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EXISTENCE OF MONOTONE, ψ-MINIMAL SOLUTIONS OF DIFFERENTIAL INCLUSIONS

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Abstract. We establish the existence of ψ-minimal, monotone solutions for a class of differential inclusions in $\mathbb{R}^n$. The criterion $\psi(\cdot)$ is continuous, convex and the preorder $P(\cdot)$ is Hausdorff continuous. The proof uses a lemma on the lower semicontinuity of the multifunction $x \to T(x)$ which we prove separately. Finally we use the main existence result to prove the existence of feedback controls that generate ψ-minimal and monotone trajectories for a class of nonlinear control systems.

1. Introduction. In this note we establish the existence of ψ-minimal monotone trajectories for a class of differential inclusions in $\mathbf{R}^n$. such solutions are important in the study of certain dynamic economic models of resource allocation (see Aubin-Cellina [2, chapter 6] and Henry [6]), in the stabilization of systems and in feedback systems [2].

Recently Falcone-Saint Pierre [4] addresed the same problem and determined conditions guaranteeing the existence of ψ-minimal solutions.

Here not only we want our solution to be ψ-minimal, but we also want it to be monotone with respect to a preorder. Despite this additional requirement on the solutions, our hypotheses on the data are weaker than those of Falcone-Saint Pierre [4]. This way we achieve a two-fold generalization of their work. Our proof makes use of a lemma concerning certgain lower semicontinuity property of the contingent cone of a given sequence of sets, which we believe is of i ndependent interest. We conclude the note with an application to finite dimensional control systems.

2. Preliminaries. Throughout this note by $P_{\mathrm{fc}}(X)$ we will denote the family of nonempty closed (convex) subsets of a Banach space X. Also if Y, Z are Hausdorff topological spaces, a multifunction $F : Y \to 2^Z \setminus \{\varnothing\}$ is said to be lower

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semicontinuous (l.s.c.), if for all $U \subseteq Z$ open, $F^{-}(U) = \{y \in Y : F(y) \cap U \neq \varnothing\}$ is open in Y. If Y, Z are metric spaces, this definition is equivalent to saying that for any $y_n \to y$, we have $F(y) \subseteq \underline{\lim} F(y_n) = \{z \in Z : \lim d_Z(z, F(y_n)) = 0\}$, where $d_Z(\cdot, \cdot)$ is the metric on Z and $d_Z(z, F(y_n)) = \inf\{\|z - z'\| : z' \in F(y_n)\}$. We will say that $F : Y \to 2^Z \setminus \{\varnothing\}$ is upper semicontinuous (u.s.c.), if for all $U \subseteq Z$ open, $F^{+}(U) = \{y \in Y : F(y) \subseteq U\}$ is open in Y (see Delahaye-Denel [3]). A multifunction $F(\cdot)$ is said to be continuous, if it is both u.s.c. and l.s.c.

If X is a Banach space, then on $P_f(X)$ we can define a generalized metric $h(\cdot, \cdot)$, known as the Hausdorff metric, by setting

$$h(A, B) = \max\Big\{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\Big\}.$$

Recall that $(P_f(X), h)$ is a complete metric space. A multifunction $F : X \to P_f(X)$ is said to be Hausdorff continuous (h-continuous), if it is continuous as a function from X into the metric space $(P_f(X), h)$.

Let X be a Banach space, $K \subseteq X$ nonempty and $x \in K$. The "Bouligand or contingent cone" to K at x defined by

$$T_K(x) = \Big\{h \in X : \underline{\lim}_{\lambda \downarrow 0} \frac{d_K(x + \lambda h)}{\lambda} = 0\Big\},$$

where for any $v \in X$, $d_K(v) = \inf\{\|v - x'\| : x' \in K\}$ (see Aubin-Cellina [2]). It is clear that this cone is closed, that $T_K(x) = T_{\bar{K}}(x)$ and furthermore if $x \in \operatorname{int} K$, then $T_K(x) = X$. In general $T_K(x)$ is not convex. However, if K is a convex set, then so is $T_K(x)$. Also note that if $\operatorname{int} K \neq \varnothing$, then $\operatorname{int} T_K(x) \neq \varnothing$ (see Aubin-Cellina [2]).

For a given multifunction $F : Y \to 2^Z \setminus \{\varnothing\}$, by the "graph of $F(\cdot)$" we will mean the set $\operatorname{Gr} F = \{(x, y) \in X \times Y : y \in F(x)\}$.

Now we will state two lemmata that we will need in the sequel and which are also of independent interest. The first gives us sufficient conditions for the intersection of two multifunctions to be l.s.c..

LEMMA 1. *If Y, Z are Hausdorff topological spaces, $G : Y \to 2^Z \setminus \{\varnothing\}$ is l.s.c., $F : Y \to 2^Z \setminus \{\varnothing\}$ has open graph and for all $y \in Y$, $G(y) \cap F(y) \neq \varnothing$, then $y \to H(y) = G(y) \cap F(y)$ is l.s.c.*

Proof. We need to show that for every $U \subseteq Z$ open $H^{-}(U) = \{y \in Y : H(y) \cap U \neq \varnothing\}$ is open in Y. Let $y \in H^{-}(U)$ and $z \in G(y) \cap F(y) \cap U$. Then $(y, z) \in \operatorname{Gr} F \cap (Y \times U)$. By hypothesis $F(\cdot)$ has an open graph. So $\operatorname{Gr} F \cap (U \times U)$ is an open subset $Y \times Z$. So we can find $U_1(y)$ (a neighborhood of y) and $V_1(z)$ (a neighborhood of z) such that $U_1(y) \times V_1(z) \subseteq \operatorname{Gr} F \cap (Y \times U)$. Note that $G(y) \cap V_1(z) \neq \varnothing$ since it contains z. Recalling that $G(\cdot)$ is l.s.c., we can find $U_2(y)$ (an open neighborhood of y) such that $G(y') \cap V_1(z) \neq \varnothing$ for all $y' \in U_2(y)$. Set $U(y) = U_1(y) \cap U_2(y)$. Then for all $y' \in U(y)$ we have $G(y') \cap V_1(z) \neq \varnothing$, while $U(y) \times V_1(z) \subseteq \operatorname{Gr} F \cap (Y \times U)$. Thus for all $y' \in U(y)$, $G(y') \cap F(y') \cap U = H(y') \cap U \neq \varnothing \implies H^-(U)$ is open $\implies H(\cdot)$ is l.s.c.. Q.E.D.

The second lemma establishes an interesting lower semicontinuity property of the Bouligand cone. Let X be a reflexive Banach space which along with its dual is strictly convex and let $K \subseteq X$ be nonempty, convex. For $x \in K$ we define the cone $S_K(x)$ spanned by $K - x$; i.e. $S_K(x) = \bigcup_{\lambda > 0} \lambda^{-1}(K - x)$. Observe that for every $x \in K$, $K \subseteq x + S_K(x) \subseteq x + T_K(x)$ and $T_K(x) = \overline{S_K(x)}$.

LEMMA 3. *If $P : X \to P_{fc}(X)$ is an h-continuous multifunction, then $x \to \hat{T} = T_{P(x)}(x)$ is l.s.c.*

Proof. Let $x_n \xrightarrow{s} x$ and let $c \in S_{P(x)}(x)$. Then by definition there exists $\lambda > 0$ such that $x + \lambda v \in P(x)$. Set $w_n = \operatorname{proj}_{P(x_n)}(x + \lambda v)$. Since $P(\cdot)$ is h-continuous, from theorem 3.33, p. 322 of Attouch [1], we have $w_n \xrightarrow{s} x + \lambda v$. Hence $v_n = (w_n - x_n)/\lambda \to v$ and clearly $x_n + \lambda v_n = w_n \in P(x_n)$. So $v_n \in T_{P(x_n)}(x_n)$ $\implies T_{P(x)}(x) \subseteq \underline{\lim} T_{P(x_n)}(x_n)$ and this implies that $\hat{T}(\cdot) = T_{P(\cdot)}(\cdot)$ is l.s.c. (see the beginning of this section).

3. Main theorem.

In this section we state and prove our result on the existence of ψ-minimal monotone trajectories.

Let K be a nonempty, closed convex subset of $\mathbf{R}^n$. A preorder $\preceq$ on K is a binary relation $x \preceq y$, which is (i) reflexive and (ii) transitive. A trajectory $x : T \to K$ of a given differential inclusion is said to be "monotone" if and only if for all $t, s \in T$, $s \leq t \implies x(t) \preceq x(s)$. It is convenient to equivalently characterize a preorder using a multifunction $P(\cdot)$ from K into 2^K. So $y \in P(x)$ if and only $y \preceq x$. Thus a trajectory $x(\cdot)$ is monotone for the given preorder if and only if for all $s, t \in T$, $t \geq s \implies x(t) \in P(x(s))$. In economic models a typical example of a preorder is defined by a family of utility functions $\{V_k(\cdot)\}_{k=1}^n$, $V_k : K \to \mathbf{R}$, $k = 1, 2, \ldots, n$. Then for all $x \in K$, $P(x) = \{y \in K : V_k(y) \leq V_k(x)$ for all $k = 1, 2, \ldots, n\}$. In this case a trajectory $x : T \to K$ is monotone if and only if for all $s, t \in T$, $t \geq s$, we have $V_k(x(t)) \leq V_k(x(s))$ for all $k = 1, 2, \ldots, n$. Haddad [5] proved that a necessary and sufficient condition for the existence of a monotone trajectory, is that for all $x \in K$, $F(x) \cap T_{P(x)}(x) \neq \varnothing$. Note that if $P(x) = K$ for all $x \in K$, then monotone trajectories are simply viable trajectories and then Haddad's condition is nothing else but the well known Nagumo tangential condition, stated in the language of contingent cones.

A trajectory $x : T \to X$ is said to be ψ-minimal, if its derivative minimizes a given criterion $\psi(\cdot)$; i.e. $\psi(\dot{x}(t)) = \min\{\psi(z) : z \in R(x(t))\}$ a.e. where $R(x) = F(x) \cap T_{P(x)}(x)$. When $\psi(x) = \|x\|$, then ψ-minimal trajectories are just slow (i.e. with minimal velocity) solutions, which appear often in dynamic economic models and in systems theory.

Let $T = [0, b]$ and let the state space be $\mathbf{R}^n$. The differential inclusion under consideration defined on $\mathbf{R}^n$, is the following:

$$\begin{aligned} &\dot{x}(t) \in F(x(t)) \quad \text{a.e.} \\ &x(0) = x_0 \in K; \qquad x(t) \in K, \quad t \in T. \end{aligned} \tag{$*$}$$

We will need the following hypotheses on the data of $(*)$.

$H(K)$: $K \subseteq \mathbf{R}^n$ is a nonempty, closed convex set.

$H(F)$: $F : K \to P_{\mathrm{fc}}(\mathbf{R}^n)$ is a multifunction s.t.

(1) $F(\cdot)$ is continuous.

(2) $|F(x)| = \sup\{\|z\| : z \in F(x)\} \leq k(1 + \|x\|)$, $k > 0$

$H(P)$: $P : K \to P_{\mathrm{fc}}(K)$ is an h-continuous preorder.

H_τ: Fore every $x \in K$, $F(x) \cap T_{P(x)}(x) \neq \varnothing$.

We will also need a hypothesis on the criterion $\psi(\cdot)$. Recall that a function $\psi \in \mathbf{R}^{\mathbf{R}^n}$ is inf-compact if and only if for every $\lambda \in \mathbf{R}$, $\{x \in \mathbf{R}^n : \psi(x) \leq \lambda\}$ is compact.

$H(\psi)$: $\psi : \mathbf{R}^n \to \mathbf{R}^n$ is convex, continuous and inf-compact.

THEOREM 3.1. *If hypotheses $H(K)$, $H(F)$, $H(P)$, H_τ, and $H(\psi)$ hold, then $(*)$ admits a ψ-minimal, monotone trajectory in K.*

Proof. Let $B_1 = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$ (i.e. the closed unit ball in $\mathbf{R}^n$). Set $F_n(x) = F(x) + B_1/n$. Then $\operatorname{int} F_n(x) = F(x) + (\operatorname{int} B)/n$, where $\operatorname{int} B_1 = \{x \in \mathbf{R}^n : \|x\| < 1\}$. Clearly for all $n \geq 1$ $\operatorname{int} F_n(\cdot)$ has an open graph. Also from Lemma 2 we know that $x \to T_{P(x)}(x)$ is l.s.c.. Hence Lemma 1 tells us that $x \to R_n(x) = \operatorname{int} F_n(x) \cap T_{P(x)}(x)$ is.l.s.c. $\implies \overline{R_n(x)} = F_n(x) \cap T_{P(x)}(x)$ is l.s.c.. Define $G_n : K \to 2^{\mathbf{R}^n}$ by $G_n(x) = \{y \in \mathbf{R}^n : \psi(y) \leq \inf_{z \in R_n(x)} \psi(z)\}$. Since $\psi(\cdot)$ is continuous and $R_n(\cdot)$ is compact valued, it is clear that $G_n(x) \neq \varnothing$. Also from the continuity and convexity hypotheses on $\psi(\cdot)$, we deduce that $G_n(\cdot)$ is $P_{\mathrm{fc}}(\mathbf{R}^n)$-valued. We claim that $G_n(\cdot)$ has a closed graph in $K \times \mathbf{R}^n$. To this end let $\{(x_m, y_m)\}_{m \geq 1} \subseteq \operatorname{Gr} G$ and assume that $(x_m, y_m) \to (x, y)$ in $\mathbf{R}^n \times \mathbf{R}^n$. By setting $\beta_n(x) = \inf\{\psi(z) : z \in R_n(x)\}$, we have:

$$\psi(y_m) \leq \beta_n(x_m) \implies \lim_{m \to \infty} \psi(y_m) = \psi(y) \leq \overline{\lim_{m \to \infty}} \beta_n(x_m).$$

But from Theorem 4, p. 51 of Aubin-Cellina [2], we know that for every $n \geq 1$, $\beta_n(\cdot)$ is u.s.c.. Hence we get

$$\psi(y) \leq \beta_n(x) \implies (x, y) \in \operatorname{Gr} G_n \implies \operatorname{Gr} G_n \text{ is closed, as claimed.}$$

Invoking theorem 1, p. 41 of Aubin-Cellina [2], we deduce that $x \to L_n(x) = F_n(x) \cap G_n(x)$ is u.s.c.. Now consider the following viabilty problem:

$$\begin{aligned} &\dot{x}_n(t) \in L_n(x_n(t)) \quad \text{a.e.} \\ &x_n(0) = x_0, \qquad x_n(t) \in K, \quad t \in T. \end{aligned} \tag{$*$}'_n$$

Observe that from the construction of $L_n(\cdot)$, we have $L_n(x) \cap T_{P(x)}(x) \neq \emptyset$ for all $x \in K$. So invoking Haddad's result [5], we get that $(*)'_n$ has a viable solution $x_n : T \to K$. Observe that

$$\|\dot{x}_n(t)\| \leq (k+1) + k\|x_n(t)\| \text{ a.e.} \implies \frac{d}{dt}\|x_n(t)\| \leq (k+1) + k\|x_n(t)\| \text{ a.e.}.$$

Integrating the above inequality and invoking Gronwall's lemma, we get $\|x_n\|_\infty \leq M < \infty$ for all $n \geq 1$. Hence $\{\dot{x}_n(\cdot)\}_{n\geq 1}$ is uniformly integrable in $L^1(T, \mathbf{R}^n) = L^1_n(T)$ and this in turn implies that $\{x_n(\cdot)\}_{n\geq 1}$ is an equicontinuous subset of $C(T, \mathbf{R}^n) = C_n(T)$. Applying the Arzela-Ascoli theorem, we deduce that $\{x_n(\cdot)\}_{n\geq 1}$ is relatively compact in $C_n(T)$. So by passing to a subsequence if necessary, we may assume that $x_n \to x$ in $C_n(T)$ and $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^1_n(T)$. Also note that by construction, for every $n \geq 1$ $x_n(\cdot)$ is the ψ-minimal trajectory of $(*)'_n$. Since by hypothesis $H(\psi)$ $\psi(\cdot)$ is inf-compact, we can apply Theorem 2.11, p. 132 of Attouch [1] and get that $\beta_n(x) \uparrow \beta(x)$ for all $x \in K$. If we set $I^A_\psi(y) = \int_A \psi(y(t))\,dt$ for all $A \in B(T) =$ Borel σ-field of T, $y(\cdot) \in L^1_n(T)$ and because $\psi(\cdot)$ is convex, we can easily check that $I^A_\psi(\cdot)$ is weakly-l.s.c. on $L^1_n(T)$. So we have:

$$\begin{aligned}\int_A \psi(\dot{x}(t))\,dt &\leq \underline{\lim} \int_A \psi(\dot{x}_n(t))\,dt \leq \overline{\lim} \int_A \beta_n(x_n(t))\,dt \\ &\leq \int_A \overline{\lim}\, \beta(x_n(t))\,dt \quad \text{(by Fatou's lemma)} \\ &\leq \int_A \beta(x(t))\,dt \quad \text{(since } \beta(\cdot) \text{ is u.s.c.)}\end{aligned}$$

Since $A \in B(T)$ was arbitrary, we deduce that $\psi(\dot{x}(t)) \leq \beta(x(t))$ a.e. On the other hand, since $\dot{x}_n \xrightarrow{w} \dot{x}$ in $L^1_n(T)$, from theorem 3.1 of [7] we have:

$$\dot{x}(t) \in \overline{\text{conv}}\, w\text{-}\overline{\lim}\{\dot{x}_n(t)\}_{n\geq 1} \subseteq \overline{\text{conv}}\, \overline{\lim}\, F_n(x_n(t)) \quad \text{a.e.}$$

But note that $\overline{\lim}\, F_n(x_n(t)) = \overline{\lim}(F(x_n(t)) + B_1/n) \subseteq \overline{\lim}\, F(x_n(t)) + \overline{\lim}\, B_1/n$. Since by hypothesis $H(F)$, $F(\cdot)$ is u.s.c. and $x_n \to x$ in $C_n(T)$, we have $\overline{\lim}\, F(x_n(t)) \subseteq F(x(t))$. Also $\overline{\lim}\, B_1/n = \{0\}$. So finally we get $\overline{\lim}\, F_n(x_n(t)) \subseteq F(x(t))$ a.e. and thus we can write that $\dot{x}(t) \in F(x(t))$ a.e. Furthermore for all $n \geq 1$ and all $t, s \in T$, $t \geq s$, we have $x_n(t) \in P(x_n(s))$. Since by hypothesis $H(P)$, $P(\cdot)$ is h-continuous, in the limit as $n \to \infty$ we get $x(t) \in P(x(s))$ for all $t, s \in T$, $t \geq s$. Therefore $x(\cdot) \in C_n(T)$ is the desired ψ-minimal, monotone trajectory of $(*)$. Q.E.D.

4. Control systems. In this section, we will apply Theorem 3.1, get feedback controls that generate monotone trajectories for a nonlinear control system.

More specifically, the problem under consideration is the following:

$(\hat{P})$: "Find a "state-control" pair $(X(\cdot), u(\cdot)) \in C_n(T) \times L^1_m(T)$ satisfying $\dot{x}(t) = f(x(t), u(t))$ a.e., $x(0) = x_0$, $u(t) \in U(x(t))$ a.e., so that the state

$x(\cdot)$ is ψ-minimal and P-monotone for a given criterion $\psi(\cdot)$ and a given preorder $P(\cdot)$ on the viablity domain K."

We will need the following hypotheses on the data of $(\hat{P})$.

$H(K)'$: $K \subseteq \mathbf{R}^n$ is nonempty, closed and convex.

$H(f)$: $f: K \times \mathbf{R}^m \to \mathbf{R}^n$ is a continuous map s.t. $|f(x,u)| \leq k(1 + \|x\| + \|u\|)$, $k > 0$.

$H(P)'$: $P: K \to P_{fc}(K)$ is an h-continuous preorder.

$H(U)$: $U: K \to P_{fc}(\mathbf{R}^m)$ continuous $|U(x)| = \sup\{\|v\| : v \in U(x)\} \leq k'\|x\|$ and $f(x, U(t)) = \bigcup_{u \in U(x)} f(x,u)$ is convex.

H_τ: $G(x) \cap \{u \in U(x) : f(x,u) \in T_{P(x)}(x) \neq \varnothing\}$ for all $x \in K$.

THEOREM 4.1. *If hypotheses $H(K)'$, $H(f)$, $H(P)'$, $H(U)$ and H'_τ hold, then there exists feedback control $u(\cdot)$ solving problem $(\hat{P})$.*

Proof. Let $F(x) = f(x, U(x))$. From hypotheses $H(f)$ and $H(U)$ we know that $F(\cdot)$ has nonempty, compact, convex values. Also for every $v \in \mathbf{R}^n$, we define: $\sigma_{F(x)}(v) = \sup\{(v, f(x,u)) : u \in U(x)\}$. (the support function of $F(x)$). From theorem 6, p. 53 of Aubin-Cellina [2], we have that $x \to \sigma_{F(x)}(v)$ continuous $\Longrightarrow$ F continuous and $|F(x)| \leq k(1 + (1 + k')\|x\|)$. Also note that because of $'_\tau$ we have $F(x) \cap T_{P(x)}(x) \neq \varnothing$ for all $x \in K$. Hence we can apply Theorem 3.1 and get a ψ-minimal, monotone trajectory for the differential inclusion $\dot{x}(t) \in F(x(t))$ a.e., $x(0) = x_0$. Then let $\Gamma(t) = \{u \in U(x(t)) : \dot{x}(t) = f(x(t), u)\}$. By redefining $\Gamma(\cdot)$ on a Lebesgue null set, we may assume that $\Gamma(t) \neq \varnothing$ for all $t \in T$. Also

$$\operatorname{Gr} \Gamma = \{(t,u) \in \operatorname{Gr} U(x(\cdot)) : \dot{x}(t) - f(x(t), u) = 0\}.$$

Since $U(\cdot)$ is u.s.c., it is measurable (see Wagner [8]) and so $\operatorname{Gr} U(x(\cdot)) \in B(T) \times B(\mathbf{R}^m)$, (where $B(T)$ and $B(\mathbf{R}^m)$ are the Borel σ-fields of T and $\mathbf{R}^m$ repsectively), while from hypothesis $H(f)$, we get that $(t,u) \to \dot{x}(t) - f(x(t), u)$ is a Caratheodory function, hence jointly measurable. Thus $\operatorname{Gr}\Gamma \in B(T) \times B(\mathbf{R}^m)$. Apply Aumann's selection theorem (see Wagner [8]), to get $u: T \to \mathbf{R}^m$ measurable s.t. $u(t) \in \Gamma(t)$ for all $t \in T$. Then $u(\cdot)$ is the desired feedbac control. Q.E.D.

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