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# TWO PROPOSITIONS ON SLOWLY VARYING FUNCTIONS

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**Abstract.** The paper contains two statements which both assert that, under certain conditions, the function $f(x) = F(R_1(x), \dots, R_m(x))$, where $F : (\mathbf{R}^+)^m \to \mathbf{R}^+$, $R_k : \mathbf{R}^+ \to \mathbf{R}^+$, $k = 1, \dots, m$, is slowly varying.

**1.** Our first theme is the following:

PROPOSITION 1. *Let $L_k$ $(k = 1, \dots, m)$ be slowly varying functions and let the real function $F$ be defined and continuous on the adherence (closure) $D$ in $(\mathbf{R}^*)^m$ of the set $E = L_1(\mathbf{R}^+) \times \cdots \times L_m(\mathbf{R}^+)$ and with the properties*

$$m = \inf_{(t_1, \dots, t_m) \in E} F(t_1, \dots, t_m) > 0, \qquad M = \sup_{(t_1, \dots, t_m) \in E} F(t_1, \dots, t_m) < +\infty.$$

*Then the function*

$$f(x) = F(L_1(x), \dots, L_m(x)) \qquad (x > 0)$$

*is slowly varying.*

Here $\mathbf{R}^*$ denotes the extended system (space) of real numbers, $\mathbf{R}^+ = (0, +\infty)$ and the continuity on $D$ means especially that, if $(\alpha_1, \dots, \alpha_n) \in D$ and $+\infty \in \{\alpha_1, \dots, \alpha_m\}$, then

$$\lim_{E \ni (t_1, \dots, t_m) \to (\alpha_1, \dots, \alpha_m)} F(t_1, \dots, t_m) = F(\alpha_1, \dots, \alpha_m)$$

exists.

*Proof.* Under the conditions of the proposition, the function $f$ is obviously defined and positive for $x > 0$, and also measurable on $\mathbf{R}^+$ if the measurability

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on $\mathbf{R}^+$ of the functions $L_k$ $(k = 1, \dots, m)$ is clamed (by the accepted definition of slowly varying function). Moreover,

$$0 < m \le f(x) \le M < +\infty \qquad (x > 0). \tag{1}$$

Suppose that $f$ is not slowly varying. This function cannot be regularly varying, because that would imply $\lim_{x\to+\infty} f(x) = +\infty$ or $\lim_{x\to+\infty} f(x) = 0$, contradicting (1). Besides, (1) implies that for no $\lambda \in (0, +\infty)$

$$\lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = +\infty \ \vee \ \lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = 0$$

in possible.

Hence, there exists $\lambda > 0$ such that $f(\lambda x)/f(x)$ oscillates as $x \to +\infty$, and consequently, for the same $\lambda$, there exist numbers $A$ and $B$ such that

$$0 < A < B < +\infty \tag{2}$$

and the sequences $(x_n)$ and $(y_n)$ of positive numbers, tending $+\infty$, for which

$$\lim_{n\to\infty} \frac{f(\lambda x_n)}{f(x_n)} = A, \qquad \lim_{n\to\infty} \frac{f(\lambda y_n)}{f(y_n)} = B. \tag{3}$$

But then there exist the elements $\alpha, \beta \in D$, the subsequence $(\bar{x}_n)$ of $(x_n)$ and the subsequence $(\bar{y}_n)$ of $(y_n)$ such that

$$\lim_{n\to\infty} L_k(\bar{x}_n) = \alpha_k, \quad \lim_{n\to\infty} L_k(\bar{y}_n) = \beta_k \qquad (k = 1, \dots, m),$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$, $\beta = (\beta_1, \dots, \beta_m)$. By the definition of slowly varying function, we have

$$\lim_{n\to\infty} L_k(\lambda\bar{x}_n) = \alpha_k, \quad \lim_{n\to\infty} L_k(\lambda\bar{y}_n) = \beta_k \qquad (k = 1, \dots, m),$$

and further

$$\lim_{n\to\infty} \frac{f(\lambda\bar{x}_n)}{f(\bar{x}_n)} = \lim_{n\to\infty} \frac{F(L_1(\lambda\bar{x}_n), \dots, L_m(\lambda\bar{x}_n))}{F(L_1(\bar{x}_n), \dots, L_m(\bar{x}_n))} = \frac{F(\alpha_1, \dots, \alpha_m)}{F(\alpha_1, \dots, \alpha_m)} = 1;$$

similarly,

$$\lim_{n\to\infty} \frac{f(\lambda\bar{y}_n)}{f(\bar{y}_n)} = \frac{F(\beta_1, \dots, \beta_m)}{F(\beta_1, \dots, \beta_m)} = 1,$$

and two last conclusions contradict (2) and (3). This proves our statement.

We can remark that simple examples show that none of the conditions concerning the function $F$ can be omitted. For example, if $m = 1$, $L_1(x) = \ln(x + 1)$, $F(x) = 2 + \sin x$, then the function $F$ is not continuous on $D = [0, +\infty]$ in the previous sense, and the function $f(x) = 2 + \sin \ln(x + 1)$ is not slowly varying. On the other hand, for $m = 1$, $L_1(x) = \ln(x + 1)$ and $F(x) = e^x$, $F$ is continuous on $D$ ($F$ can be considered as a function whose values belong to $\mathbf{R}^*$), but (1) is not satisfied; in this case $f(x) = x + 1$ is not a slowly varying function.

**2.** Besides the previous proposition, one can give the following one, in some sense similar, but really incomparable to it.

PROPOSITION 2. *Let $R_k(x)$ $(k = 1, \dots, m)$ be regularly varying functions tending to infinity with $x$, and let the function $F : (\mathbf{R}^+)^m \to \mathbf{R}^+$ be continuous and slowly varying, in Karamata-Bajšanski sense* [**1**]. *Then the function $f(x) = F(R_1(x), \dots, R_m(x))$ $(x > 0)$ is slowly varying.*

*Proof.* First, recall that, by the definition given by Karamata and Bajšanski in [**1**], the function $F : (\mathbf{R}^+)^m \to \mathbf{R}^+$ is slowly varying if

$$\lim_{\min\{x_1,\dots,x_m\}\to+\infty} \frac{F(\lambda_1 x_1, \dots, \lambda_m x_m)}{F(x_1, \dots, x_m)} = 1 \quad \text{for each } \lambda = (\lambda_1, \dots, \lambda_m) \in (\mathbf{R}^+)^m$$

and that the theorem on uniform convergence in [**1**] implies that, for such a function and for $0 < \alpha_k < \beta_k < +\infty$ $(k = 1, ..., m)$,

$$\lim_{\min\{x_1,\dots,x_m\}\to+\infty} \frac{F(\lambda_1 x_1, \dots, \lambda_m x_m)}{F(x_1, \dots, x_m)} = 1$$
$$\text{uniformly for } \lambda = (\lambda_1, \dots, \lambda_m) \in \prod_{k=1}^{m} [\alpha_k, \beta_k]$$

Under the hypotheses of the proposition, the function $f(x)$ is defined and measurable on $\mathbf{R}^+$.

Let $\lambda > 0$ be arbitrarily chosen. By hypothesis,

$$R_k(x) = x^{\rho_k} L_k(x), \quad \text{with } \rho_k \geq 0 \ (k = 1, \dots, m),$$

where $L_k(x)$ $(k = 1, \dots, m)$ are slowly varying functions. Then, for $x$ large enough, we have

$$1/2 \leq L_k(\lambda x)/L_k(x) \leq 2 \qquad (k = 1, \dots, m),$$

and hence, on account of what we have supposed on $R_k$ and by the just mentioned uniform convergence property of $F$,

$$\lim_{x\to+\infty} \frac{f(\lambda x)}{f(x)} = \lim_{x\to+\infty} \frac{F(R_1(\lambda x), \dots, R_m(\lambda x))}{F(R_1(x), \dots, R_m(x))}$$
$$= \lim_{x\to+\infty} \frac{F\left(\dfrac{R_1(\lambda x)}{R_1(x)} \cdot R_1(x), \dots, \dfrac{R_m(\lambda x)}{R_m(x)} \cdot R_m(x)\right)}{F(R_1(x), \dots, R_m(x))}$$

$$= \lim_{x\to+\infty} \frac{F\left(\lambda^{p_1}\dfrac{L_1(\lambda x)}{L_1(x)}\cdot R_1(x),\dots,\lambda^{p_m}\dfrac{L_m(\lambda x)}{L_m(x)}\cdot R_m(x)\right)}{F(R_1(x),\dots,R_m(x))} = 1.$$

So the function $f(x)$ is slowly varying.

Finally, we note that special cases of Propositions 1 and 2 were given in [**2**, Theorem II, 1°, 2°, 3°]).

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