dt\\ &= \Big(\int_{I_1} + \int_{I_2} + \int_{I_3}\Big) g_x(t)K_n(x,t)\,dt = R_1 + R_2 + R_3, \quad \text{say.}\end{aligned}$$

Suppose $\lambda_n(x,t) = \int_0^t K_n(x,u)\,du$. We first estimate R_1. Let $y = x - (x/\sqrt{n})$. Using partial integration, we get:

$$\begin{aligned}R_1 = \int_0^y g_x(t)K_n(x,t)\,dt &= \int_0^y g_x(t)\,d_t(\lambda_n(x,t))\\ &= g_x(y+)\lambda_n(x,y) - \int_0^y \lambda_n(x,t)\,d_t(g_x(t)).\end{aligned}$$

Since $|g_x(y+)| = |g_x(y+) - g_x(x)| \le \mathrm{V}_{y+}^x(g_x)$, then by using (2.3) of Lemma 3, we obtain

$$
\begin{aligned}
R_1 &\le \mathbf{V}_{y+}^{x}(g_x)\lambda_n(x,y) + \int_0^y \lambda_n(x,t)\, d_t\big(-\mathbf{V}_t^x(g_x)\big) \\
&\le \mathbf{V}_{y+}^{x}(g_x)\frac{3x}{n(x-y)^2} + \frac{3x}{n}\int_0^y \frac{1}{(x-t)^2}\, d_t\big(-\mathbf{V}_t^x(g_x)\big).
\end{aligned}
$$

Integrating by parts, we have

$$
\int_0^y \frac{1}{(x-t)^2}\, d_t\big(-\mathbf{V}_t^x(g_x)\big) = \frac{-\mathbf{V}_{y+}^x(g_x)}{(x-y)^2} + \frac{\mathbf{V}_0^x(g_x)}{x^2} + 2\int_0^y \frac{\widetilde{\mathbf{V}}_t^x(g_x)\, dt}{(x-t)^3},
$$

where $\widetilde{\mathbf{V}}_t^x(g_x)$ is the normalized form of $\mathbf{V}_t^x(g_x)$ and $\widetilde{\mathbf{V}}_t^x(g_x) = \mathbf{V}_t^x(g_x)$. We have

$$
\begin{aligned}
R_1 &\le \mathbf{V}_{y+}^x(g_x)\frac{3x}{n(x-y)^2} + \frac{3x}{n}\left[-\frac{\mathbf{V}_{y+}^x(g_x)}{(x-y)^2} + \frac{\mathbf{V}_0^x(g_x)}{x^2} + 2\int_0^y \mathbf{V}_t^x(g_x)\,\frac{dt}{(x-t)^3}\right] \\
&= \frac{3x}{n}\left[\frac{\mathbf{V}_0^x(g_x)}{x^2} + 2\int_0^y \mathbf{V}_t^x(g_x)\,\frac{dt}{(x-t)^3}\right].
\end{aligned}
$$

Replacing the variable y in the last integral by $x-(x/\sqrt{n})$ we get

$$
\begin{aligned}
\int_0^{x-(x/\sqrt{n})} \mathbf{V}_t^x(g_x)\,\frac{dt}{(x-t)^3} &= \frac{1}{2x^2}\int_1^n \mathbf{V}_{x-(x/\sqrt{n})}^x(g_x)\, dt \\
&\le \frac{1}{2x^2}\sum_{k=1}^n \mathbf{V}_{x-(x/\sqrt{k})}^x(g_x).
\end{aligned}
$$

Hence

$$
\begin{aligned}
|R_1| &\le \frac{3x}{nx^2}\left(\mathbf{V}_0^x(g_x) + \sum_{k=1}^n \mathbf{V}_{x-(x/\sqrt{k})}^x(g_x)\right) \\
&= \frac{3}{nx}\left(\mathbf{V}_0^x(g_x) + \sum_{k=1}^n \mathbf{V}_{x-(x/\sqrt{k})}^x(g_x)\right), \quad \text{or}
\end{aligned}
$$

$$
|R_1| \le \frac{6}{nx}\sum_{k=1}^n \mathbf{V}_{x-(x/\sqrt{k})}^x(g_x). \tag{3.4}
$$

Similarly, using (2.4) of Lemma 3, we obtain

$$
|R_3| < \frac{6}{nx}\sum_{k=1}^n \mathbf{V}_x^{x+(x/\sqrt{k})}(g_x). \tag{3.5}
$$

Now we estimate R_2. For $t \in I_2$, we have

$$
|g_x(t)| = |g_x(t) - g_x(x)| \le \mathbf{V}_{x-(x/\sqrt{n})}^{x+(x/\sqrt{n})}(g_x).
$$

Therefore

$$
|R_2| \le \mathbf{V}_{x-(x/\sqrt{n})}^{x+(x/\sqrt{n})}(g_x)\int_{x-(x/\sqrt{n})}^{x+(x/\sqrt{n})} d_t\lambda_n(x,t).
$$

Since $\int_a^b d_t\lambda_n(x,t) \le 1$ for all $(a,b) \subset [0,\infty)$, then

$$|R_2| \le \mathbf{V}_{x-(x/\sqrt{n})}^{x+(x/\sqrt{n})}(g_x). \tag{3.6}$$

Combining (3.4), (3.5) and (3.6), we get

$$\begin{aligned} |S_n(g_x;x)| &\le \frac{6}{nx}\sum_{k=1}^{n} \mathbf{V}_{x-(x/\sqrt{k})}^{x+(x/\sqrt{k})}(g_x) + \mathbf{V}_{x-(x/\sqrt{n})}^{x+(x/\sqrt{n})}(g_x) \\ &\le \frac{7}{nx}\sum_{k=1}^{n} \mathbf{V}_{x-(x/\sqrt{k})}^{x+(x/\sqrt{k})}(g_x). \end{aligned} \tag{3.7}$$

The theorem follows from (3.2), (3.3) and (3.7).

Remark. We shall show that our estimate is essentially the best possible. If f is continuous at x then in (3.1), we have

$$|S_n(f;x) - f(x)| \le \frac{7}{nx}\sum_{k=1}^{n} \mathbf{V}_{x-(x/\sqrt{k})}^{x+(x/\sqrt{k})}(f). \tag{4.1}$$

Consider the function $f(t) = |t-x|$ $(x > 0)$ on $[0,\infty)$. From (2.2) for any small $\delta > 0$ and n sufficiently large, we obtain

$$\begin{aligned} S_n(|t-x|;x) &= n\sum_{k=0}^{\infty} p_k(nx)\left(\int_{|t-x|\le\delta} + \int_{|t-x|>\delta}\right) p_k(nt)|t-x|\,dt \\ &\le n\sum_{k=0}^{\infty} p_k(nx)\left(\frac{\delta}{n} + \frac{1}{\delta}\int_0^\infty (t-x)^2 p_k(nt)\,dt\right), \quad \text{i.e.} \end{aligned}$$

$$S_n(|t-x|;x) \le \delta + 3x/(n\delta) \tag{4.2}$$

and

$$\begin{aligned} S_n(|t-x|;x) &\ge n\sum_{k=0}^{\infty} p_k(nx)\int_{|t-x|\le\delta} |t-x| p_k(nt)\,dt \\ &\ge \frac{n}{\delta}\sum_{k=0}^{\infty} p_k(nx)\int_{|t-x|\le\delta} (t-x)^2 p_k(nt)\,dt \\ &\ge \frac{x}{n\delta} - \frac{1}{\delta}\left[n\sum_{k=0}^{\infty} p_k(nx)\int_{|t-x|>\delta} (t-x)^2 p_k(nt)\,dt\right]. \end{aligned}$$

Again by using [3]

$$n\sum_{k=0}^{\infty} p_k(nx)\int_{|t-x|>\delta} (t-x)^2 p_k(nt)\,di \le \frac{1}{\delta^2} S_n((t-x)^4;x) \le \frac{C_1}{\delta^2 n^3},$$

where C_1 is a constant. Hence,

$$S_n(|t-x|;x) \geq \frac{x}{n\delta} - \frac{C_1}{n^3\delta^3}. \tag{4.3}$$

Choosing $\delta = 3\sqrt{C_1/(nx)}$, by using (4.2) and (4.3) we get

$$\frac{x^{3/2}}{3\sqrt{nC_1}}\left(1-\frac{1}{9n}\right) \leq S_n(|t-x|;x) \leq \frac{3C_1+x^2}{\sqrt{nC_1 x}}. \tag{4.4}$$

On the other hand from (4.1), since $\mathrm{V}_{x-\beta}^{x+\beta}(f) = \alpha+\beta$, we have

$$|S_n(f;x)-f(x)| \leq \frac{7}{nx}\sum_{k=1}^{n} \mathrm{V}_{x-(x/\sqrt{k})}^{x+(x/\sqrt{k})}(f) \leq \frac{7}{nx}\sum_{k=1}^{n}\frac{x}{\sqrt{k}} \leq \frac{14}{\sqrt{n}}. \tag{4.5}$$

Thus it is shown that (3.1) cannot be asymptotically improved (by comparing (4.5) and (4.4)).

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