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SOME CHARACTERIZATIONS OF ALMOST CONVERGENCE FOR SINGLE AND DOUBLE SEQUENCES

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Abstract. At the 1985 Dubrovnik conference, D. Butkovic, H. Kraljevic, and N. Sarapa showed that ac, the space of almost convergent sequences, is the intersection of the convergence domains of the matrices obtained by shifting the rows of the Cesàro matrix. In this paper we show that ac can be realized as the intersection of the convergence domains of another small class of matrices. This result and the corresponding result of Butkovic, et al is then extended to almost convergent double sequences.

In [**4**] the second author demonstrated that the set, ac, of almost convergent sequences, can be obtained as the intersection of the convergence domains of a certain class of matrices called "hump" matrices. In [**1**] the authors showed that ac is the intersection of the convergence domains of a much smaller class of matrices called, by them, generalized Cesàro matrices. In this paper we show that ac is the intersection of another small class of matrices. This result, and the corresponding theorem in [**1**] are then extended to almost convergent double sequences.

Let A be an infinite matrix of real or complex numbers, x an infinite sequence of real or complex numbers. The convergence domain of A, denoted by c_A, consists of all sequences x for which $\sum_{k=1}^{\infty} a_{nk}x_k$ is defined for each n, and $Ax = \left\{\sum_{k=1}^{\infty} a_{nk}x_k\right\}$ is a convergent sequence. We write the limit of Ax, or the A-limit of x, as $\lim_A x = \lim_n \sum_{k=1}^{\infty} a_{nk}x_k$, and A is called regular if $\lim_A x = \lim x$ for each convergent sequence x. A sequence x is said to be almost convergent, with limit t, if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=p}^{p+n-1} x_k = t, \qquad p = 0, 1, 2, \dots ,$$

uniformly in p. Lorentz [**2**] defined this concept and obtained necessary and sufficient conditions for an infinite matrix to contain ac in its convergence domain. These conditions are the standard Silverman-Toeplitz conditions for regularity plus the condition $\lim_n \sum_{k=1}^{\infty} |a_{nk} - a_{n,k+1}| = 0$. Such matrices are called strongly regular.

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One of the best known strongly regular matrices is C, the Cesàro matrix of order 1. C is a lower triangular matrix with entries $c_{nk} = (1/n)$, $1 \le k \le n$. From [1], a matrix A is called a generalized Cesàro matrix if it is obtained from C by shifting rows. Let $p : \mathbf{N} \to \mathbf{N}$. Then A is defined by

$$a_{nk} = \begin{cases} 1/n, & \text{if } p(n) \le k < p(n) + n, \\ 0, & \text{otherwise.} \end{cases}$$

G is the set of all such matrices obtained by using all possible functions p. The following result appears in [1].

THEOREM BSK. $\mathrm{ac} = \bigcap_{A \in G} c_A$.

For each $n \in \mathbf{N}$ define

$$\alpha_n = \inf_{k>0} \frac{1}{n} \sum_{j=k}^{k+n-1} x_j, \qquad \beta_n = \sup_{k>0} \frac{1}{n} \sum_{j=k}^{k+n+1} x_j.$$

THEOREM 1. (i) $\{\alpha_{2^n}\}$ *is nondecreasing*, (ii) $\{\beta_{2^n}\}$ *nonincreasing*, (iii) $\alpha_n \le \beta_n$ *for each* n, *and* (iv) $x \in ac$ *iff* $\lim_p(\beta_{2^p} - \alpha_{2^p}) = 0$.

To prove (i),

$$\begin{aligned} \alpha_{2^{n+1}} &= \inf_{k>0} \frac{1}{2^{n+1}} \sum_{j=k}^{k+2^{n+1}-1} x_j \\ &= \inf_{k>0} \frac{1}{2^{n+1}} \left[\frac{1}{2^n} \left(\sum_{j=k}^{k+2^n-1} x_j \right) 2^n + \frac{1}{2^n} \left(\sum_{j=k+2^n}^{k+2^{n+1}-1} x_j \right) 2^n \right] \\ &\ge \frac{1}{2^{n+1}} [2^n \alpha_{2^n} + 2^n \alpha_{2^n}] = \alpha_{2^n}. \end{aligned}$$

Condition (ii) is proved in a similar manner, and (iii) is trivial.

To establish (iv), first observe that, from the definition of almost convergence, $x \in \mathrm{ac}$ iff $\lim_n(\beta_n - \alpha_n) = 0$. Since $x \in \mathrm{ac}$ implies $\lim_p(\beta_{2^p} - \alpha_{2^p}) = 0$, it remains only to prove the converse.

Suppose $\lim_p(\beta_{2^p} - \alpha_{2^p}) = 0$. For each n choose p to satisfy $2^p \le n < 2^{p+1}$. We may write n in a dyadic representation of the form $n = \sum_{i=0}^{p} n_i 2^i$, where each n_i is 0 or 1, $i = 0, 1, \ldots, p-1$, and $n_p = 1$. Then

$$\begin{aligned} \frac{1}{n} \sum_{i=k}^{n+k-1} x_i &= \frac{1}{n} \left[\frac{1}{2^p} \left(\sum_{i=k}^{k+2^p-1} x_i \right) 2^p + \frac{1}{2^{p-1}} \left(\sum_{i=k+2^p}^{k+2^p+n_{p-1}2^{p-1}} x_i \right) 2^p + \cdots \right] \\ &\ge \frac{1}{n} \left[2^p \alpha_{2^p} + 2^{p-1} \alpha_{2^{p-1}} + \cdots + 2^0 \alpha_{2^0} \right] \ge \frac{1}{n} \sum_{j=0}^{p} n_j 2^j \alpha_j, \end{aligned}$$

since each $n_j = 0, 1$, and hence

$$\alpha_n \geq \frac{1}{n} \sum_{j=0}^{p} n_j 2^j \alpha_{2^j}.$$

Similarly,

$$\beta_n \leq \frac{1}{n} \sum_{j=0}^{p} n_j 2^j \beta_{2^j},$$

and thus

$$0 \leq \beta_n - \alpha_n \leq \frac{1}{n} \sum_{j=0}^{p} n_j 2^j \left(\beta_{2^j} - \alpha_{2^j}\right).$$

Let B be the lower triangular matrix with nonzero entries $b_{nk} = n_k 2^k / n$. Then B is regular matrix, so that $\lim_p \left(\beta_{2^p} - \alpha_{2^p}\right) = 0$ implies $\lim_n (\beta_n - \alpha_n) = 0$.

Let A be a matrix such that there exists a $p : \mathbf{N} \to \mathbf{N}$ for which

$$a_{nk} = \begin{cases} 2^{-n}, & \text{if } p(n) \leq k < p(n) + 2^n, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\bar{G}$ denote the set of all such matrices for all such functions p.

COROLLARY. $\mathrm{ac} = \bigcap_{A \in \bar{G}} c_A = \bigcap_{A \in G} c_A$.

From Theorem 1, for each A in $\bar{G}$ c_A contains ac, so $\mathrm{ac} \subseteq \bigcap_{A \in \bar{G}} c_A$. The proof of the converse is similar to the corresponding proof of Theorem BSK of [**1**], and is therefore omitted. The equality between the first and the third term constitute the statement of Theorem BSK.

We now extend Theorem 1 and the Corollary to double sequences. Let $x = \{x_{jk}\}$ $j, k = 1, 2, \ldots,$ be a double sequence, $A = (a_{jk}^{mn})$ a doubly infinite matrix. The A-means of x are defined by

$$y_{mn} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}^{mn} x_{jk},$$

provided these exist in the sense of Pringsheim's convergence; i.e.,

$$y_{mn} = \lim_{M,N \to \infty} \sum_{j=1}^{M} \sum_{k=1}^{N} a_{jk}^{mn} x_{jk}.$$

A sequence x is A-summable to the limit t if the A-means exist for each $m, n = 1, 2, \ldots$; and $\lim_{m,n \to \infty} y_{mn} = t$.

A is called bounded regular if every bounded convergent sequence x is A-summable to the same limit, and the A-means are also bounded. Necessary and sufficient conditions for a matrix to be bounded regular were obtained by Robison [**5**] and are

(i) $$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|a_{jk}^{mn}| \le C < \infty \qquad (m,n=1,2,\dots)$$

(ii) $$\lim_{m,n\to\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}^{mn} = 1$$

(iii) $$\lim_{m,n\to\infty}\sum_{j=1}^{\infty}|a_{jk}^{mn}| = 0 \qquad (k=1,2,\dots) \quad \text{and}$$

(iv) $$\lim_{m,n\to\infty}\sum_{k=1}^{\infty}|a_{jk}^{mn}| = 0 \qquad (j=1,2,\dots)$$

Clearly either (iii) or (iv) implies that $\lim_{m,n\to\infty} a_{jk}^{mn} = 0$ $(j,k=1,2,\dots)$. The double Cesàro matrix has entries

$$c_{jk}^{mn} = \begin{cases} 1/mn, & 1 \le j \le m,\ 1 \le k \le n, \\ 0, & \text{otherwise.} \end{cases}$$

A shifted double Cesàro matrix, $C^{(p,q)}$ has entries

$${}^{pq}c_{jk}^{mn} = \begin{cases} 1/mn, & p \le j < p+m \text{ and } q \le k < q+n, \\ 0, & \text{otherwise.} \end{cases}$$

where p,q may be any of the positive integers.

The application of $C^{(p,q)}$ to a sequence x is equivalent to applying C to the shifted sequence $\{x_{jk}\}$, $j=p,p+1,\dots$, $k=q,q+1,\dots$. If x converges to t, then the $C^{(p,q)}$ limit of x is also t; i.e.,

$$\lim_{m,n\to\infty}\frac{1}{mn}\sum_{j=p}^{p+m-1}\sum_{k=q}^{q+m-1}x_{jk} = t.$$

If this convergence is uniform with respect to p and q, then x is called almost convergent, or ac_2. In a recent paper **[3]** the authors developed some properties of almost convergent double sequences, including the following result.

THEOREM MR. *A matrix A is strongly regular iff it is bounded regular and, in addition, satisfies*

(v) $$\lim_{m,n\to\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\Delta_{10}a_{jk}^{mn}| = 0,$$

(vi) $$\lim_{m,n\to\infty}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|\Delta_{01}a_{jk}^{mn}| = 0,$$

where $\Delta_{10}a_{jk}^{mn} = a_{jk}^{mn} - a_{j+1,k}^{mn}$ *and* $\Delta_{01}a_{jk}^{mn} = a_{jk}^{mn} - a_{j,k+1}^{mn}$.

In **[3]** is was also shown that every ac_2 sequence is bounded, even though convergent double sequences need not be bounded. It was also shown that, if a convergent double sequence is bounded, then it is almost convergent.

A matrix A will be called generalized double Cesàro if there exists functons $p, q : \mathbf{N} \to \mathbf{N}$ such that

$$(1)\qquad a_{jk}^{mn} = \begin{cases} 1/mn, & p(m) \le j < p(m) + m \text{ and } q(n) \le k < q(n) + n, \\ 0, & \text{otherwise.} \end{cases}$$

In other words, a generalized double Cesàro matrix is the result of shifting the entries of a double Cesàro matrix given by the integer-valued functions p and q. In principle it would be possible to consider two functions $p(m,n)$ and $q(m,n)$. However, since the nonzero entries $1/mn$ separate, it is sufficient to consider the shift functions separately as well.

Let c_A denote the set of bounded double sequences that are A-summable, and let G_2 denote the set of all generalized double Cesàro matrices.

THEOREM 2. $\mathrm{ac}_2 = \bigcap_{A \in G_2} c_A$.

Since each A in G_2 is strongly regular, ac_2 is contained in the intersection.

To prove the converse suppose there exists an $x \in c_A$ for every A in G_2, but $x \notin \mathrm{ac}_2$.

We shall first show that $\lim_A x = \lim_B x$ for every x in the intersection. Suppose that there were two matrices A and B in G_2 with $\lim_A x \ne \lim_B x$. Define a matrix D by

$$d_{jk}^{mn} = \begin{cases} a_{jk}^{mn} & \text{if } m \equiv n \pmod 2 \\ b_{jk}^{mn} & \text{if } m \not\equiv n \pmod 2). \end{cases}$$

Then $D \in G_2$, but $x \notin c_D$, a contradiction. To complete the proof we shall need the following result from [**3**].

LEMMA 1. *A bounded double sequence x is not almost convergent iff there exist numbers t and $\varepsilon > 0$ and four strictly increasing sequences of positive integers $\{m_r\}$, $\{n_r\}$, $\{p_r\}$, $\{q_r\}$, $r = 1, 2, \ldots$, such that*

$$(2)\qquad \frac{1}{m_r n_r} \sum_{j=p_r}^{p_r+m_r-1} \sum_{k=q_r}^{q_r+n_r-1} x_{jk} \begin{cases} \ge t + \varepsilon & \text{if } r \text{ is odd} \\ \le t - \varepsilon & \text{if } r \text{ is even.} \end{cases}$$

Now define a matrix A in G by (1), where

$$p(m) = \begin{cases} p_r, & \text{if } m = m_r \text{ for some } r = 1, 2, \ldots, \\ m & \text{otherwise;} \end{cases}$$

$$q(m) = \begin{cases} q_r, & \text{if } m = n_r \text{ for some } r = 1, 2, \ldots, \\ n & \text{otherwise.} \end{cases}$$

Then (2) shows that the A-limit of x does not exist, which is a contradiction.

The above proof, together with Lemma 1, serves to motivate introducing a class $\bar{G}_2$ of generalized arithmetic means defined as follows: Let $m, n : \mathbf{N} \to \mathbf{N}$

be strictly increasing functions and $p, q : \mathbf{N} \to \mathbf{N}$ arbitrary functions. Then $A = (a_{jk}^r) \in \bar{G}_2$ if

$$a_{jk}^r = \begin{cases} (m(r)n(r))^{-1}, & \text{if } p(r) \le j < p(r) + m(r) \text{ and } q(r) \le k < q(r) + n(r), \\ 0, & \text{otherwise.} \end{cases}$$

In other words, the A-means

$$y_r = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{jk}^r x_{jk}$$

form an ordinary sequence of arithmetic means over the rectangular grids $[p(r), p(r) + m(r) - 1] \times [q(r), q(r) + n(r) - 1]$.

It then follows that

$$\mathrm{ac}_2 = \bigcap_{A \in G_2} c_A = \bigcap_{A \in \bar{G}_2} c_A.$$

Now define

$$\alpha_{mn} = \inf_{i,l} \left(\frac{1}{mn} \sum_{j=i}^{i+m-1} \sum_{k=l}^{l+n-1} x_{jk} \right), \qquad \beta_{mn} = \sup_{i,l} \left(\frac{1}{mn} \sum_{j=i}^{i+m-1} \sum_{k=l}^{l+n-1} x_{jk} \right).$$

Then

$$\max\{\alpha_{2^{p-1},2^q}, \alpha_{2^p,2^{q-1}}\} \le \alpha_{2^p,2^q}, \quad \text{and} \quad \min\{\beta_{2^{p-1},2^q}, \beta_{2^p,2^{q-1}}\} \ge \beta_{2^p,2^q}.$$

If $2^p \le m < 2^{p+1}$ and $2^q \le n < 2^{q+1}$ then, as in Theorem 1, with $m = \sum_{j=0}^{p} m_j 2^j$, $n = \sum_{k=0}^{q} n_k 2^k$, each $m_j, n_k = 0, 1$,

$$\alpha_{mn} \ge \frac{1}{mn} \sum_{j=0}^{p} \sum_{k=0}^{q} m_j 2^j n_k 2^k \alpha_{2^j,2^k}, \quad \text{and} \quad \beta_{mn} \le \frac{1}{mn} \sum_{j=0}^{p} \sum_{k=0}^{q} m_j 2^j n_k 2^k \beta_{2^j,2^k}.$$

Therefore

$$0 \le \beta_{mn} - \alpha_{mn} \le \frac{1}{mn} \sum_{j=0}^{p} \sum_{k=0}^{q} m_j 2^j n_k 2^k \left(\beta_{2^j,2^k} - \alpha_{2^j,2^k}\right).$$

Suppose $\lim_{j,k\to\infty} \left(\beta_{2^j,2^k} - \alpha_{2^j,2^k}\right) = 0$. Fix an $\varepsilon > 0$ and choose p_0, q_0 so that, for $p \ge k_0$, $q \ge j_0$, $\beta_{2^p,2^q} - \alpha_{2^p,2^q} < \varepsilon$.

Then

$$\beta_{mn} - \alpha_{mn} \le \frac{1}{mn} \left(\sum_{j=0}^{p_0-1} \sum_{k=0}^{q_0-1} + \sum_{j=p_0}^{p} \sum_{k=0}^{q_0-1} + \sum_{j=0}^{p_0-1} \sum_{k=q_0}^{q} + \sum_{j=p_0}^{p} \sum_{k=q_0}^{q} \right) m_j 2^j n_k 2^k \times$$
$$\times \left(\beta_{2^j,2^k} - \alpha_{2^j,2^k}\right) < \frac{1}{mn} \left(2^{p_0} 2^{q_0} \beta_{11} + m 2^{q_0} \beta_{11} + n 2^{p_0} \beta_{11} + mn\varepsilon\right)$$

$$\leq \frac{2^{p_0}2^{q_0}}{2^p2^q}\beta_{11} + \frac{2^{q_0}}{2^q}\beta_{11} + \frac{2^{p_0}}{2^p}\beta_{11} + \varepsilon < 4\varepsilon,$$

if p and q are large enough, and $\lim_{m,n\to\infty}(\beta_{mn} - \alpha_{mn}) = 0$.

Define $A \in \bar{\bar{G}}_2$ if there exist functions $p, q : \mathbf{N} \to \mathbf{N}$ such that

$$a_{jk}^{mn} = \begin{cases} 2^{-m}2^{-n}, & \text{if } p(m) \leq j < p(m) + 2^m \text{ and } q(n) \leq k < q(n) + 2^n, \\ 0, & \text{otherwise.} \end{cases}$$

Then the following result is true.

THEOREM 3. $\mathrm{ac}_2 = \bigcap_{A\in\bar{\bar{G}}_2} c_A$.

LEMMA 2. *Let y be a double sequence. Then the Pringsheim limit*

$$\lim_{m,n\to\infty} y_{mn} = t \tag{3}$$

exists iff for all strictly increasing functions $m, n : \mathbf{N} \to \mathbf{N}$ the ordinary limit

$$\lim_{r\to\infty} y_{m(r),n(r)} = t \tag{4}$$

exists.

Clearly (3) implies (4). To prove the converse, suppose the limit in (3) does not exist. Then for every t there exists an $\varepsilon > 0$ such that, for each N, there exist integers m and n for which

$$|y_{mn} - t| \geq \varepsilon. \tag{5}$$

For $N = 1$, denote the corresponding values of m and n for which (5) is true by $m(1)$, $n(1)$, respectively. Pick $N_2 = \max\{m(1), n(1)\} + 1$ and denote a pair of values of m and n satisfying (5) by $m(2)$, $n(2)$, respectively. Continuing in this way we obtain an infinite sequence of values $\{m(r), n(r)\}$ for which (5) is true, and, therefore, the limit in (4) does not exist.

Lemma 2 thus motivates the blending of the classes $\bar{G}_2$ and $\bar{\bar{G}}_2$ in the following way.

A matrix $A \in \bar{\bar{\bar{G}}}_2$ if there exist functions $p, q, m, n : \mathbf{N} \to \mathbf{N}$, m and n strictly increasing, such that

$$a_{jk}^{r} = \begin{cases} 2^{-m(r)}2^{-n(r)}, & \text{if } p(r) \leq j < p(r) + 2^{m(r)} \text{ and } q(r) \leq k < q(r) + 2^{n(r)}, \\ 0, & \text{otherwise.} \end{cases}$$

Then we have the following result.

THEOREM 4. $\mathrm{ac}_2 = \bigcap_{A\in\bar{\bar{\bar{G}}}_2} c_A$.

Finally, we remark that, if the ordinary limits of $\{y_r\}$ exist, then the limits must be the same, say t, and, for each $A \in \bar{G}_2$, the $\lim x = \lim_r y_r = t$ is uniform.

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