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λ-REGULAR OSCILLATION AND APPLICATIONS

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In the Proceedings of the International Congress of Mathematicians, 1950, (Vol. II) there is a short note of J. Karamata which contains (i) a new concept "regularity of convergence" and (ii) a test for convergence of Fourier series of continuous functions. Karamata never published a proof of his convergence test, and as he himself observed in several conversations, it is practically impossible to understand his result because of peculiar misprint at an essential place.[1] In this paper we intend to prove Karamata's convergence test, investigate its relationship to the classical tests for convergence of Fourier series, and answer some basic questions about "regularity of convergence".

Notation. For any real-valued function f defined on an interval $[\alpha, \beta]$ we set

$$\mathrm{Osc}[f;\alpha,\beta] = \sup\{|f(x) - f(y)| : x, y \in [\alpha,\beta]\}.$$

For a real-valued function f defined on $(0, c)$ we set

$$\Omega_f(\delta) = \limsup_{x\to 0+} \mathrm{Osc}[f; x, x(1+\delta)]. \tag{1}$$

Obviously, Ω_f is an extended real-valued function defined on $(0,\infty)$, non-negative, increasing and subadditive. These properties imply that either Ω_f is identically 0 or identically ∞ or $0 < \Omega_f(\delta) < \infty$ for every $\delta > 0$. Moreover,

$$\text{If } \liminf_{\delta\to 0+} \Omega_f(\delta)/\delta = 0 \text{ then } \Omega_f(\delta) = 0 \text{ for all } \delta > 0. \tag{2}$$

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[1] In the sentence
"Nous dirons que $f(x)$ est regulière d'ordre $\lambda(t)$ au voisinage du point x, si en posant

$$\varphi(t) = f(x+t) + f(x-t) - 2f(x)$$

on a

$$\varphi(t') = O[\lambda(t)] \quad \text{pour } t \le t' \le t\lambda(t),\ t \to 0."$$

the last line should be replaced by

$$\varphi(t') - \varphi(t) = O[\lambda(t)] \quad \text{pour } t \le t' \le t\lambda(t),\ t \to 0.$$

(This holds because for $0 < \delta < 1$ we have $\Omega_f(1) \le \Omega_f(2[1/\delta]\delta) \le 2\delta^{-1}\Omega_f(\delta)$.)

One way of ordering functions according to their oscillation would give the following hierarchy:

(i) f tends to a finite limit as $x \to 0+$ (iii) $\Omega_f(\delta) = O(\delta)$, $\delta \to 0+$

(ii) Ω_f is identically 0 (iv) $\Omega_f(\delta) \to 0$, $\delta \to 0+$.

Condition (iv) is well known from Tauberian theory, such functions f are said to be slowly oscillating (in the multiplicative sense) as $x \to 0+$. Condition (iii) is naturaly singled out: this is, in view of (2), the fastest that Ω_f can tend to 0 without becoming identically 0.

λ-regular oscillation. Let λ be a continuous and strictly increasing function on $[0,a)$ with $\lambda(0) = 0$.

Definition. f oscillates regularly as $x \to 0+$ means: (i) f is defined on $(0,c)$ for some $c > 0$ and (ii) there exist $\gamma > 0$, $K > 0$ such that

(3) $$\mathrm{Osc}[f; x, x(1+\lambda(x))] \le K\lambda(x) \quad \text{for } 0 < x < \gamma.$$

In this definition we depart somewhat from Karamata: we do not assume that f tends to a finite limit as $x \to 0+$ nor that λ is a function from the logarithmic-exponential class; accordingly we change the terminology and speak about "λ-regular oscillation" instead of "regular convergence of order λ".

Ordering according to the regularity of oscillation is quite different from the ordering of functions according to their oscillations: any differentiable function f such that $f'(x) = O(1/x)$, $x \to 0+$ oscillates λ-regularly for any λ. This class includes functions like

$$\sqrt{|\log x|} \cdot \sin\big(\sqrt{|\log x|}\big)$$

which oscillate between $-\infty$ and $+\infty$; on the other hand a function which converges as $x \to 0+$, like $\sqrt{x}\sin(1/x)$ does not oscillate λ-regularly for $\lambda(x) = x$.

The oscillation of a λ-regularly oscillating function f on an arbitrary interval $[u,v]$ can be estimated as follows.

THEOREM 1. *f oscillates λ-regularly if and only if there exist $\delta > 0$, $C > 0$ such that*

(4) $$\mathrm{Osc}[f; u, v] \le C \max\{v/u - 1, \lambda(u)\} \quad \textit{for } 0 < u < v \le \delta.$$

Proof. To see that (4) implies (3) it is sufficient to choose $v = u(1+\lambda(u))$. Hence we have only to show that (3) implies (4). If $v \le u(1+\lambda(u))$ we have

$$\mathrm{Osc}[f; u, v] \le \mathrm{Osc}[f; u, u(1+\lambda(u))] \le K\lambda(u) = K\max\{v/u - 1, \lambda(u)\}.$$

If $u(1+\lambda(u)) < v$ let the sequence $v_m < \ldots < v_1 < v_0$ be defined by $v_0 = v$ and $(1+\lambda(v_n))v_n = v_{n-1}$, $n = 1, \ldots, m$, where $v_m \le u < v_{m-1}$. This is clearly possible since $(1+\lambda(x))x$ is a strictly increasing continuous function. We have then

$$\mathrm{Osc}[f; u, v] \le \sum_{k=1}^{m} \mathrm{Osc}[f; v_k, v_{k-1}] \le K\sum_{k=1}^{m} \lambda(v_k) \le 2K\sum_{k=1}^{m-1} \lambda(v_k)$$

since $\lambda(v_m) < \lambda(v_{m-1})$. Hence

$$\begin{aligned}\operatorname{Osc}[f;u,v] &\leq 2K\sum_{k=1}^{m-1}\left(\frac{v_{k-1}-v_k}{v_k}\right) \leq 2K\left(\frac{v_0-v_{m-1}}{u}\right)\\ &\leq 2K\left(\frac{v}{u}-1\right) = 2K\max\left\{\frac{v}{u}-1,\lambda(u)\right\}\end{aligned}$$

and the theorem is proved.

COROLLARY. *If* $\lambda_1(x) = O(\lambda_2(x))$ *as* $x \to 0+$ *and* f *oscillates* λ_1*-regularly as* $x \to 0+$, *then* f *oscillates* λ_2*-regularly as* $x \to 0+$.

This follows from Theorem 1, with $\lambda(u) = \lambda_1(u)$, by choosing $v = u(1+\lambda_2(u))$.

THEOREM 2. *The following two conditions are equivalent*:

(5) *there exists* λ *such that* f *oscillates* λ*-regularly as* $x \to 0+$

and

(6) $$\Omega_f(\delta) = O(\delta), \qquad \delta \to 0+ .$$

To prove that (6) implies (5), let us observe that by (6) there exists $K > 0$ such that $\Omega_f(1/n) \leq K/n$ for $n = 1,2,3,\ldots$, and so, by the definition of Ω_f there exists a strictly decreasing sequence (x_n) such that

$$\operatorname{Osc}[f;x,x(1+1/n)] \leq 2K/n \quad \text{for } 0 < x \leq x_n.$$

Let λ be defined by $\lambda(x_n) = 1/n$, λ linear on $[x_{n+1},x_n]$, $n = 1,2,3,\ldots$. Then for $x_{n+1} \leq x < x_n$ we have

$$\operatorname{Osc}[f;x,(1+\lambda(x))x] \leq \operatorname{Osc}[f;x,x(1+1/n)] \leq 2K/n \leq 4K\lambda(x).$$

So for $x \leq x_1$, $\operatorname{Osc}[f;x,x(1+\lambda(x))] \leq 4K\lambda(x)$, and consequently, f oscillates λ-regularly as $x \to 0+$.

Conversely, if (5) holds, then by Theorem (1), with $u = x$ and $v = x(1+\delta)$, we have

$$\operatorname{Osc}[f;x,x(1+\delta)] \leq C\max\{\delta,\lambda(x)\}$$

and it follows that

$$\Omega_f(\delta) = \limsup_{x\to 0}\operatorname{Osc}[f;x,x(1+\delta)] \leq C\delta$$

since $\lambda(x) \to 0$ as $x \to 0+$. Hence, f satisfies condition (6).

Karamata's test for convergence of Fourier series. Let f be a 2π-periodic and integrable function. The convergence of the Fourier series of f at x_0 depends of the behaviour of

$$\varphi(t) = \big(f(x_0+t)+f(x_0-t)-2f(x_0)\big)/2$$

as $t \to 0+$. One of the most widely used convergence tests is the following corollary of Dini's test:

If

$$\varphi(t) = O(\lambda(t)), \quad t \to 0 \tag{7}$$

where λ is a continuous, monotonically increasing function with $\lambda(0) = 0$, such that

$$\int_{0+} \frac{\lambda(t)}{t}\, dt < \infty$$

then the Fourier series of f converges at x_0.

Karamata has generalized this test by replacing condition (7) with

$$\begin{gathered} \varphi(t) \to 0, \qquad t \to 0+ \\ \varphi(t) \text{ oscillates } \lambda\text{-regularly as } t \to 0+. \end{gathered} \tag{8}$$

We state Karamata's test as follows:

THEOREM 3. *If f is continuous at x_0 and $\varphi(t)$ oscillates λ-regularly as $t \to 0+$ and if*

$$\int_{0+} \frac{\lambda(t)}{t}\, dt < \infty$$

then the Fourier series of f converges at x_0.

Proof. We shall show for δ sufficiently small and $n \geq N(\delta)$

$$\left| \int_0^\delta \varphi(t) \frac{\sin nt}{t}\, dt \right| \leq (10 + 2K) \sup_{0 \leq t \leq \delta} \sqrt{|\varphi(t)|} + K \int_{0+}^\delta \frac{\lambda(t)}{t}\, dt \tag{9}$$

where K is defined in (3). From inequality (9) it follows that

$$\limsup_{n \to \infty} \left| \int_0^\delta \varphi(t) \frac{\sin nt}{t}\, dt \right| \to 0 \quad \text{as } \delta \to 0+ \tag{10}$$

and the convergence of the Fourier series of f at the point x_0 is established.

Since $\varphi(t) \to 0$ as $t \to 0+$, we may assume that $|\varphi(t)| \leq 1$ for $0 < t < \delta$. We define $(t_{k,n})$ by

$$t_{1,n} \sqrt{\sup_{0 \leq t \leq t_{1,n}} |\varphi(t)|} = 1/n$$

and, for $k \geq 1$, $t_{k+1,n} = t_{k,n}(1 + \lambda(t_{k,n}))$. Clearly, for $n \geq N(\delta)$ we have $0 < t_{1,n} < \delta$. Obviously, the sequence $(t_{k,n})$ is an increasing sequence and, since $t_{k+1,n} \geq t_{k,n} \cdot (1 + \lambda(t_{1,n}))$, it follows that $t_{k,n} \to \infty$ $(k \to \infty)$, for every fixed $n = 1, 2, 3, \ldots$.

Choose m such that $t_{m,n} \leq \delta \leq t_{m+1,n}$. We have then

$$\begin{aligned} \int_0^\delta \varphi(t) \frac{\sin nt}{t}\, dt &= \left(\int_0^{t_{1,n}} + \sum_{k=1}^{m-1} \int_{t_{k,n}}^{t_{k+1,n}} + \int_{t_{m,n}}^\delta \right) \varphi(t) \frac{\sin nt}{t}\, dt \\ &= A_n(\varphi) + B_n(\varphi) + C_n(\varphi). \end{aligned}$$

Since $|\sin nt| \le nt$ we obtain first

$$(11)\qquad |A_n(\varphi)| \le \int_0^{t_{1,n}} |\varphi(t)| \frac{|\sin t|}{t}\, dt \le n t_{1,n} \sup_{0\le t\le t_{1,n}} |\varphi(t)| \le \sqrt{\sup_{0\le t\le t_{1,n}} |\varphi(t)|} \le \sqrt{\sup_{0\le t\le \delta} |\varphi(t)|}.$$

On the other hand, since

$$\log(t_{m+1,n}/t_{m,n}) = \log(1+\lambda(t_{m,n})) \le \lambda(t_{m,n}) \le \lambda(\delta),$$

we have

$$(12)\qquad |C_n(\varphi)| \le \int_{t_{m,n}}^{\delta} |\varphi(t)| \frac{dt}{t} \le \sup_{0\le t\le\delta} |\varphi(t)| \int_{t_{m,n}}^{t_{m+1,n}} \frac{dt}{t} \le \lambda(\delta) \sup_{0\le t\le\delta} |\varphi(t)| \le \sqrt{\sup_{0\le t\le\delta} |\varphi(t)|}.$$

Next

$$B_n(\varphi) = \sum_{k=1}^{m-1} \varphi(t_{k,n}) \int_{t_{k,n}}^{t_{k+1,n}} \frac{\sin nt}{t}\, dt + \sum_{k=1}^{m-1} \int_{t_{k,n}}^{t_{k+1,n}} \big(\varphi(t) - \varphi(t_{k,n})\big) \frac{\sin nt}{t}\, dt = B_n^*(\varphi) + B_n^{**}(\varphi).$$

Since φ oscillates λ-regularly, we have, by (3),

$$|\varphi(t) - \varphi(t_{k,n})| \le K\lambda(t_{k,n}) \le K\lambda(t) \quad \text{for } t_{k,n} \le t \le t_{k+1,n}$$

and so

$$(13)\qquad |B_n^{**}(\varphi)| \le \sum_{k=1}^{m-1} \int_{t_{k,n}}^{t_{k+1,n}} |\varphi(t) - \varphi(t_{k,n})| \frac{dt}{t} \le K \sum_{k=1}^{m-1} \int_{t_{k,n}}^{t_{k+1,n}} \frac{\lambda(t)}{t}\, dt \le K \int_{0+}^{\delta} \frac{\lambda(t)}{t}\, dt.$$

To estimate $B_n^*(\varphi)$ we use summation by parts:

$$B_n^*(\varphi) = \sum_{k=1}^{m-1} \varphi(t_{k,n}) \int_{t_{k,n}}^{\pi} \frac{\sin nt}{t}\, dt - \sum_{k=2}^{m} \varphi(t_{k-1,n}) \int_{t_{k,n}}^{\pi} \frac{\sin nt}{t}\, dt$$
$$= \varphi(t_{1,n}) \int_{t_{1,n}}^{\pi} \frac{\sin nt}{t}\, dt - \varphi(t_{m,n}) \int_{t_{m,n}}^{\pi} \frac{\sin nt}{t}\, dt + \sum_{k=2}^{m} \big(\varphi(t_{k,n}) - \varphi(t_{k-1,n})\big) \int_{t_{k,n}}^{\pi} \frac{\sin nt}{t}\, dt.$$

Since $|\varphi(t_{k,n}) - \varphi(t_{k-1,n})| \le K\lambda(t_{k-1,n})$, using the estimates

$$\left| \int_x^{\pi} \frac{\sin nt}{t}\, dt \right| \le 5 \quad \text{and} \quad \left| \int_x^{\pi} \frac{\sin nt}{t}\, dt \right| \le \frac{2}{nx},$$

we have

$$|B_n^*(\varphi)| \le 5|\varphi(t_{1,n})| + 5|\varphi(t_{m,n})| + \frac{2K}{n}\sum_{k=2}^{m}\frac{\lambda(t_{k-1,n})}{t_{k,n}}.$$

Since $\lambda(t_{k-1,n}) = (t_{k,n} - t_{k-1,n})/t_{k-1,n}$ we have

$$\begin{aligned}|B_n^*(\varphi)| &\le 10 \sup_{0\le t\le\delta}|\varphi(t)| + \frac{2K}{n}\sum_{k=2}^{m}\frac{\lambda(t_{k-1,n})}{t_{k,n}} \\ &\le 10 \sup_{0\le t\le\delta}|\varphi(t)| + \frac{2K}{n}\sum_{k=2}^{m}\left(\frac{1}{t_{k-1,n}} - \frac{1}{t_{k,n}}\right) \\ &\le 10 \sup_{0\le t\le\delta}|\varphi(t)| + \frac{2K}{nt_{1,n}}.\end{aligned}$$

Hence

$$|B_n^*(\varphi)| \le (10+2K)\sqrt{\sup_{0\le t\le\delta}|\varphi(t)|}. \tag{14}$$

The proof of Theorem 3 follows now from inequalities (11)–(14).

Relation between Karamata's test and classical tests for convergence of Fourier series. It is easy to see that none of the four classical tests (Jordan, Dini, Young and de la Vallée-Poussin) is stronger than Karamata's test. For example, Dini's test fails for $1/\log|x|$, both Jordan and de la Vallée-Poussin tests fail for $\sin(\log|x|)/\log|x|$ and Young's test fails for $|x|^{1/2}\sin(1/x)$ (see N. K. Bari, *Trigonometric series*, Moscow 1961, Ch. III, §§2–5.). However, since the first two functions satisfy condition $f'(x) = O(1/x)$, $x \to 0+$, they oscillate λ-regularly for any λ; the third function oscillates λ-regularly for $f(x) = |x|^{1/2}$. Hence, all three functions satisfy conditions of Karamata's test.

The converse is also true: Karamata's test is not stronger than any of these four tests. Since Young and de la Vallée-Poussin tests are stronger than either Dini's or Jordan's test (see ibid.), it is sufficient to show that Karamata's test is not stronger than any of these two. To construct examples we shall need the following two lemmas.

LEMMA 1. *Let f oscillate λ-regularly as $x \to 0+$ and let the sequences (u_n), (v_n) satisfy*

$$v_{n+1} < u_n < v_n, \qquad v_n \to 0+. \tag{15}$$

If

$$u_n\frac{\operatorname{Osc}[f;u_n,v_n]}{v_n-u_n} \to \infty \quad \text{as } n\to\infty \tag{16}$$

then, for some $\alpha > 0$ and $N > 0$, we have $\lambda(u_n) > \alpha\operatorname{Osc}[f;u_n,v_n]$ for all $n > N$.

Proof. By Theorem 1

$$\mathrm{Osc}[f; u_n, v_n] \le C \max\left\{ \frac{v_n - u_n}{u_n}, \lambda(u_n) \right\}.$$

From (16) it follows that

$$\mathrm{Osc}[f; u_n, v_n] > C \frac{v_n - u_n}{u_n} \quad \text{for } n \ge N.$$

The last two inequalities imply

$$\mathrm{Osc}[f; u_n, v_n] \le C\lambda(u_n) \quad \text{for } n \ge N$$

which proves the lemma.

LEMMA 2. *Let the sequences* (u_n), (v_n) *satisfy* (15) *and let the function* f *satisfy*

$$\sum_{n=1}^{\infty} \mathrm{Osc}[f; u_n, v_n] \log\left(\frac{u_{n-1}}{u_n}\right) = \infty \tag{17}$$

and

$$u_n \frac{\mathrm{Osc}[f; u_n, v_n]}{v_n - u_n} \to \infty \quad (n \to \infty). \tag{18}$$

Then, if f *oscillates* λ*-regularly as* $x \to 0+$, *we have*

$$\int_{0+} \frac{\lambda(u)}{u}\, du = \infty.$$

Proof. If f oscillates λ-regularly as $x \to 0+$, from Lemma 1 and (18) it follows that

$$\lambda(u_n) \ge \alpha\, \mathrm{Osc}[f; u_n, v_n] \quad \text{for } n \ge N. \tag{19}$$

On the other hand,

$$\int_0^{u_{N-1}} \frac{\lambda(u)}{u}\, du = \sum_{n=N}^{\infty} \int_{u_n}^{u_{n-1}} \frac{\lambda(u)}{u}\, du \ge \sum_{n=N}^{\infty} \lambda(u_n) \int_{u_n}^{u_{n-1}} \frac{du}{u}$$
$$\ge \sum_{n=N}^{\infty} \lambda(u_n) \log\left(\frac{u_{n-1}}{u_n}\right).$$

It follows then from (19) that

$$\int_0^{u_{N-1}} \frac{\lambda(u)}{u}\, du \ge \alpha \sum_{n=N}^{\infty} \mathrm{Osc}[f; u_n, v_n] \log\left(\frac{u_{n-1}}{u_n}\right)$$

and the lemma follows from (17).

Examples. We construct here continuous functions f and g satisfying at the point $x_0 = 0$ conditions of Dini's and Jordan's tests respectively, but neither one of them satisfying conditions of Karamata's test.

Let

$$v_{n+1} < u_n < v_n, \qquad v_n \to 0+ \quad (n \to \infty) \tag{20}$$

and let

$$c_n \to 0 \quad (n \to \infty). \tag{21}$$

Let f and g be even and continuous functions defined on $[-\pi, \pi]$ as follows:

$$f(x) = \begin{cases} 0 \text{ for } v_{n+1} \le x \le u_n \text{ and for } v_1 \le x \le \pi \\ c_n \text{ for } x = (u_n + v_n)/2 \\ \text{linear on } [u_n, (u_n + v_n)/2] \text{ and } [(u_n + v_n)/2, v_n] \end{cases}$$

$$g(x) = \begin{cases} c_1 \text{ for } u_1 \le x \le \pi \\ c_n \text{ for } v_{n+1} \le x \le u_n \\ \text{linear on } [u_n, v_n] \text{ for } n \ge 2. \end{cases}$$

Since

$$\int_0^\pi \frac{|f(u)|}{u}\,du = \sum_{n=1}^\infty \int_{u_n}^{v_n} \frac{|f(u)|}{u}\,du \le \sum_{n=1}^\infty |c_n| \int_{u_n}^{v_n} \frac{du}{u} = \sum_{n=1}^\infty |c_n| \log\left(\frac{v_n}{u_n}\right),$$

f will satisfy conditions of Dini's test provided

$$\sum_{n=1}^\infty |c_n| \log\left(\frac{u_{n-1}}{u_n}\right) < \infty. \tag{22}$$

Since $\mathrm{Osc}[f; u_n, v_n] = |c_n|$, the function f, by Lemma 2, will not satisfy Karamata's test if, in addition to conditions (20), (21), (22), the sequences (u_n), (v_n) and (c_n) satisfy conditions

$$\sum_{n=1}^\infty |c_n| \log\left(\frac{v_n}{u_n}\right) = \infty \quad \text{and} \quad u_n \frac{|c_n|}{v_n - u_n} \to \infty \quad (n \to \infty).$$

For this it is sufficient to take $u_n = 2^{-n}$, $v_n = 2^{-n}(1 + n^{-2})$ and $c_n = 1/n$.

If, in addition to (20), (21), we have also

$$\sum_{n=2}^\infty |c_n - c_{n-1}| < \infty, \tag{23}$$

then g will be of bounded variation, and even absolutely continuous on $[-\pi, \pi]$. However, since

$$\mathrm{Osc}[g; u_n, v_n] = |c_{n-1} - c_n|,$$

the function g, by Lemma 2, will not satisfy Karamata's test if in addition to (20), (21), (23), the sequences (u_n), (v_n) and (c_n) satisfy conditions

$$\sum_{n=1}^{\infty} |c_{n-1} - c_n| \log\left(\frac{u_{n-1}}{u_n}\right) = \infty \quad \text{and} \quad u_n \frac{|c_{n-1} - c_n|}{v_n - u_n} \to \infty \quad (n \to \infty).$$

For this it is sufficient to take $u_n = 2^{-n^2}$, $v_n = (1 + n^{-3})2^{-n^2}$ and $c_n = (n+1)^{-1}$.

Finally, we have the following result:

THEOREM 4. *If the function φ satisfies conditions of Karamata's test, then there exist functions g and s such that $\varphi = g + s$, g satisfies conditions of Dini's test and s satisfies conditions of Young's test.*

Theorem 4 provides clearly a new proof of Karamata's test. From Theorem 4 it follows also that if a function satisfies conditions of Karamata's test, then it satisfies conditions of Lebesgue's test.

Proof. Without loss of generality we assume that $x_0 = 0$, that φ is an even function with $\varphi(0) = 0$ and that φ vanishes outside $[-\delta, \delta]$.

By assumption there exists a positive monotone and continuous function λ on $[0, \delta]$ such that

$$\int_{0+}^{\delta} \frac{\lambda(t)}{t}\, dt < \infty \tag{24}$$

and

$$|\varphi(u') - \varphi(u)| \le K\lambda(u) \quad \text{for } u < u' \le u\lambda(u). \tag{25}$$

Let the sequence (u_n) be defined by

$$u_0 = \delta, \qquad u_n + \lambda(u_n)u_n = u_{n-1}. \tag{26}$$

Let s be defined by $s(u) = \varphi(u_n)$ for $u \in [u_n, u_{n-1})$ and g by $g(u) = \varphi(u) - s(u)$. For $u \in [u_n, u_{n-1})$ we have then

$$|g(u)| = |\varphi(u) - s(u)| = |\varphi(u) - \varphi(u_n)|. \tag{27}$$

Since $u_n \le u < u_{n-1} = u_n(1 + \lambda(u_n))$ we obtain from (25) and (27) that $|g(u)| \le K\lambda(u)$. It follows then from (24) that g satisfies conditions of Dini's test.

To show that s satisfies conditions of Young's test we need to show that $us(u)$ is of bounded variation on some interval $[0, \delta]$ and that the total variation of $us(u)$ on $[0, t]$ is $O(t)$ as $t \to 0+$.

The function s has the jump $\varphi(u_n) - \varphi(u_{n+1})$ at u_n, so $us(u)$ has the jump $u_n(\varphi(u_n) - \varphi(u_{n+1}))$ at u_n. Since $s(u) = \varphi(u_n)$ on $[u_n, u_{n-1})$, the variation of $us(u)$ on that semi-open interval is $(u_{n-1} - u_n)\varphi(u_n)$. So, if $u_N < t \le u_{N-1}$, we have

$$\begin{aligned} \mathrm{Var}_{[0,t]}(us(u)) \le (t - u_N)|\varphi(u_N)| &+ \sum_{n \le N} (u_n - u_{n+1})|\varphi(u_{n+1})| \\ &+ \sum_{n \le N} u_n |\varphi(u_n) - \varphi(u_{n+1})|. \end{aligned} \tag{28}$$

Because of (25) and (26) we have

$$|\varphi(u_n) - \varphi(u_{n+1})| \leq K\lambda(u_{n+1}) \leq K\lambda(u_n).$$

So, in view of (26) we get

$$\sum_{n\leq N} u_n|\varphi(u_n) - \varphi(u_{n+1})| \leq K \sum_{n\leq N} u_n\lambda(u_n) = K \sum_{n\leq N}(u_{n-1} - u_n) \tag{29}$$
$$= Ku_{N-1} \leq 2Ku_N \leq 2Kt,$$

for t sufficiently small, since $u_{N-1}/u_N = 1 + \lambda(u_N) \to 1$ as $N \to \infty$.

By assumption $\varphi(u) \to 0$ as $u \to 0$. So, $|\varphi(u_n)| \leq M$ for all n. Thus

$$(t - u_N)|\varphi(u_N)| + \sum_{n\leq N}(u_n - u_{n+1})|\varphi(u_{n+1})| \leq Mt. \tag{30}$$

From (28), (29) and (30) we deduce that

$$\mathrm{Var}_{[0,t]}(us(u)) = O(t), \qquad t \to 0$$

which proves Theorem 4.

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