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BARWISE COMPLETENESS THEOREMS FOR LOGICS WITH INTEGRALS

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Abstract. We prove the Barwise completeness theorem for the logic $L_{\mathcal{A} \int_1 \int_2}$ for both absolutely continuous and singular cases.

0. Let $\mathcal{A}$ be a countable admissible set and $\omega \in \mathcal{A}$. The logics $L^a_{\mathcal{A} \int_1 \int_2}$ and $L^s_{\mathcal{A} \int_1 \int_2}$ are similar to the standard logic $L_{\mathcal{A} \int}$ (see [3]). The only difference is that two types of integral operators $\int_1 \dots dx$ and $\int_2 \dots dx$ are allowed.

A biprobability model for L is a structure $\mathfrak{M} = (M, R_i^{\mathfrak{M}}, c_j^{\mathfrak{M}}, \mu_1, \mu_2)$ where μ_k is a countably additive probability measure on M such that each singleton is measurable, each $R_i^{\mathfrak{M}}$ is $\mu_k^{(n_i)}$-measurable ($k = 1, 2$), and each $c_j^{\mathfrak{M}} \in M$. (The measure $\mu_k^{(n)}$ is the restriction of the completion of μ_k^n to the σ-algebra generated by the measurable rectangles and the diagonal sets $\{\mathbf{x} \in M^n : x_i = x_j\}$.)

We shall see the difference in semantics for logics $L^a_{\mathcal{A} \int_1 \int_2}$ and $L^s_{\mathcal{A} \int_1 \int_2}$ using the following types of models.

Definition 1. (a) A biprobability model for $L^a_{\mathcal{A} \int_1 \int_2}$ logic is a biprobability structure $\mathfrak{M}$ such that μ_1 is absolutely continuous with respect to μ_2, i.e. $\mu_1 \ll \mu_2$.
(b) A biprobability model for $L^s_{\mathcal{A} \int_1 \int_2}$ logic is a biprobability structure $\mathfrak{M}$ such that μ_1 is singular with respect to μ_2.

A graded biprobability structure for L is a structure

$$\mathfrak{M} = (M, R_i^{\mathfrak{M}}, c_j^{\mathfrak{M}}, \mu_n^k)_{i \in I, j \in J, n \in \mathrm{N}, k=1,2}$$

such that:

(a) each μ_n^k is a countable additive probability measure on M^n;

(b) each n-placed relation $R_i^{\mathfrak{M}}$ is μ_n^k-measurable and the identity relation is μ_2^k-measurable;

(c) $\mu_n^k \times \mu_m^k \subseteq \mu_{m+n}^k$.

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(d) each μ_n^k is preserved under permutation of $\{1,2,\ldots,n\}$;

(e) $(\mu_n^k : n \in \mathbf{N})$ has the Fubini property: If B is μ_{m+n}^k-measurable then

(1) For each $\mathbf{x} \in M^m$, the section $B_{\mathbf{x}} = \{\mathbf{y} : B(\mathbf{x},\mathbf{y})\}$ is μ_n^k-measurable.

(2) The function $f(\mathbf{x}) = \mu_n^k(B_{\mathbf{x}})$ is μ_m^k-measurable.

(3) $\int f(\mathbf{x})\, d\mu_m^k = \mu_{m+n}^k(B)$.

Let us introduce two sorts of auxiliary models.

Definition 2. (a) A graded biprobability structure for $L^a_{\mathcal{A}\int_1\int_2}$ logic is a graded biprobability structure $\mathfrak{M}$ such that $\mu_n^1 \ll \mu_n^2$ for each $n \in \mathbf{N}$. (b) A graded biprobability structure for $L^s_{\mathcal{A}\int_1\int_2}$ logic is a graded biprobability structure $\mathfrak{M}$ such that $\mu_n^1 \perp \mu_n^2$ for each $n \in \mathbf{N}$.

We shall prove the Barwise completeness theorem for logics $L^a_{\mathcal{A}\int_1\int_2}$ and $L^s_{\mathcal{A}\int_1\int_2}$, that means that the set of all sentences ψ of $L_{\mathcal{A}\int_1\int_2}$ which are valid in all biprobability models is Σ_1 over $\mathcal{A}$.

1. In order to prove the main result for the absolutely continuous case, let us introduce one more logic $L^a_{\mathcal{A}\int_1\int_2 X}$, as in [5]. Biprobability models for the logic $L^a_{\mathcal{A}\int_1\int_2 X}$ are of the form

$$\mathfrak{M} = \left(M, R_i^{\mathfrak{M}}, c_j^{\mathfrak{M}}, [X \geq r]^{\mathfrak{M}}, [X \leq r]^{\mathfrak{M}}, \mu_k\right)_{i\in I,\, j\in J,\, r\in \mathbf{Q},\, k=1,2}$$

(a countable number of new unary relation symbols $[X \geq r]$, $[X \leq r]$, for $r \in \mathbf{Q}$, are added to the language of $L^a_{\mathcal{A}\int_1\int_2}$), where $X^{\mathfrak{M}} : M \to \mathbf{R}_+$ is the Radon-Nikodym derivative of μ_1 with respect to μ_2.

The axioms and rules of inference for the logic $L^a_{\mathcal{A}\int_1\int_2 X}$ are those of $L_{\mathcal{A}\int}$ as listed in [3], — remark that both integral operators $\int_1$ and $\int_2$ can play the role of $\int$, together with the following axioms:

Axioms of continuity:

A_1
$$\bigwedge_n \bigvee_m \bigvee_k \int_i G_k\left(\int_j \tau(\mathbf{x},\mathbf{y})\, d\mathbf{x}\right) d\mathbf{y} < \frac{1}{n}, \quad (i,j = 1,2)$$

where $d\mathbf{x} = dx_1 \ldots dx_p$ and

$$G_k(s) = \begin{cases} 1 & \text{if } r - 1/m + 1/k \leq s \leq r - 2/k \\ 0 & \text{if } s \leq r - 1/m \text{ or } s \geq r - 1/k \\ \text{linear for } r - 1/m \leq s \leq r - 1/m + 1/k \text{ or } r - 2/k \leq s \leq r - 1/k. \end{cases}$$

A_2
$$\bigwedge_n \bigvee_m \bigvee_k \int_i H_k\left(\int_j \tau(\mathbf{x},\mathbf{y})\, d\mathbf{x}\right) d\mathbf{y} < \frac{1}{n}, \quad (i,j = 1,2)$$

where

$$H_k(s) = \begin{cases} 1 & \text{if } r + 2/k \leq s \leq r + 1/m - 1/k \\ 0 & \text{if } s \leq r + 1/k \text{ or } s \geq r + 1/m \\ \text{linear for } r + 1/k \leq s \leq r + 2/k \text{ or } r + 1/m - 1/k \leq s \leq r + 1/m. \end{cases}$$

Axioms of random variable $(i = 1, 2)$:

A3
$$\bigvee_k \int_i J_k(\mathbf{1}[X < r](x), \mathbf{1}[X \geq s](x))\, dx \geq 1 \qquad (r \geq s)$$

where

$$J_k(s,t) = \begin{cases} 1 & \text{if } s \geq 2/k \text{ or } t \geq 2/k \\ 0 & \text{if } s \leq 1/k \text{ and } t \leq 1/k \\ \text{linear for other cases.} \end{cases}$$

A4
$$\bigvee_p \bigvee_k \int_i J_k\left(\mathbf{1}[X \leq r](x), \sum_{n=1}^{p} \mathbf{1}[X \geq r + 1/n](x)\right) dx \geq 1.$$

A5
$$\bigvee_p \bigvee_k \int_i J_k\left(\sum_{n=1}^{p} \mathbf{1}[X < r - 1/n](x), \mathbf{1}[X \geq r](x)\right) dx \geq 1$$

A6
$$\bigvee_p \bigvee_k \int_i L_k\left(\sum_{n=1}^{p} \mathbf{1}[X \geq -n](x), \sum_{n=1}^{p} \mathbf{1}[X \leq n](x)\right) dx \geq 1$$

where

$$L_k(s,t) = \begin{cases} 1 & \text{if } s \geq 2/k \text{ and } t \geq 2/k \\ 0 & \text{if } s \leq 1/k \text{ or } t \leq 1/k \\ \text{linear for other cases.} \end{cases}$$

Radon-Nikodym axioms $(n \in \mathbf{N};\ s_0, \ldots, s_n \in \mathbf{Q})$:

A7
$$\left(\bigwedge_{0 \leq i \leq n-1} \int_2 \tau(x) \cdot \mathbf{1}[X < s_{i+1}](x) \cdot \mathbf{1}[X \geq s_i](x)\, dx \geq r_i\right)$$
$$\Longrightarrow \left(\int_1 \tau(x)\, dx \geq \sum_{i=0}^{n-1} r_i s_i\right), \qquad (s_0 \leq s_1 \leq \ldots \leq s_n).$$

A8
$$\left(\bigwedge_{0 \leq i \leq n-1} \int_2 \tau(x) \cdot \mathbf{1}[X < s_{i+1}](x) \cdot \mathbf{1}[X \geq s_i](x)\, dx \leq r_i\right)$$
$$\Longrightarrow \left(\int_1 \tau(x) \cdot \mathbf{1}[X \leq m](x)\, dx \leq \sum_{i=0}^{n-1} r_i s_{i+1}\right),$$
$$(s_0 \leq s_1 \leq \ldots \leq s_n = m,\ m \in \mathbf{N}).$$

We shall first prove the completenses theorem for the logic $L^a_{\mathcal{A} \int_1 \int_2 X}$. The soundness theorem holds because all the axioms represent known properties of random variables and the Radon-Nikodym derivative. The method of proof is similar to the completeness proof of Hoover [2]. It uses Loeb's construction in [4] which corresponds to the Daniell integral. We first state a simple case of Loeb's result as a lemma.

LEMMA (Loeb [3]). *In an ω_1-saturated nonstandard universe, let L be an internal vector lattice of functions from an internal set M into* $^*\mathbf{R}$ *(the set of hyper real numbers), and let I be an internal positive linear functional on L, such that $1 \in L$ and $I(1) = 1$. Then there is a complete probability measure μ on M such that for each finitely bounded $\phi \in L$, the standard part of ϕ is integrable with respect to μ and its integral is equal to the standard part of $I(\phi)$.*

THEOREM 1 (Completeness theorem for $L^a_{\mathcal{A} \int_1 \int_2 X}$). *Every sentence which is consistent in $L^a_{\mathcal{A} \int_1 \int_2 X}$ has a biprobability model.*

Proof. Let ψ be a sentence which is consistent in $L^a_{\mathcal{A} \int_1 \int_2 X}$. Our plan is to use a Henkin construclion to bulid a weak model of ψ, i.e. a structure $\mathfrak{M} = \left(M, R_1^{\mathfrak{M}}, [X \geq r]^{\mathfrak{M}}, [X \leq r]^{\mathfrak{M}}, c_j^{\mathfrak{M}}, I_k\right)_{i \in I, j \in J, r \in \mathbf{Q}, k=1,2}$ where I_k is a positive linear real function on the set of terms of $L^a_{\mathcal{A} \int_1 \int_2 X}$ with at most one free varible and parametars from M. Let C be a countable set of new constant symbols. We use the Henkin construction to obtain a maximal consistent set Φ of sentences of $K^a_{\mathcal{A} \int_1 \int_2 X}$, where $K = L \cup C$, such that:

(1) $\psi \in \Phi$

(2) if $\left(\int_k \tau(x)\, dx > 0\right) \in \Phi$, then $(\tau(c) > 0) \in \Phi$ for some $c \in C$.

The witness properties (1) and (2) are obtained using the rule of generalization. Since Φ is complete and contains all the axioms of $K^a_{\mathcal{A} \int_1 \int_2 X}$ the Henkin theory Φ induces a weak structure $\mathfrak{M}$ with universe $M = C$, such that every sentence in Φ holds in $\mathfrak{M}$.

The next step is the construction of the graded biprobability model. This structure is formed in the non-standard universe. Using Lemma we obtain probability measures μ_1 and μ_2 on $^*M$ such that for each $*$-term $\tau(x)$, the standard part $^*I_k(\tau)$ is the integral $\int^0 \tau(b)^{\mathfrak{M}}\, d\mu_k(b)$. Define measures μ_n^k using iterated integrals. This graded biprobability model $\widehat{\mathfrak{M}} = ({}^*M, {}^*R_i, {}^*[X \geq r], {}^*[X \leq r], {}^*c_j, \mu_n^k)$ can be used to produce a biprobability model $(\mathcal{N}, \nu_k)$ of ψ (see [3]).

Let $X^{\mathcal{N}}(a) = \sup\{r \mid [X \leq r]^{\mathcal{N}}(a)\}$, $a \in \mathbf{N}$. Radon-Nikodym axioms guarantee that $\nu_1(B) = \int_B X^{\mathcal{N}}\, d\nu_2$, holds for any measurable set B, i.e. $\nu_1 \ll \nu_2$.

A biprobability model $\mathfrak{M}$ for $L^a_{\mathcal{A} \int_1 \int_2}$ can be expanded to a model of $L^a_{\mathcal{A} \int_1 \int_2 X}$ simply by taking $X^{\mathfrak{M}} = d\mu_1/d\mu_2$ and defining $[X \leq r]^{\mathfrak{M}}(a)$ if $X^{\mathfrak{M}}(a) \leq r$. The set of all valid $L^a_{\mathcal{A} \int_1 \int_2 X}$ sentences is Σ_1 definable, for example by the formula $\exists P\,(P$ is a proof of $\psi)$.

As a consequence of the preceding, we obtain our main result.

THEOREM 2 (Barwise Completeness Theorem for $L^a_{\mathcal{A} f_1 f_2}$). *The set of all valid sentences of the logic* $L^a_{\mathcal{A} f_1 f_2}$ *is* Σ_1 *definable over* $\mathcal{A}$.

2. In order to prove the Barwise completeness theorem for singular case, let us introduce one more logic $L^s_{\mathcal{A} f_1 f_2 R}$. That means that one more unary relation symbol R is added to the language of $L^s_{\mathcal{A} f_1 f_2}$.

Axioms and rules of inference for the logic $L^s_{\mathcal{A} f_1 f_2 R}$ are those of $L_{\mathcal{A} f}$ together with the following axiom:

B1 $$\int_1 1(R(x))\,dx = 0 \ \wedge \ \int_2 1(R(x))\,dx = 1.$$

The completeness theorem for $L^s_{\mathcal{A} f_1 f_2 R}$ is easy to prove. The proof makes use of the Loeb-Hoover-Keisler construction as in the absolutely continuous case. The only remark is that for the set $B = \{a \in \mathrm{N} \mid R^{\mathcal{N}}(a)\}$ we obtain $\nu_1(B) = 0$ and $\nu_2(B) = 1$, i.e. $\nu_1 \perp \nu_2$.

A biprobability model $\mathfrak{M}$ can be expanded to a model of $L^s_{\mathcal{A} f_1 f_2 R}$ simply by taking $R^{\mathfrak{M}}(a)$ iff $a \in B$, where B is the set given by the condition $\mu_1 \perp \mu_2$. As a consequence of the preceding, we obtain the Barwise completeness theorem for $L^s_{\mathcal{A} f_1 f_2}$.

THEOREM 3. *The set of all valid sentences of the logic* $L^s_{\mathcal{A} f_1 f_2}$ *is* Σ_1*-definable over* $\mathcal{A}$.

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