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ON MULTIDIMENSIONAL GENERALIZATIONS OF REGULARLY VARYING FUNCTIONS AND TAUBERIAN AND ABELIAN THEOREMS

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Abstract. Admissible and strictly admissible functions are introduced. They are multidimensional generalizations of regularly varying Karamata's functions and dominated varying Feller's functions. Some multidimensional Tauberian theorems related to these functions are considered.

1. Some facts and definitions. Let Γ be a closed convex acute-angled solid cone in $\mathbf{R}^n$ with vertex at zero. For an acute-angled cone Γ, the dual cone is $\Gamma^* = \{y \in \mathbf{R}^n : (y,t) \geq 0,\, t = (t_1, \dots, t_n) \in \Gamma\}$; $\operatorname{int}\Gamma^* = C \neq \varnothing$. By $\mathcal{S}'_\Gamma$ we denote the subspace of generalized functions of slow growth with support in the cone Γ.

The Laplace transform

$$L[f] = \big(f(t), e^{i(z,t)}\big) = \tilde{f}(z), \qquad z = x + iy, \quad x \in \mathbf{R}^n, \quad y \in C$$

realizes an isomorphism of the convolution algebra $\mathcal{S}'_\Gamma$ on the algebra $H(C)$ of functions which are holomorphic in the tube domain $T^C = \mathbf{R}^n + iC$ and satisfy the estimate $|f(z)| \leq M(1+|z|)^a/[\Delta_C(y)]^b$ for some constants M, a and b.

The cone C is said to be *regular* if the Cauchy kernel $\mathcal{K}_C(z) = L[\theta_\Gamma]$, $z \in T^C$, is the divisor of unity in the algebra $H(C)$. Here $\theta_\Gamma(y)$ is the characteristic function of Γ. For a regular cone Γ we define the family of generalized functions θ_Γ^α in $\mathcal{S}'_\Gamma$ by the formula

$$L(\theta_\Gamma^\alpha) = [\mathcal{K}_C(z)]^\alpha, \qquad z \in T^C, \quad -\infty < \alpha < +\infty.$$

It is clear that $\theta_\Gamma^\alpha * \theta_\Gamma^\beta = \theta_\Gamma^{\alpha+\beta}$. Let $f \in \mathcal{S}'_\Gamma$. The distribution $f^{(-\alpha)} = \theta_\Gamma^\alpha * f$ is called the primitive of f of order $\alpha > 0$ (the derivative of order $-\alpha$ if $\alpha < 0$). Note that the N-th primitive of $f(t)$ is the generalization of the N-th Cesaro mean value with respect to the cone Γ.

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2. Admissible and strictly admissible functions. The set of all non-singular automorphisms of Γ is denoted by $\mathcal{A}(\Gamma)$. If $U \in \mathcal{A}(\Gamma)$, then $V = (U^T)^{-1} \in \mathcal{A}(\Gamma^*)$. Let $\{U_k \in \mathcal{A}(\Gamma),\ k \in I\}$ be a family of automorphisms of Γ. We assume that $I \in \mathbf{R}^1$ and that infinity is the limit point of the set I.

Definition 1. The distribution $u \in \mathcal{S}'_\Gamma$ is called q-strictly admissible for the family $\{U_k,\ k \in I\}$ if the following conditions hold

1. the primitive of u of order q is a locally summable positive function: $u^{(-q)} > 0$, $t \in \operatorname{int}\Gamma$, $u^{(-q)} \in L^1_{\mathrm{loc}}$;

2. there exists a vector $t_0 \in \operatorname{int}\operatorname{pr}\Gamma$ such that

$$\frac{u^{(-q)}(U_k t)}{u^{(-q)}(U_k t_0)} \overset{t\in K}{\rightrightarrows} g(t), \qquad k \to \infty, \quad k \in I,$$

where K is any compact set in $\operatorname{int}\Gamma$ and $g(t)$ is a continuous positive function in $\operatorname{int}\Gamma$.

3. there exists a k_0 such that

$$\frac{u^{(-q)}(U_k t)}{u^{(-q)}(U_k t_0)} \le \psi(t), \qquad k > k_0, \quad k \in I, \quad t \in \operatorname{int}\Gamma,$$

where $\psi(t)$ is some tempered function in Γ.

In the case of one variable, when the cone $\Gamma = [0, \infty]$ and the family of automorphisms is just the dilatation group, 0-strictly admissible functions are nothing else but regularly varying functions. In the case of several variables for the family of dilatations of the cone the description of the class of such functions was given by Yakimiv [**2**]. Thus the q-strictly admissible functions are generalizations of regularly varying functions for the arbitrary family $\{U_k \in \mathcal{A}(\Gamma),\ k \in I\}$.

Definition 2. A distribution $u \in \mathcal{S}'_\Gamma$ is called q-admissible for the cone Γ if for any family of non-singular linear automorphisms of Γ $\{U_k \in \mathcal{A}(\Gamma),\ k \in I\}$ there exists a subsequence $\{U_{k_m},\ k_m \to \infty,\ m \to \infty\}$ such that $u(t)$ is q-strictly admissible for this subsequence.

In the case of one variable and the family of dilatations the admissible functions are so-called dominated varying functions considered by Feller [**3**]. In the multidimensional case such functions are studied carefully in [**1**]. As an example we give here some sufficient conditions for the function to be admissible for the cone $\Gamma = \mathbf{R}^n_+$.

Let $\Gamma = \bar{\mathbf{R}}^n_+$ be the positive octant in $\mathbf{R}^n$. Let also $u(t)$ be a positive and continuously differentiable function in $\mathbf{R}^n_+$ satisfying the conditions

$$-1 < a < \frac{t_j \cdot \partial u(t)/\partial t_j}{u(t)} < b, \qquad t_j > 0, \quad j = 1, \dots, n.$$

Then $u(t)$ is q-admissible for $\mathbf{R}^n_+$ for any $q > 0$.

These conditions are multidimensional generalization of Keldysh type Tauberian conditions.

3. The comparison of Tauberian theorems for measures and holomorphic functions with non-negative imaginary part.

THEOREM 1. *Let $f = [d\mu]$ where $d\mu$ is a non-negative tempered measure with support in Γ and u is 1-admissible for the cone Γ. Let for some $-\infty < \gamma_0 \leq 1$ and any constant C_1*

$$\tilde{f}(iy)/\tilde{u}(iy) \to 1 \text{ when } \Delta_C(y) \to 0, \quad |y| \leq C_1 \Delta_C^{\gamma_0}(y), \quad y \in C. \tag{1}$$

Then for any $N \geq 1$ and any $C_2 > 0$

$$f^{(-N)}(t)/u^{(-N)}(t) \to 1 \text{ when } |t| \to \infty, \quad \Delta_\Gamma(t) \geq C_2 |t|^{\gamma_0}, \quad t \in \operatorname{int}\Gamma. \tag{2}$$

THEOREM 2. *Let $f \in \mathcal{S}'_\Gamma$ and $\operatorname{Im}\tilde{f}(z) > 0$, $z \in T^C$ and let $u(t)$ be a q-admissible function for the cone Γ. If for any $C_1 > 0$ the relation (1) holds then for any C_2 and for any $N \geq q$, $N > 3$ the relation (2) holds as well. Here $-\infty < \gamma_0 \leq 1$.*

Remark. Theorem 2 will be true if one assumes only that $\tilde{f}(z)$ has a bounded argument in T^C instead of the condition $\operatorname{Im}\tilde{f}(z) > 0$. But in this case the constant N in (2) should be enlarged.

Now we shall discuss the corresponding Abelian theorems.

4. Abelian theorems. In [4] we have shown that there exist a function $f(t_1, t_2)$ and an admissible function $u(t_1, t_2)$ for the cone $\Gamma = \mathbf{R}^2_+$ such that for any $N > 0$ and $C > 0$

$$f^{(-N)}(t)/u^{(-N)}(t) \to 1 \text{ when } |t| \to \infty, \quad \Delta_\Gamma(t) \geq C$$

so that the condition (2) holds with $\gamma_0 = 0$. On the other hand the condition (1) fails.

This example shows that the corresponding Abelian theorems require some additional (so-called "Abelian") conditions. For instance if $f(t)$ is a non-negative measure the Abelian theorem is true. In other words the condition (2) with $N = 1$ implies the condition (1).

To conclude we consider a version of an Abelian comparison theorem which does not require any additional condition.

THEOREM 3. *Let $f(t) \in \mathcal{S}'_\Gamma$ and $u(t)$ be a q-admissible function for the cone Γ. If there exists a number N_0 such that for any $N \geq N_0$ and any constant C_2*

$$f^{(-N)}(t)/u^{(-N)}(t) \to 1 \text{ when } |t| \to \infty, \qquad \Delta_\Gamma(t) \geq C_2 |t|^{\gamma_0}, \quad t \in \operatorname{int}\Gamma.$$

with some $-\infty < \gamma_0 < 1$, then for any $\gamma > \gamma_0$ and $C_1 > 0$

$$\tilde{f}(iy)/\tilde{u}(iy) \to 1 \text{ when } \Delta_C(y) \to 0, \qquad |y| \leq C_1 \Delta_C^{\gamma}(y), \quad y \in C.$$

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