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KARAMATA'S ITERATION THEOREM AND NORMED REGULARLY VARYING SEQUENCES IN A HISTORICAL LIGHT

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Abstract. Karamata's Iteration Theorem is discussed from the standpoint of significance of its results in anticipating later work. It is applied to provide necessary and sufficient conditions in terms of a normalized regularly varying of iterates in two settings, one of which involves Abel's functional equation. These are analytical extensions of arguments used in the theory of critical branching stochastic processes.

1. Karamata's Iteration Theorem

To motivate our subsequent discussion, and to make the statement and proof available in English, we give a translation for Theorem III of Karamata [**6**]. The equation numbering (1)–(16) is ours; Karamata's numbering is (6)–(17) for our equation numbers (1)–(12).

THEOREM III (Karamata, [**6**]). *Let the function $f(x)$ be defined in the neighbourhood of the point $x = +0$ and suppose that in the neighbourhood of this point the function $x - f(x)$ is regularly varying, i.e.*

$$f(x) = x - a(x)x^k L(x) \tag{1}$$

where

$$a(x) \to a \neq 0, \qquad x \to +0 \tag{2}$$

and

$$xL'(x) = o\{L(x)\}, \quad x \to +0. \tag{3}$$

If

$$a > 0 \quad and \quad k > 1 \tag{4}$$

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and x_0 is sufficiently small that

$$f(x) > 0 \quad \textit{for } 0 < x \le x_0$$

(5) *and*

$$\inf_{\xi \le x \le x_0} \{x - f(x)\} > 0 \quad \textit{for any } 0 < \xi < x_0,$$

then from

(6) $$x_{n+1} = f(x_n), \qquad n = 0, 1, 2, \ldots$$

it follows that

(7) $$x_n \sim a^* n^{-k^*} L^*(1/n), \qquad n \to \infty,$$

where

(8) $$k^* = 1/(k-1) \quad \textit{and } a^* = (k^*/a)^{k^*}$$

and $x^{k^*} L^*(x)$ is the inverse function of $x^{k-1} L(x)$.

Proof. From (5) and Theorem II it follows that

(9) $$x_n \to 0 \qquad \text{and} \quad n \to \infty.$$

If we put $f(x) = x - h(x)$ and $k = 1 + q$, then from (4), $q > 0$ and from (1) and (2)

$$h(x) = a(x) x^{1+q} L(x) \sim a x^{1+q} L(x), \qquad x \to +0$$

so that by (6) and (9)

(10) $$x_n - x_{n+1} = h(x_n) \sim a x_n^{1+q} L(x_n), \qquad n \to \infty.$$

Consequently

(11) $$x_{n+1} \sim x_n \quad \text{as} \quad n \to \infty.$$

Further

$$x_n^q L(x_n) - x_{n+1}^q L(x_{n+1}) = (x_n^q - x_{n+1}^q) L(x_n) + x_{n+1}^q \{L(x_n) - L(x_{n+1})\}$$

(12) $$= q(x_n - x_{n+1}) \xi_n^{q-1} L(x_n) + x_{n+1}^q (x_n - x_{n+1}) L'(\eta_n)$$

where ξ_n and η_n are in the interval (x_{n+1}, x_n). Inasmuch as x_n and x_{n+1} are asymptotically equal, it follows that $x_n \sim \xi_n \sim \eta_n \sim x_{n+1}$, $n \to \infty$, and hence, on account of the properties of slowly varying functions

$$L(x_n) \sim L(\eta_n) \sim L(x_{n+1}), \qquad n \to \infty.$$

From these relations and (10), we obtain for the first summand on the right of (12)

$$q(x_n - x_{n+1}) \xi_n^{q-1} L(x_n) \sim a q x_n^{2q} (L^2(x_n))$$
$$\sim a q x_n^q x_{n+1}^q L(x_n) L(x_{n+1}), \qquad n \to \infty,$$

and for the second summand

$$x_{n+1}^q(x_n - x_{n+1})L'(\eta_n) \sim a x_n^{q+1} x_{n+1}^q L(x_n) L'(\eta_n)$$
$$\sim a x_n^q x_{n+1}^q L(x_n) L(x_{n+1}) \eta_n L'(\eta_n)/L(\eta_n), \quad n \to \infty.$$

Thus, with a view to (3), the relation (12) reduces to

$$\frac{1}{x_{n+1}^q L(x_{n+1})} - \frac{1}{x_n^q L(x_n)} \to aq \quad \text{as } n \to \infty \tag{13}$$

whence

$$1/x_n^q L(x_n) \sim aqn \quad \text{as } n \to \infty, \tag{14}$$

or

$$x_n^q L(x_n) \sim 1/aqn \quad \text{as } n \to \infty. \tag{15}$$

Since $x^{k^*} L^*(x)$ denotes the inverse function of the function $x^q L(x) = x^{k-1} L(x)$, we obtain finally (see for example [**5**, pp. 116–120]), keeping in mind (8),

$$x_n \sim \frac{1}{(aqn)^{k^*}} L^*\left(\frac{1}{aqn}\right) \sim a^* n^{-k^*} L^*(1/n), \quad n \to \infty. \quad \square \tag{16}$$

In the formulation (1) for $f(x)$, on account of condition (2), Karamata takes $L(x)$ without loss of generality to be of the form

$$L(x) = \exp\left\{\int_x^1 \frac{\varepsilon(t)}{t}\, dt\right\} \tag{17}$$

where $\varepsilon(t)$ is a continuous function, with $\varepsilon(t) \to 0$ as $x \to 0+$, so (3) follows immediately. The function $x^r L(x)$ is then continuous and strictly monotone for any $r > 0$ in the neighbourhood of zero, and if g denotes its inverse in the neighbourhood of the origin, it is readily shown that $xg'(x)/g(x) \to 1/r$ as $x \to 0+$, whence it follows immediately that $g(x) = x^{1/r} L^*(x)$ where L^* is also of form (17) (apart, possibly from a multiplicative constant), and in particular slowly varying. This is pointed out on p. 48 of Karamata's paper, and already encapsulates the essence of the notion of conjugate pairs of slowly varying functions ascribed to De Bruijn [**3**]. The reference to pp. 116–120 of Karamata's textbook [**5**] for justification of the step from (15) to (16) is to involve an implied discussion of the consequences of the uniform convergence theorem for the monotone regularly varying function $x^k L(x)$ in relation to its inverse, as might be expected.

The condition (5) ensures that in (6) $x_n \downarrow 0$ as $n \to \infty$, which is the content of Karamata's earlier Theorem II (in which he also points out that (5) can be replaced by $0 < f(x) < x$ for $0 < x \leq x_0$, for $f(x)$ continuous on $0 \leq x \leq x_0$). This Theorem II can be generalized (in both its versions) along the lines of Tasković [**16**], by supposing that $0 < f(x) < x$ for $0 < x \leq x_0$ and $\limsup_{x \to \xi + 0} f(x) < \xi$ for $0 < \xi < x_0$, to obtain $x_n \downarrow 0$ as $n \to \infty$.

It is of particular interest that the motivation for Karamata's Iteration Theorem was a careful reading (indeed, he remarks on an implicit monotonicity assumption) of Pólya and Szegő [**12**, p. 31, problems 173 and 174]. As is now well-known, later sections of the same book had led him to the general theory of regularly varying functions, away from the monotonicity assumptions by which, upto that time, it had been constrained. Karamata indicates a proof of the Pólya-Szegő result, under essentially more restrictive conditions than his own but with a more refined conclusion, in a work (not seen by the present author) of A. Ostrowski of 1940, and acknowledges [**6**, p. 47] that his own proof of Theorem III is a modification of Ostrowski's, whose theorem he states as Theorem I.

The situation described in Karamata's Iteration Theorem relates to the critical branching process. It has been evident since the early (1938) paper of Kolmogorov [**7**] that simple branching processes theory could be developed in terms of the theory of iteration and related functional equations; how this could be done using simple real-variable notions of monotonicity/convexity in the manner of M. Kuczma was set out by the present author [**13**].

The results of Karamata's Iteration Theorem anticipate results in branching process theory substantially. For example, the case $L(x) \equiv 1$ was first treated in the branching process context in a little-known paper of Nagaev [**11**] of (1961) following on from Kolmogorov's result; and Karamata's Iteration Theorem in the more restricted context of branching process theory was not rediscovered [**14**] until 1968. The reader may consult [**13**] for details.

In the sequel we will use Karamata's Iteration Theorem in a general analytical setting to show that the notion of a normed regularly varying sequence can be used to express necessary and sufficient conditions. In the course of this discussion we extend somewhat the results of Slack [**14**], [**15**], described also in Bingham, Goldie and Teugels [**1**], which occur in a probabilistic setting.

2. Regularly Varying Sequences

A regularly varying sequence $\{\theta_n\}$ is a sequence of positive terms such that there exists a sequence of positive terms $\{c(n)\}$ such that

$$\theta(n) \sim Kc(n) \qquad \text{where} \tag{18}$$

$$n\big(1 - \big(c(n-1)/c(n)\big)\big) \to \rho \tag{19}$$

where ρ is finite [**4**], [**2**], [**1**]. In analogy to the (Zygmund) class of normalised regularly varying functions [**1**], we call a sequence of positive terms $\{c(n)\}$ satisfying (19) a normalized regularly varying sequence of index ρ. We notice from the lines (10) and (15) of Karamata's proof that as $n \to \infty$, $n(1 - x_{n+1}/x_n) \to q^{-1}$ whence it follows (since $x_{n+1}/x_n \to 1$), that the sequence $\{x_n\}$ is a normalized regularly varying sequence of index $(-q)^{-1}$.

THEOREM 1. *Suppose $f(x)$ is defined on $(0,1]$ with $0 < x - f(x)$, and $x - f(x)$ is non-decreasing with x. Put $x_0 = 1$, $x_{n+1} = f(x_n)$, $n \geq 0$, $= f_{n+1}(1)$, where f_n is the n^{th} functional iterate of f.*

Then $\{x_n\}$ is a regularly varying sequence of index $(-q)^{-1}$ for some $q > 0$, as $n \to \infty$, if and only if

$$f(x) = x - x^{1+q}\mathcal{L}(x) \tag{20}$$

where $\mathcal{L}$ is a slowly varying function as $x \to 0+$.

Proof. The sufficiency of (20) follows directly from the proof of Karamata's Iteration Theorem, since the conditions of that theorem are satisfied, as indicated above.

Assuming now that $\{x_n\}$ is a regularly varying sequence of index $(-q)^{-1}$ implies $x_n = n^{-1/q}\mathcal{L}^*(n)$ for some $\mathcal{L}^*(x)$ slowly varying as $x \to \infty$ [**4**]. Consequently, $\mathcal{L}^*(x) = a^*(x)L^*(x)$ where $a^*(x) \to a^* > 0$ as $x \to \infty$, $xL^{*\prime}(x)/L^*(x) \to 0$ as $x \to \infty$, $L^*(x)$ being continuously differentiable, with $x^{-1/q}L^*(x)$ strictly decreasing and continuous as $x \to \infty$. Write $\Phi(x) = x^{-q}L(x)$ for the inverse of $x^{-1/q}L^*(x)$; this is defined and strictly monotone near the origin. By the uniform convergence theorem, since L is slowly varying, $n = \Phi(x_n/a^*(n)) \sim (a^*)^q\Phi(x_n)$, so $\Phi(x_n) \sim n(a^*)^{-q}$ as $n \to \infty$.

Let x be in the neighbourhood of 0 ($x \le x_0 = 1$) but arbitrary. Choose $n \equiv n(x)$ such that

$$x_n = f_n(1) \ge x > f_{n+1}(1) = x_{n+1}. \tag{21}$$

Since $x - f(x)$ is nondecreasing on $(0, 1]$

$$x_n - x_{n+1} \ge x - f(x) \ge x_{n+1} - x_{n+2}. \tag{22}$$

From (21), providing x is sufficiently small

$$\Phi(x_n) \le \Phi(x) \le \Phi(x_{n+1})$$

and from (22)

$$(x_{n+1} - x_{n+2})\Phi(x_n) \le (x - f(x))\Phi(x) \le (x_n - x_{n+1})\Phi(x_{n+1})$$

$$\text{i.e.}\quad (x_{n+1} - x_{n+2})\Phi(x_n) \le (x - f(x))x^{-q}L(x) \le (x_n - x_{n+1})\Phi(x_{n+1})$$

whence by (21)

$$\frac{x_{n+1} - x_{n+2}}{x_n}\Phi(x_n) \le \frac{x - f(x)}{x}x^{-q}L(x) \le \frac{x_n - x_{n+1}}{x_{n+1}}\Phi(x_{n+1})$$

i.e.

$$\frac{x_{n+2}}{x_n}\left(\frac{x_{n+1}}{x_{n+2}} - 1\right)\Phi(x_n) \le \frac{x - f(x)}{x}x^{-q}L(x) \le \left(\frac{x_n}{x_{n+1}} - 1\right)\Phi(x_{n+1})$$

whence from the fact that the sequence $\{x_n\}$ is normalized regularly varying $(x - f(x))x^{-q-1}L(x) = a(x)$, where $a(x) \to q^{-1}(a^*)^{-q}$ as $x \to 0+$. □

Elements of the above argument occur in Slack [**15**] and [**1**].

THEOREM 2. *Suppose $f(x)$ satisfies the prior conditions of Theorem 1, and the sequence $\{x_n\}$ is defined as in that theorem. Suppose $u(x)$, $x \in (0,1]$ is a non-increasing, convex solution of Abel's functional equation*

$$u(f(x)) = u(x) + 1, \qquad x \in (0,1] \tag{23}$$

where $u(1) = u(1-) = 0$.

Then $\{x_n\}$ is a regularly varying sequence of index $(-q)^{-1}$ for some $q > 0$ as $n \to \infty$ if and only if

$$u(x) \sim q^{-1}x^{-q}/\mathcal{L}(x), \quad \textit{as } x \to 0+ \tag{24}$$

for some $q > 0$ and some slowly varying function at 0, $\mathcal{L}$.

Proof. Iterating (23) at $x = 1$ yields $u(x_n) = n$. From (21), and the monotonicity of u

$$n = u(x_n) \le u(x) \le u(x_{n+1}) = n + 1. \tag{25}$$

Now assume $\{x_n\}$ is a regularly varying sequence of index $(-q)^{-1}$; it follows that $x_n/x_{n+1} \to 1$, and from (21) that $x_{n+1}/x \to 1$ as $x \to 0+$. Now, from Theorem 1, and from Karamata's Iteration Theorem, $x_n^q\mathcal{L}(x_n) \sim (qn)^{-1}$, where $\mathcal{L}(x) = a(x)L(x)$, so by the uniform convergence theorem, $x^q\mathcal{L}(x) \sim (qn)^{-1} \sim (qu(x))^{-1}$, the last asymptotic equality following from (25). Thus (24) holds. The ideas of the above argument occur in a more restricted probabilistic setting in Slack [**14**]. This completes the proof of necessity of (24).

Consider now the implication of the assumed convexity property of u. Since $x_{n-1} > x_n$, by convexity

$$u_+(x_n)(x_{n-1} - x_n) \le u(x_{n-1}) - u(x_n) \le (x_{n-1} - x_n)u_-(x_{n-1}) \tag{26}$$

where the left and right derivatives u_+ and u_- are non-decreasing, almost surely equal and for $x \in (0,1]$

$$u(x) = -\int_x^1 u_+(t)\,dt = -\int_x^1 u_-(t)\,dt. \tag{27}$$

(See [**8**, Chapter I, §1]). Further, by (23), $u(x_n) - u(x_{n-1}) = 1$, and $u(x_n) = n$, whence, in turn

$$\begin{gathered} u_+(x_n)(x_{n-1} - x_n) \le -1 \le (x_{n-1} - x_n)u_-(x_{n-1}) \\ n\left(1 - \frac{x_{n-1}}{x_n}\right)\frac{x_n u_+(x_n)}{u(x_n)} \ge 1 \ge n\left(1 - \frac{x_{n-1}}{x_n}\right)\frac{x_n u_-(x_n)}{u(x_n)}. \end{gathered} \tag{28}$$

Now assume (24) holds. Then by the *monotone density* theorem (Landau [**10**]; see Bingham, Goldie and Teugels [**1**, pp. 38–40]) from (27), as $x \to 0+$, $-xu_+(x)/u(x) \sim q \sim -xu_-(x)/u(x)$ whence, since $x_n \downarrow 0$, that $\{x_n\}$ is normalised regularly varying follows from (28). □

Note that in the above proof, convexity of u is not required in the first part, and monotonicity of u is not required in the second part.

Kuczma [9] has shown that (23) has a convex solution unique to an additive constant if f is additionally: concave in $(0,1]$ and satisfies the conditions $f'(x) > 0$ and $\lim_{x\to+0} f'(x) = 1$.

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