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ON PASCU TYPE α-CLOSE-TO-STAR FUNCTIONS

R. Paravatham and S. Srinivasan

Abstract. We introduce a new class of functions $CS(\alpha)$ and study some of its properties. Let $CS(\alpha)$ denote the class of holomorphic functions f in the unit disc E, with $f(0) = 0 = f'(0)-1$ and $f(z) \neq 0$, $\alpha zf'(z) + (1-\alpha)f(z) \neq 0$ in E satisfying

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)}\right\} d\theta > -\pi$$

whenever $0 \leq \theta_1 < \theta_2 \leq \theta_1 + 2\pi$, $z = re^{i\theta}$, $0 < r < 1$ and α is a non negative real number. The functions in $CS(\alpha)$ unify the class CS^* of Reade ($\alpha = 0$) [4] and the class C of Kaplan ($\alpha = 1$) [3]. Though the class $P(\alpha)$ of Bharati [1] also unifies these two classes, the class $CS(\alpha)$ is quite different from the class $P(\alpha)$.

Introduction. Let f be a holomorphic function in the open unit disc E with $f(0) = 0 = f'(0) - 1$. Let the class of functions be denoted by A.

Let $CS(\alpha)$ denote the class of functions f in A with $f(z) \neq 0$ and $\alpha zf'(z) + (1-\alpha)f(z) \neq 0$ in $E - \{0\}$ satisfying

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)}\right\} d\theta > -\pi$$

whenever $0 \leq \theta_1 < \theta_2 \leq \theta_1 + 2\pi$, $z = re^{i\theta}$, $0 < r < 1$ and α non negative real number. The functions in the class $CS(\alpha)$ will be called α-close-to-star functions.

When $\alpha = 0$ this class $CS(\alpha)$ reduces to the class CS^* of Reade [4] and when $\alpha = 1$ this is the class C of Kaplan [3]. Thus the classes $CS(\alpha)$ provide a continuous passage from the class CS^* to the class C.

Bharati [1] introduced the class $P(\alpha)$ — the class of functions f analytic in the unit disc E with $f(0) = 0$, $f(z) \neq 0$ for $z \neq 0$ and $f'(z) \neq 0$ in E satisfying the condition

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{\alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1-\alpha)\frac{zf'(z)}{f(z)}\right\} d\theta > -\pi$$

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whenever $0 \leq \theta_1 < \theta_2 \leq \theta_1 + 2\pi$, $z = re^{i\theta}$, $0 < r < 1$ and α is a positive real number. In spite of the fact that the class $P(\alpha)$ also unifies the classes of close-to-starlike ($\alpha = 0$) and close-to-convex ($\alpha = 1$) functions, this class $CS(\alpha)$ is quite different from $P(\alpha)$.

In this paper, we study some properties of the class $CS(\alpha)$. In particular employing a lemma due to Blezu and Pascu, we derive the invariant property of this class $CS(\alpha)$ under the transform by certain integral operator.

First let us state without proof two results which will be used in the sequel.

LEMMA 1. Blezu and Pascu [2]. *Let q be holomorphic in E with $q(0) = 1$. If for every $r \in (0,1)$, $\gamma \in [0,1]$, a, θ_1, θ_2 with* $\operatorname{Re} a \geq 0$, $0 \leq \theta_1 < \theta_2 \leq \theta_1 + 2\pi$

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{ q(z) + \frac{zq'(z)}{a + q(z)} \right\} d\theta > -\pi\gamma,$$

then $\int_{\theta_1}^{\theta_2} \operatorname{Re} q(z)\, d\theta > -\pi\gamma$, $z = re^{i\theta}$.

LEMMA 2 [5]. *Let ϕ be a convex univalent function and g be a starlike univalent function. Then for any function F holomorphic in E with* $\operatorname{Re} F(z) > 0$ *in E we have* $\operatorname{Re}\big((\phi * Fg)(z)/(\phi * g)(z)\big) > 0$, *where $*$ stands for Hadamard product or convolution.*

As a result of Theorem 3 in [4] we can define the class $CS(\alpha)$ in another equivalent form namely $f \in CS(\alpha)$ if and only if $\{\alpha z f'(z) + (1-\alpha) f(z)\} \in CS^*$ of Reade. That is there exists a $g(z)$ in S^* — the class of starlike univalent functions — such that

$$\operatorname{Re}\big\{ \big(\alpha z f'(z) + (1-\alpha) f(z)\big)/g(z) \big\} > 0 \quad \text{in } E.$$

THEOREM 1 (Integral representation). *Any $f \in A$ is in the class $CS(\alpha)$ if and only if there exists a $g \in S^*$ and $p \in P$ — the Caratheodory class — such that*

$$\begin{aligned} f(z) &= \frac{1}{\alpha z^{1/\alpha - 1}} \int_0^z t^{1/\alpha - 2} p(t) g(t)\, dt \quad && \textit{for } \alpha \neq 0 \\ &= p(z) g(z) && \textit{for } \alpha = 0. \end{aligned}$$

Proof. $f \in CS(\alpha) \iff \alpha z f'(z) + (1-\alpha) f(z) \in CS^*$. Hence there is a $g(z) \in S^*$ such that

$$\operatorname{Re}\big\{ \big(\alpha z f'(z) + (1-\alpha) f(z)\big)/g(z) \big\} > 0 \quad \text{in } E,$$

or $\alpha z f'(z) + (1-\alpha) f(z) = p(z) g(z)$, where $g \in S^*$ and $p \in P$ — the Caratheodory class of functions. If $\alpha \neq 0$, then multiplying the above equation by $\alpha^{-1} z^{1/\alpha - 2}$ and integrating with respect to z, we get

$$z^{1/\alpha - 1} f(z) = \frac{1}{\alpha} \int_0^z t^{1/\alpha - 2} p(t) g(t)\, dt$$

or

$$f(z) = \frac{1}{\alpha z^{1/\alpha - 1}} \int_0^z t^{1/\alpha - 2} p(t) g(t)\, dt.$$

If $\alpha = 0$ then $f(z) = p(z)g(z)$ where $p \in P$ and $g \in S^*$. The converse is immediate.

THEOREM 2 (Coefficient estimate). *If* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *is an* α*-close-to-star function, then*

$$a_n \leq n^2/(\alpha n + 1 - \alpha) \quad \textit{for } \alpha \geq 0$$

This result is sharp for $\alpha \geq 0$.

Proof. Since $f \in CS(\alpha)$ there exists a starlike function $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$ and an analytic function $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ with $\operatorname{Re} p(z) > 0$ in E such that

$$\alpha z f'(z) + (1 - \alpha) f(z) = p(z) g(z);$$

or

$$\begin{aligned} z + \sum_{n=2}^{\infty} (\alpha n + 1 - \alpha) a_n z^n \\ &= (1 + p_1 z + \cdots + p_n z^n + \cdots)(z + g_2 z^2 + \cdots + g_n z^n + \cdots) \\ &= z + (p_1 + g_2) z^2 + \cdots + (p_{n-1} p_{n-2} g_2 + \cdots + g_{n-1} p_1 + g_n) z^n. \end{aligned}$$

Now by comparing the coefficients on both sides we get

$$(\alpha n + 1 - \alpha) a_n = p_{n-1} + p_{n-2} g_2 + \cdots + g_{n-1} p_1 + g_n.$$

Using the fact that $|g_n| \leq n$, $\forall n \geq 2$, and $|p_n| \leq 2$, $\forall n \geq 1$, we get

$$(\alpha n + 1 - \alpha)|a_n| \leq 2(1 + 2 + \cdots + n - 1) + n = n^2.$$

Thus $|a_n| \leq n^2/(\alpha n + 1 - \alpha)$.

The bounds are attained by the function

$$f(z) = \frac{1}{\alpha z^{1/\alpha - 1}} \int_0^z \frac{t^{1/\alpha - 1}(1 + t)}{(1 - t)^3}\, dt, \quad \text{for } \alpha > 0.$$

For $\alpha = 0$ the extremal function is the Robertson function starlike in one direction.

Remark. We get the sharp inequality $|a_n| \leq n$ for the Kaplan's class C (proved in [4]), as a special case of our theorem with $\alpha = 1$. Similarly when $\alpha = 0$ we get the sharp inequality $|a_n| \leq n^2$ for the class CS^* proved in [4] as a particular case of our theorem.

THEOREM 3. *Let* $f \in CS_\alpha$. *Then* $f \in CS^*$ *for* $0 \leq \alpha \leq 1$.

Proof. Let $q(z) = zf'(z)/f(z)$. Then

$$z(zf'(z))' = zq'(z)f(z) + zq(z)f'(z).$$

Now
$$\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)} = \frac{zq'(z)}{q(z) + (1/\alpha) - 1} + q(z).$$
Since $f \in CS_\alpha$ we have for $0 \le \theta_1 < \theta_2 \le \theta_1 + 2\pi$, $z = re^{i\theta}$, $r \in (0,1)$
$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{\frac{\alpha z(zf'(z))' + (1-\alpha)zf'(z)}{\alpha zf'(z) + (1-\alpha)f(z)}\right\} d\theta > -\pi$$
or
$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{\frac{zq'(z)}{q(z) + (1/\alpha) - 1} + q(z)\right\} d\theta > -\pi.$$
An application of Lemma 1 gives
$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\, q(z)\, d\theta > -\pi \quad \text{for } 0 \le \theta_1 < \theta_2 \le \theta_1 + 2\pi,\ z = re^{i\theta}, \quad r \in (0,1), \text{ and } 0 \le \alpha \le 1.$$

Thus for $0 \le \alpha \le 1$, $f \in CS^*$.

THEOREM 4. *Let f be an α-close-to-star function. Then F defined by*
$$F(z) = \frac{1}{\alpha z^{1/\alpha - 1}} \int_0^z t^{1/\alpha - 2} f(t)\, dt$$
is also an α-close-to-star function for $0 < \alpha \le 1$.

Proof. It is clear that F is holomorphic in E with $F(0) = 0 = F'(0) - 1$. Now differentiation of the above integral with respect to z gives
$$z^{1/\alpha - 1}F'(z) + (1/\alpha - 1)z^{1/\alpha - 2}F(z) = \alpha^{-1} z^{1/\alpha - 2} f(z);$$
or
$$\alpha zF'(z) + (1-\alpha)F(z) = f(z) \quad \text{and} \quad \alpha z(zF'(z))' + (1-\alpha)zF'(z) = zf'(z).$$
Thus
$$\frac{\alpha z(zF'(z))' + (1-\alpha)zF'(z)}{\alpha zF'(z) + (1-\alpha)F'(z)} = \frac{zf'(z)}{f(z)}.$$
Now
$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{\frac{\alpha z(zF'(z))' + (1-\alpha)zF'(z)}{\alpha zF'(z) + (1-\alpha)F(z)}\right\} d\theta = \int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} d\theta$$
with $0 \le \theta_1 < \theta_2 \le 2\pi + \theta_1$, $z = re^{i\theta}$, $r \in (0,1)$.

Since $f \in CS^*$ for $0 < \alpha \le 1$ (by the previous theorem) we have for $0 \le \theta_1 < \theta_2 \le \theta_1 + 2\pi$, $z = re^{i\theta}$, $r \in (0,1)$
$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{\frac{\alpha z(zF'(z))' + (1-\alpha)zF'(z)}{\alpha zF'(z) + (1-\alpha)F(z)}\right\} d\theta > -\pi,$$

or $F \in CS(\alpha)$ for $0 < \alpha \leq 1$.

THEOREM 5. *Let ϕ be a convex univalent function and $f \in CS(\alpha)$. Then $F = \phi * f \in CS(\alpha)$.*

Proof. Let $f \in CS(\alpha)$. Then there is a $g \in S^*$ such that

$$\mathrm{Re}\,\frac{\alpha z f'(z) + (1-\alpha)f(z)}{g(z)} > 0 \quad \text{in } E.$$

Let ϕ be a convex univalent function and let further $F(z) = \phi * f$. Then

$$\begin{aligned}\alpha z F'(z) + (1-\alpha)F(z) &= \alpha\phi(z) * zf'(z) + (1-\alpha)\phi(z) * f(z)\\ &= \phi(z) * [\alpha z f'(z) + (1-\alpha)f(z)]\end{aligned}$$

$$\begin{aligned}\mathrm{Re}\left\{\frac{\alpha z F'(z) + (1-\alpha)F(z)}{(\phi * g)(z)}\right\} &= \mathrm{Re}\left\{\frac{\phi(z) * [\alpha z f'(z) + (1-\alpha)f(z)]}{(\phi * g)(z)}\right\}\\ &= \mathrm{Re}\left\{\frac{\phi(z) * \dfrac{\alpha z f'(z) + (1-\alpha)f(z)}{g(z)}g(z)}{(\phi * g)(z)}\right\}.\end{aligned}$$

It is well known that $G(z) = (\phi * g)(z)$ is in S^* whenever $g \in S^*$ and ϕ is convex univalent in E. Now since $f \in CS_\alpha$

$$\mathrm{Re}\left\{\frac{\alpha z f'(z) + (1-\alpha)f(z)}{g(z)}\right\} > 0 \quad \text{in } E.$$

Hence an application of Lemma 2 gives

$$\mathrm{Re}\left\{\frac{\alpha z F'(z) + (1-\alpha)F(z)}{G(z)}\right\} > 0,$$

where $G(z) = (\phi * g)(z) \in S^*$ which means $F \in CS(\alpha)$.

As a corollary we have: if $\alpha = 1$ then $f \in C$. Thus from the theorem above we get $\phi * f \in C$ whenever ϕ is a convex univalent function. This result had been conjectured and proved by Ruscheweyh and Sheil Small in [5].

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Ramanujan Institute
University of Madras
Madras – 600 005
India

Department of Mathematics
Presidency College
Madras – 600 005
India

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