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O-REGULAR VARIATION AND UNIFORM CONVERGENCE

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Abstract. We consider some kinds of regularly varying functions. For one of them we prove a more precise form of the Uniform Convergence Theorem.

Introduction

Basic problems of the theory of regular variation are:

0. Definition	3. Representation	6. Abelian theorems
1. Index	4. Characterization	7. Tauberian theorems
2. Uniformity	5. Growth	8. Mercerian theorems.

Of course, this division is not disjunctive.

By Definition problem we understand the problem of the definition of certain types of regular variations for mathematical objects which are not *real functions in one real variable*. On this meeting of IWAA it was seen that, even in simple cases, there is no unique way to a solution.

By Characterization problem we understand the problem of finding logical equivalents to a given type of regular variation. An example of this kind is [Karamata 1930]:

A measurable function $R : [c,\infty[\to]0,\infty[$ ($c > 0$ is a real number) is regularly varying of index ρ ($\in \mathbf{R}$) if and only if

$$\lim_{x\to\infty} \frac{1}{R(x)} \int_x^\infty \left(\frac{x}{t}\right)^\tau R(t)\,\frac{dt}{t} = \frac{1}{\tau-\rho} \quad \text{for a real number } \tau > \rho.$$

We shall consider Problems 1 and 2 for the classical (Karamata) case of O-regular variation.

First was introduced the notion of regular variation [Karamata 1930], then O-regular variation [Avakumović 1936], [Karamata 1936], then (between them) the notion of extended regular variation [Matuszewska 1964]. We shall see that there are intermediate regular variations which might be of some interest.

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Assumptions and notation

$I :=]0,\infty[$, $I_- :=]0,1[$, $I_+ = [1,\infty]$, $a \geq 0$ is a real number,
$K := I \cap [a,\infty[\to I$ is measurable,
$k(t) := \overline{\lim_{x\to\infty}} \dfrac{K(tx)}{K(x)} < \infty$ for all t in I,
$\check{k}(t) := k(1/t)$ for all t in I,
R is a regularly varying function of index ρ ($\in \mathbf{R}$),
$r(t) := \lim_{x\to\infty} \dfrac{R(tx)}{R(x)} = t^\rho$ for all t in I,
$x \vee y := max(x,y)$ for all x and y in $\overline{\mathbf{R}} = \mathbf{R} \cup \{-\infty,\infty\}$.

Note that the condition "$k(t) < \infty$ for all t in I" means: K is an O-regularly varying function.

Index function

Definition 1. k is the *index function* of the (O-regularly varying) function K.

First, somewhat rough, way of founding the theory of O-regularly varying functions is the selection of those properties of the function K which depend on the corresponding properties of the function k. In such an approach, we do not distinguish between two functions of O-regular variation whose index functions are equal.

From the definition of the function k, we have immediately

$$k(1) = 1 \tag{k1}$$

and then, since $k(t) < \infty$ for all t in I,

$$k(st) \leq k(s)k(t) \quad \text{for all } s \in I \text{ and } t \in I. \tag{k2}$$

This implies $1 = k(1) \leq k(t)k(1/t)$ and further $k(t) > 0$ for all t in I. Therefore, we shall assume that $k : I \to I$. Such an approach has certain consequences. For example "The function k is bounded on the set $T \subseteq I$" means "There are m and M in I such that $m \leq k(t) \leq M$ for all $t \in T$".

From the Uniform Convergence Theorem,

$$\overline{\lim_{x\to\infty}} \sup_{t\in T} \frac{K(tx)}{K(x)} < \infty \quad \text{for every compact nonempty set } T \subseteq I. \tag{UCT}$$

and $\sup_{t\in T} \overline{\lim_{x\to\infty}} \dfrac{K(tx)}{K(x)} \leq \overline{\lim_{x\to\infty}} \sup_{t\in T} \dfrac{K(tx)}{K(x)}$, it follows that the function k is locally bounded from above on I. The function k is locally bounded from below on I since $1 \leq k(t)\sup_{t\in T} k(1/t)$ for all $T \subseteq I$ and $t \in T$. Consequently,

(k3) The function k is locally bounded on I.

The function $k : I \to I$ is submultiplicative, locally bounded and $k(1) = 1$. The general theory of submultiplicative functions (as the theory of subadditive functions) can be found in the well-known book of Hille and Phillips.

For traditional reasons, we shall consider the following "case $(\,.\,,.\,)$" based on the substructure $(I,\ .\,,\leq,1)$ of the real number field: the domain of the function K ($\operatorname{dom} K$) is the neighborhood of the point ∞ in I and K takes the values in I. If the operation of multiplication is substituted with the operation of addition, then the intervals I, I_-, I_+ are substituted respectively with intervals $]-\infty,\infty[$, $]-\infty,0[$, $]0,\infty[$. Hence, we have three more cases: $(\,.\,,+)$, $(+,\,.\,)$, $(+,+)$; in these cases the term $K(tx)/K(x)$ is transformed respectively into:

$$K(tx)-K(x), \qquad K(t+x)/K(x), \qquad K(t+x)-K(x).$$

All these four cases are isomorphic. In fact, they become a single case when the real numbers are interpreted appropriately. Namely, real numbers can be considered as the structure $(I,\ .\,,*,\leq,1,e)$, where $x*y=\exp((\log x)(\log y))$. The most common case would be the case $(+,+)$ since we are adapted to work with real functions with one real variable, in which case the field of reals are fixed.

Rules of computing 1. *Suppose $\delta \in I$, $p \in I_+$, and assume K_* is a function of O-regular variation, and k_* is its index function.*

Then

(a)

function	K	$1/K$	K^δ	RK
index function	k	$\check{k}$	k^δ	rk

(b) *If $K_*(x) \asymp K(x)$ as $x \to \infty$ ($K_*(x)/K(x)$ is bounded for sufficiently large x), then $k_* \asymp k$ on I (k_*/k is bounded on I).*

(c) *If $K_*(x) \underline{\asymp} K(x)$ as $x \to \infty$ ($K_*(x) \sim cK(x)$ as $x \to \infty$ for some $c \in I$), then $k_* = k$.*

(d) *If $K(px) = K(x)$ for all x in $\operatorname{dom} K$, then $k(pt) = k(t)$ for all $t \in I$ and $k(t) = \sup\{K(tu)/K(u) : 1 \leq u/(1 \vee a) < p\}$ for all $1 \leq t < p$.*

Proof. We shall prove only the second statement in (d) — the others are infered immediately from the definition of the index function.

Suppose $K(px) = K(x)$ for all $x \in \operatorname{dom} K$, $J := [1 \vee a, p(1 \vee a)[$, $1 \leq t < p$ and $f(x) := K(tx)/K(x)$. Then $f(px) = f(x)$ for all x in $\operatorname{dom} K$, thus $f([x,\infty[) = f(J)$. This implies $\sup_{y\geq x} f(y) = \sup_{u\in J} f(u)$. We conclude that

$$\varlimsup_{y\to\infty} f(y) = \lim_{x\to\infty}\sup_{y\geq x} f(y) = \sup_{u\in J} f(u). \quad \square$$

We see that the function k is O-regularly varying if k is measurable, since $\displaystyle\varlimsup_{t\to\infty}\frac{k(st)}{k(t)} \leq k(s)$ for all $s \in I$. This follows from (k2). It is an interesting question when the equality holds in this inequality. One such case is $K = R$, the other is given by

Rule of computing 2. *Let $h : I \to I$, $p \in I_+$ and*

(i) $h(st) \le h(s)h(t)$ *for all* $s \in I$ *and* $t \in I$,

(ii) $h(1) = 1$,

(iii) $h(pt) = h(t)$ *for all* $t \in I$.

Then

$$\overline{\lim_{t\to\infty}} \frac{h(st)}{h(t)} = h(s) \quad \text{for all } s \text{ in } I.$$

Proof. For all $s \in I$, from (i) it follows $\overline{\lim\limits_{t\to\infty}} \dfrac{h(st)}{h(t)} \le h(s)$, and from (iii) and (ii) it follows that

$$\overline{\lim_{t\to\infty}} \frac{h(st)}{h(t)} = \sup_{1\le u<p} \frac{h(su)}{h(u)} \ge \frac{h(s1)}{h(1)} = h(s). \quad \square$$

Indices

Certain properties of the function k can be described by some real numbers — we shall call them indices of the function K (equally well we could call them indices of the function k). We shall consider here four such numbers which describe the behavior of the function k in four significant "points": $0+$, $1-$, $1+$ and ∞. If $K = R$, all these numbers become equal to ρ, and vice versa. We shall consider also the property "the function k is continuous" without assigning to it a number although it is possible:

Lemma 1. *Let* $\overline{\lim}_{t\to 1} k(t) = 1$. *Then the function k is continuous.*

Proof. Let us suppose $s \in I$. Applying the operation $\underline{\lim}_{t\to s}$ to $k(s) \le k(t)k(s/t)$, we obtain

$$k(s) \le \underline{\lim_{t\to s}}\, k(t)\, \overline{\lim_{t\to s}}\, k\left(\frac{s}{t}\right) = \underline{\lim_{t\to s}}\, k(t).$$

Applying the operation $\overline{\lim}_{t\to s}$ to $k(t) \le k(s)k(t/s)$, we obtain $\overline{\lim}_{t\to s} k(t) \le k(s)$, thus $\lim_{t\to s} k(t) = k(s)$. $\square$

Definition 2. The *index* of the function K is the quadruple $\operatorname{index} K := (\underset{\sim}{\kappa}, \underline{\kappa}, \bar{\kappa}, \tilde{\kappa})$ which consists of:

the lower Matuszewska index

$$\underset{\sim}{\kappa} := \inf_{t\in I_-} \frac{\log k(t)}{\log t}$$

the lower Karamata index

$$\underline{\kappa} := \sup_{t\in I_-} \frac{\log k(t)}{\log t}$$

the upper Matuszewska index

$$\tilde{\kappa} := \sup_{t\in I_+} \frac{\log k(t)}{\log t}$$

the upper Karamata index

$$\bar{\kappa} := \inf_{t\in I_+} \frac{\log k(t)}{\log t}.$$

It is known (for example: [Aljančić and Aranđelović 1977]) that

(1) $-\infty < \underline{\kappa} \leq \bar{\kappa} < \infty$,

(2) $\underline{\kappa} = \lim_{t\to 0+} \dfrac{\log k(t)}{\log t}$, $\bar{\kappa} = \lim_{t\to\infty} \dfrac{\log k(t)}{\log t}$,

(3) $k(t) \geq t^{\underline{\kappa}} \vee t^{\bar{\kappa}}$ for all $t \in I$,

(4) $k(t) \preceq t^{\sigma} \vee t^{\tau}$ on I for all real $\sigma < \underline{\kappa}$ and $\tau > \bar{\kappa}$ (for all $\sigma < \underline{\kappa}$ and $\tau > \bar{\kappa}$ there is a real $M \geq 1$ with the property $k(t) \leq M(t^{\sigma} \vee t^{\tau})$ for all t in I).

Applying the operation $\underline{\lim}_{t\to 1}$ to (3), we obtain

$$1 \leq \underline{\lim}_{t\to 1} k(t). \tag{5}$$

From the definition it also follows that

$$-\infty \leq \underset{\sim}{\kappa} \leq \underline{\kappa} \quad \text{and} \quad \bar{\kappa} \leq \tilde{\kappa} \leq \infty, \tag{6}$$

$$\underset{\sim}{\kappa} = \underline{\kappa} \quad \text{implies} \quad k(t) = t^{\underline{\kappa}} \text{ for all } t \in I_{-}, \tag{7}$$

$$\tilde{\kappa} = \bar{\kappa} \quad \text{implies} \quad k(t) = t^{\bar{\kappa}} \text{ for all } t \in I_{+}.$$

The index of the function K belongs to $\overline{\mathbf{R}} \times \mathbf{R} \times \mathbf{R} \times \overline{\mathbf{R}}$ and, by (1) and (6), it is an increasing sequence. Indices are ordered by the relation $\leq'$ in the following way:

Definition 3. Assume $\operatorname{index} K = (\underset{\sim}{\kappa}, \underline{\kappa}, \bar{\kappa}, \tilde{\kappa})$ and let K_* be an O-regularly varying function with index $K_* = (\underset{\sim}{\kappa}_*, \underline{\kappa}_*, \bar{\kappa}_*, \tilde{\kappa}_*)$. Now we define

$$\operatorname{index} K \leq' \operatorname{index} K_* \quad \text{if and only if} \quad \underset{\sim}{\kappa} \leq \underset{\sim}{\kappa}_*, \quad \underline{\kappa} \geq \underline{\kappa}_*, \quad \tilde{\kappa} \geq \tilde{\kappa}_*, \quad \bar{\kappa} \leq \bar{\kappa}_*.$$

If the inequalities from the previous definition are written as $[\underset{\sim}{\kappa}, \underline{\kappa}] \supseteq [\underset{\sim}{\kappa}_*, \underline{\kappa}_*]$ and $[\bar{\kappa}, \tilde{\kappa}] \supseteq [\bar{\kappa}_*, \tilde{\kappa}_*]$, we see that the maximal elements are those indices of index K with the property $\underset{\sim}{\kappa} = \underline{\kappa}$ and $\bar{\kappa} = \tilde{\kappa}$. If the inequalities are written as $[\underset{\sim}{\kappa}, \tilde{\kappa}] \supseteq [\underset{\sim}{\kappa}_*, \tilde{\kappa}_*]$ and $[\underline{\kappa}, \bar{\kappa}] \subseteq [\underline{\kappa}_*, \bar{\kappa}_*]$, we see that minimal elements are those indices of index K_* which have the property $\underset{\sim}{\kappa}_* = -\infty$, $\underline{\kappa}_* = \bar{\kappa}_*$ and $\tilde{\kappa}_* = \infty$.

The bigger indices with respect to $\leq'$ are indices of those O-regularly varying functions which are closer to functions of regular variation. If $K = R$, then $\log k(t)/\log t = \rho$ for all $t \neq 1$ in I, thus $\underset{\sim}{\kappa} = \underline{\kappa} = \rho = \bar{\kappa} = \tilde{\kappa}$.

RULES OF COMPUTING 3. *Let $\delta \in I$, and K_* be an O-regularly varying function with the index $(\underset{\sim}{\kappa}_*, \underline{\kappa}_*, \bar{\kappa}_*, \tilde{\kappa}_*)$. Then*

(a)

function	K	$1/K$	K^{δ}	RK
lower Matuszewska index	$\underset{\sim}{\kappa}$	$-\tilde{\kappa}$	$\delta\underset{\sim}{\kappa}$	$\rho + \underset{\sim}{\kappa}$
lower Karamata index	$\underline{\kappa}$	$-\bar{\kappa}$	$\delta\underline{\kappa}$	$\rho + \underline{\kappa}$
upper Karamata index	$\bar{\kappa}$	$-\underline{\kappa}$	$\delta\bar{\kappa}$	$\rho + \bar{\kappa}$
upper Matuszewska index	$\tilde{\kappa}$	$-\underset{\sim}{\kappa}$	$\delta\tilde{\kappa}$	$\rho + \tilde{\kappa}$

(b) *If $K(x) \asymp K_*(x)$ as $x \to \infty$, then $\underset{\sim}{\kappa} = \underset{\sim}{\kappa}_*$ and $\bar{\kappa} = \bar{\kappa}_*$.*

(c) *If the function k is bounded on I, then $\underset{\sim}{\kappa} = 0 = \bar{\kappa}$.*

(d) *If $\check{k} = k$, then $\underset{\sim}{\kappa} = -\tilde{\kappa}$ and $\underset{\sim}{\kappa} + \bar{\kappa} = 0$.*

Proof. (a) is obtained from Rules of computing 1(a) and Definition 2. (b) is obtained from Rules of computing 1(b) and (2). (c) follows from (b) since "k is bounded on I" means "$k \asymp 1$ on I" and Karamata indices of the constant function 1 are 0. (d) $\check{k}$ is the index function of the function $1/K$, so the statement follows from (a). □

PROPOSITION 1. *Let $\delta \in I_+$. Then*

$$\text{(a)}\quad \underset{\sim}{\kappa} = \inf_{1/\delta \le t < 1} \frac{\log k(t)}{\log t}, \qquad \text{(b)}\quad \underset{\sim}{\kappa} = \sup_{0<t\le 1/\delta} \frac{\log k(t)}{\log t},$$

$$\text{(c)}\quad \tilde{\kappa} = \sup_{1<t\le \delta} \frac{\log k(t)}{\log t}, \qquad \text{(d)}\quad \bar{\kappa} = \inf_{\delta\le t<\infty} \frac{\log k(t)}{\log t}.$$

Proof. The implications (c) $\Longrightarrow$ (a) and (d) $\Longrightarrow$ (b) are obtained from Rules of computing 1(a) and 3(a) by taking $t = 1/s$.

(c) Let us consider the mapping $t \mapsto (n, \theta)$ of the set I_+ into $\mathbf{N}\times]1,\delta]$ ($\mathbf{N} = \{0,1,2,\dots\}$) determined by conditions $\delta^n < t \le \delta^{n+1}$ and $t = \delta^n\theta$. For $t \in I_+$ we have $k(t) = k(\delta^n\theta) \le k^n(\delta)k(\theta)$, thus

$$\frac{\log k(t)}{\log t} \le \frac{n\log k(\delta) + \log k(\theta)}{n\log\delta + \log\theta} =: \varphi_\theta(n).$$

The function φ_θ is monotonous, so

$$\begin{aligned}\varphi_\theta(n) &\le \sup_{n\ge 0}\varphi_\theta(n) = \sup(\varphi_\theta(0), \varphi_\theta(\infty))\\ &= \sup\left(\frac{\log k(\theta)}{\log\theta}, \frac{\log k(\delta)}{\log\delta}\right) \le \sup_{1<s\le\delta}\frac{\log k(s)}{\log s},\end{aligned}$$

therefore

$$\sup_{t\in I_+}\frac{\log k(t)}{\log t} \le \sup_{1<s\le\delta}\frac{\log k(s)}{\log s}.$$

(d) Consider the map $t \mapsto (n,\theta)$ of the set I_+ into $\mathbf{N}^* \times [\delta,\infty[$ ($\mathbf{N}^* = \{1,2,3,\dots\}$) determined by the conditions $t^{n-1} < \delta \le t^n = \theta$. For t in I we have $k^n(t) \ge k(t^n) = k(\theta)$, thus

$$\frac{\log k(t)}{\log t} = \frac{\log k^n(t)}{\log t^n} \ge \frac{\log k(\theta)}{\log\theta} \ge \inf_{\delta\le s<\infty}\frac{\log k(s)}{\log s}.$$

We conclude

$$\inf_{t\in I_+}\frac{\log k(t)}{\log t} \ge \inf_{\delta\le s<\infty}\frac{\log k(s)}{\log s}. \quad \square$$

If we take into account the definition

$$\bar{\kappa} = \inf_{t\in I_+} \frac{\log k(t)}{\log t} = \max\left\{u \in \mathbf{R} \,\middle|\, \frac{\log k(t)}{\log t} \geq u \quad \text{for all } t \in I_+ \right\},$$

we have

$$\begin{aligned} \bar{\kappa} &= \max\{u \in \mathbf{R} \mid k(t) \geq t^u \ \text{ for all } t \in I_+\} \\ &= \max\left\{u \in \mathbf{R} \,\middle|\, \inf_{t\in I_+} k(t)t^{-u} \geq 1\right\}. \end{aligned}$$

In a similar way one can obtain the other equalities

$$\begin{aligned} \underset{\sim}{\kappa} &= \max\{u \in \overline{\mathbf{R}} \mid k(t) \leq t^u \text{ on } I_-\} \\ &= \max\left\{u \in \overline{\mathbf{R}} \,\middle|\, \sup_{t\in I_-} k(t)t^{-u} \leq 1\right\}, \\ \underline{\kappa} &= \min\{u \in \mathbf{R} \mid k(t) \geq t^u \text{ on } I_-\} \\ &= \min\left\{u \in \mathbf{R} \,\middle|\, \inf_{t\in I_-} k(t)t^{-u} \geq 1\right\}, \\ \tilde{\kappa} &= \min\{u \in \overline{\mathbf{R}} \mid k(t) \leq t^u \text{ on } I_+\} \\ &= \min\left\{u \in \overline{\mathbf{R}} \,\middle|\, \sup_{t\in I_+} k(t)t^{-u} \leq 1\right\}. \end{aligned} \tag{8}$$

In particular

$$\begin{aligned} &k(t) \leq t^{\underset{\sim}{\kappa}} \text{ for all } t \in I_-, \qquad k(t) \geq t^{\underline{\kappa}} \text{ for all } t \in I_-, \\ &k(t) \leq t^{\tilde{\kappa}} \text{ for all } t \in I_+, \qquad k(t) \geq t^{\bar{\kappa}} \text{ for all } t \in I_+. \end{aligned} \tag{9}$$

We can obtain some information of, in general nonmeasurable, function k using the following transformations of k

$$\begin{aligned} \varphi &: \mathbf{R} \to \overline{\mathbf{R}}, \qquad \varphi(u) = \sup_{t\in I_-} k(t)t^{-u} \\ \psi &: \mathbf{R} \to \overline{\mathbf{R}}, \qquad \psi(u) = \sup_{t\in I_+} k(t)t^{-u}. \end{aligned}$$

They give us certain relations beteween k and power functions.

For example, from (9) we have

$$\varphi(u) \geq \sup_{t\in I_-} t^{\underline{\kappa}-u} = \begin{cases} 1 & \text{for } u \leq \underline{\kappa} \\ \infty & \text{for } u > \underline{\kappa} \end{cases}, \qquad \psi(u) \geq \sup_{t\in I_+} t^{\bar{\kappa}-u} = \begin{cases} \infty & \text{for } u < \bar{\kappa} \\ 1 & \text{for } u \geq \bar{\kappa} \end{cases}.$$

If $u < \underline{\kappa}$, then $k(t) \preceq t^u$, i.e. $k(t)t^{-u} \preceq 1$ on I_-, so $\varphi(u) < \infty$.

If $u > \bar{\kappa}$, then $k(t) \preceq t^u$, i.e. $k(t)t^{-u} \preceq 1$ on I_+, so $\psi(u) < \infty$.

These relations and (8) imply

(10)
$$\underset{\sim}{\kappa} = \sup\{u \in \mathbf{R} \mid \varphi(u) = 1\}, \qquad \underline{\kappa} = \sup\{u \in \mathbf{R} \mid \varphi(u) < \infty\},$$
$$\tilde{\kappa} = \inf\{u \in \mathbf{R} \mid \psi(u) = 1\}, \qquad \bar{\kappa} = \inf\{u \in \mathbf{R} \mid \psi(u) < \infty\}.$$

From the Rules of computing 1(a) we get the following table

$K(x)$	$1/K(x)$	$K^\delta(x)$	$R(x)K(x)$
$k(t)$	$k(1/t)$	$k^\delta(t)$	$r(t)k(t)$
$\varphi(u)$	$\psi(-u)$	$\varphi^\delta(u/\delta)$	$\varphi(u-\rho)$
$\psi(u)$	$\varphi(-u)$	$\psi^\delta(u/\delta)$	$\psi(u-\rho)$

which tells us that it suffices to consider one of the functions φ and ψ, say φ — the other is of the form $\varphi(-u)$. If $\check{k} = k$, then $\psi(u) = \varphi(-u)$ for all u in $\mathbf{R}$.

Functions $\varphi = \sup_{t\in I_-} f_t$ and $\psi = \sup_{t\in I_+} f_t$ inherit some properties of the functions $f_t(u) = k(t)t^{-u}$. We have

(12) The function φ is increasing. The function ψ is decreasing. The function φ is logarithmically convex and therefore convex on the interval $]-\infty, \underline{\kappa}[$. The same holds for ψ and the interval $]\bar{\kappa}, \infty[$.

Furthermore,

(13)
$$\varphi(\underline{\kappa}-) = \varphi(\underline{\kappa}) \quad \text{and} \quad \psi(\bar{\kappa}+) = \psi(\bar{\kappa}),$$

since

$$\varphi(\underline{\kappa}-) = \sup_{u<\underline{\kappa}} \varphi(u) = \sup_{u<\underline{\kappa}} \sup_{t\in I_-} k(t)t^{-u}$$
$$= \sup_{t\in I_-} \sup_{u<\underline{\kappa}} k(t)t^{-u} = \sup_{t\in I_-} k(t)t^{-\underline{\kappa}} = \varphi(\underline{\kappa}).$$

(14)
$$\varphi(-\infty) = \varlimsup_{t\to 1-} k(t) \quad \text{and} \quad \psi(\infty) = \varlimsup_{t\to 1+} k(t).$$

We shall prove the first equality. For $u \in \mathbf{R}$ and $\delta \in I_-$,

$$\varphi(u) = \sup_{0<t<1} k(t)t^{-u} = \sup\Big(\sup_{0<t\le\delta} k(t)t^{-u}, \sup_{\delta<t<1} k(t)t^{-u}\Big),$$

and further for $u \le c < \underline{\kappa}$

$$0 \le \sup_{0<t\le\delta} k(t)t^{-u} = \sup_{0<t\le\delta} k(t)t^{-c}t^{c-u} \le \varphi(c) \sup_{0<t\le\delta} t^{c-u} = \varphi(c)\delta^{c-u} \to 0$$

as $u \to -\infty$, so

$$\varphi(-\infty) = \lim_{u\to-\infty} \sup_{\delta<t<1} k(t)t^{-u} = \inf_{u\le c} \sup_{\delta<t<1} k(t)t^{-u}$$

(because the function $\sup_{\delta<t<1} k(t)t^{-u}$ is increasing); applying the operation $\inf_{\delta\in I_-}$ to the above equality, we obtain

$$\varphi(-\infty) = \inf_{u\le c} \inf_{\delta\in I_-} \sup_{\delta<t<1} k(t)t^{-u} = \inf_{u\le c} \varlimsup_{t\to 1-} k(t)t^{-u} = \inf_{u\le c} \varlimsup_{t\to 1-} k(t) = \varlimsup_{t\to 1-} k(t).$$

This finishes the proof.

Examples

We shall describe several properties of the function K which can be represented by the properties of the function k. In each but the last case the property in question will be weaker than the previous one. For each property will be given a separate example of K, and the corresponding k, $\underset{\sim}{\kappa}$, $\underline{\kappa}$, $\bar{\kappa}$, $\tilde{\kappa}$, φ and ψ. Finally, we shall mention a few more examples of K which determine certain types of O-regular variation.

Example 0. $K = R$. In this case we have

$$k(t) = r(t) = t^\rho \text{ for all } t \in I, \quad \underset{\sim}{\kappa} = \underline{\kappa} = \rho = \bar{\kappa} = \tilde{\kappa},$$
$$\varphi(u) = 1 \text{ for } u \le \rho \quad \text{and} \quad \psi(u) = 1 \text{ for } u \ge \rho.$$

Example 1. $\underset{\sim}{\kappa} = \underline{\kappa}$ and $\bar{\kappa} = \tilde{\kappa}$. In this case we have

$$k(t) = t^{\underline{\kappa}} \vee t^{\bar{\kappa}} \text{ for all } t \in I,$$
$$\varphi(u) = 1 \text{ for } u \le \underline{\kappa} \quad \text{and} \quad \psi(u) = 1 \text{ for } u \ge \bar{\kappa}.$$

If σ and τ are real numbers, $\sigma < \tau$ and $k(t) = t^\sigma \vee t^\tau$, then $\underset{\sim}{\kappa} = \underline{\kappa} = \sigma$ and $\bar{\kappa} = \tilde{\kappa} = \tau$.

LEMMA 2. *Suppose f and g are real functions defined in a neighborhood of the positive infinity in* $\mathbf{R}$, *and $g(\infty) = 0$. Then*

$$\sin(f + g + o(g)) = \sin f + g \cos f + o(g),$$

where $o(g)$ denotes $o(g(x))$ as $x \to \infty$.

Proof. $\sin(f + g + o(g)) = \sin f \cos(g + o(g)) + \cos f \sin(g + o(g)) = (\sin f) \cdot (1 + o(g)) + (\cos f) \cdot (g + o(g)) = \sin f + g \cos f + o(g)$. □

Example 1A. $K_1(x) = x^{(1/\sqrt{2}) \sin \log \log x}$.

First observe that $\sqrt{2} \log K_1(x) = \log x \cdot \sin \log \log x$. For $x \in I$ and $t \in I$ taking substitution $y = \log x$ and $u = \log t$, we obtain $\log tx = u + y = y(1 + u/y)$, $\log \log tx = \log y + u/y + o(u/y)$ as $y \to \infty$, $\sqrt{2} \log K_1(x) = y \sin \log y$,

$$\begin{aligned}\sqrt{2} \log K_1(tx) &= (u + y) \sin(\log y + u/y + o(u/y)) \quad \text{as } y \to \infty \\ &= (u + y)(\sin \log y + (u/y) \cos \log y + o(u/y)) \\ &= u \sin \log y + y \sin \log y + u \cos \log y + o(1) \\ &= \sqrt{2} \log K_1(x) + \sqrt{2} u \sin(\pi/4 + \log y) + o(1), \quad \text{hence}\end{aligned}$$

$$\log \frac{K_1(tx)}{K_1(x)} = \log t \cdot \sin\left(\frac{\pi}{4} + \log \log x\right) + o(1) \quad \text{as } x \to \infty. \tag{15}$$

Applying the operation $\overline{\lim}_{x \to \infty}$ to (15), we obtain

$$\log k_1(t) = |\log t| = \log t^{-1} \vee \log t = \log(t^{-1} \vee t),$$

and therefore $k_1(t) = t^{-1} \vee t$, $\underset{\sim}{\kappa}_1 = \underline{\kappa}_1 = -1$ and $\bar{\kappa}_1 = \tilde{\kappa}_1 = 1$.

If σ and τ are real numbers, $\sigma \leq \tau$, then the function $K(x) = x^{(\tau+\sigma)/2} K_1^{(\tau-\sigma)/2}(x)$ has the index function $k(t) = t^{(\tau+\sigma)/2}(t^{-1} \vee t)^{(\tau-\sigma)/2} = t^\sigma \vee t^\tau$, thus $\underset{\sim}{\kappa} = \underline{\kappa} = \sigma$ and $\bar{\kappa} = \tilde{\kappa} = \tau$.

Example 2. $\underset{\sim}{\kappa} > -\infty$ and $\tilde{\kappa} < \infty$. Functions K with this property are called *extended regularly varying functions*.

PROPOSITION 2. *Let $c \in I$, $h : \mathbf{R} \to \mathbf{R}$ be mapping with the period c, decreasing on the interval $]0, c[$, $\int_0^c |h| < \infty$, $H(y) = \int_0^y h$ for $y \in \mathbf{R}$, and $\chi(u) = \overline{\lim}_{y\to\infty}(H(u+y) - H(y))$ for $u \in \mathbf{R}$. Then $\chi(u) = H(u)$ for all $u \in \mathbf{R}$,*

$$\lim_{u\to 0-} \frac{\chi(u)}{u} = h(c-), \qquad \lim_{u\to 0+} \frac{\chi(u)}{u} = h(0+)$$
$$\lim_{u\to -\infty} \frac{\chi(u)}{u} = \lim_{u\to\infty} \frac{\chi(u)}{u} = \frac{1}{c} H(c) = \frac{1}{c}\int_0^c h.$$

Proof. First observe that

$$H(y+c) = \int_0^y h + \int_y^{y+c} h = H(y) + \int_0^c h = H(y) + H(c),$$

and thus $H(y + cn) = H(y) + nH(c)$ for all $y \in \mathbf{R}$ and $n \in \mathbf{Z}$. In particular,

$$H(y) = H(y - c[y/c] + c[y/c]) = H(y - c[y/c]) + [y/c]H(c),$$

in which case $0 \leq y - c[y/c] < c$.

The function $y \mapsto H(u+y) - H(y)$, $u \in \mathbf{R}$, has the period c; therefore $\chi(u) = \sup_{0 \leq y < c}(H(u+y) - H(y))$.

Since $H(u + c + y) - H(y) = H(u+y) - H(y) + H(c)$, we have $\chi(u + c) = \chi(u) + H(c)$, so the function $\chi - H$ has the period c. We shall prove $\chi - H = 0$ on $[0, c[$, and therefore $\chi - H = 0$ on $\mathbf{R}$. So suppose $0 \leq u < c$ and $0 \leq y < c$. First we have $H(u+y) - H(y) = \int_y^{u+y} h$. If $u + y \leq c$ then

$$\int_y^{u+y} h = \int_0^u h(v+y)\,dv \leq \int_0^u h(v)\,dv = H(u)$$

since h is decreasing on $]0, c[$. If $c < u + y$ then $\int_y^{u+y} h = \int_y^c h + \int_c^{u+y} h$, therefore from

$$\int_c^{u+y} h = \int_0^{u+y-c} h(v+c)\,dv = \int_0^{u+y-c} h(v)\,dv \quad (h \text{ has the period } c)$$

and

$$\int_y^c h = \int_{y+u-c}^u h(v-u+c)\,dv \leq \int_{u+v-c}^u h(v)\,dv \quad (h \text{ is decreasing on }]0,c[),$$

we infer $\int_y^{u+y} h \le \int_0^u h = H(u)$. Hence we proved $\chi(u) \le H(u)$. On the other hand, $\chi(u) \ge H(u+0) - H(0) = H(u)$.

From $\chi(u) = H(u) = [u/c]H(c) + H(u - c[u/c]) = [u/c]H(c) + O(1)$ on $\mathbf{R}$ it follows that

$$\lim_{u\to-\infty} \frac{\chi(u)}{u} = \lim_{u\to\infty} \frac{\chi(u)}{u} = \frac{1}{c}H(c).$$

For $0 < u < c$ we also have

$$h(u) \le \frac{1}{u}\int_0^u h(v)\,dv \le h(0+), \qquad h(c-) \le \frac{1}{u}\int_{-u}^0 h(c+v)\,dv \le h(c-u),$$

and

$$\frac{1}{u}\int_{-u}^c h(c+v)\,dv = \frac{1}{-u}\int_0^{-u} h(v)\,dv = \frac{\chi(-u)}{-u},$$

and these relations imply

$$\lim_{u\to0+} \frac{\chi(u)}{u} = \sup_{0<u<c} \frac{\chi(u)}{u} = h(0+), \qquad \lim_{u\to0-} \frac{\chi(u)}{u} = \inf_{0<u<c} \frac{\chi(u)}{u} = h(c-). \quad \square$$

Example 2A. $K_2(x) = \exp H(\log x)$, where H is the function from Proposition 2. We have $k_2 = K_2$, $\underset{\sim}{\kappa}_2 = h(c-)$, $\underset{\sim}{\kappa}_2 = c^{-1}H(c) = \bar{\kappa}_2$, $\tilde{\kappa}_2 = h(0+)$, $K_2(px) = K_2(p)K_2(x)$ for $p = e^c$ and all $x \in I$.

The function K_2 is extended regularly varying if and only if the function h is bounded.

If θ, σ, τ are real numbers, $0 < \theta < 1$, $\sigma < \tau$, $c = 1$, and if $h(y) = \begin{cases} \tau & \text{for } 0 \le y < \theta \\ \sigma & \text{for } \theta \le y < 1 \end{cases}$, then the function K_2 has the indices: $\underset{\sim}{\kappa}_2 = \sigma$, $\underset{\sim}{\kappa}_2 = \theta\tau + (1-\theta)\sigma = \bar{\kappa}_2$, $\tilde{\kappa}_2 = \tau$.

LEMMA 3. *Let $\alpha \in \mathbf{R}$, and $\beta \in \mathbf{R} \setminus \pi\mathbf{Q}$. Then the accumulation points of the set $e^{i(\alpha+\beta\mathbf{N})}$ in $\mathbf{C}$ form the unit circle $\mathbf{U} := \{z \in \mathbf{C} \mid |z| = 1\}$.*

Proof. The set $\beta\mathbf{Z} + 2\pi\mathbf{Z}$ is dense in $\mathbf{R}$ as nonmonogenic subgroup of the additive group of reals $\mathbf{R}$.

If σ and τ are real numbers and $0 < \tau - \sigma < 2\pi$, then in the interval $J :=]\sigma, \tau[$ there are infinitely many elements of the set $\beta\mathbf{Z} + 2\pi\mathbf{Z}$, therefore there are infinitely many points of the set $e^{i(\beta\mathbf{Z}+2\pi\mathbf{Z})} = e^{i\beta\mathbf{Z}}$ in the set e^{iJ} (observe that the map $t \mapsto e^{it}$ of $\mathbf{R}$ into $\mathbf{U}$ is 1-1 into J). So $(e^{i\beta\mathbf{Z}})' = \mathbf{U}$, where X' denotes the set of accumulation points of $X \subseteq \mathbf{C}$. Further, suppose $A = e^{i\beta\mathbf{N}}$ and $B = e^{-i\beta\mathbf{N}}$. Then $B = 1/A$, thus $B' = 1/A'$, hence $A' \cup B' = (A \cup B)' = \mathbf{U}$ and $1 \in A' \cap B'$ (if $1 \in A'$, then $1/1 \in 1/A' = B'$). The mapping $z \mapsto e^{i\alpha}z$ of $\mathbf{U}$ into itself is a homeomorphism, therefore $(e^{i(\alpha+\beta\mathbf{N})})' = (e^{i\alpha}A)' = e^{i\alpha} \cdot A'$ ($= \mathbf{U}$ when $A' = \mathbf{U}$). In particular, for $j \in \mathbf{N}$ and $\alpha = \beta j$ we have $e^{i\beta j} \cdot A' = (e^{i\beta(j+\mathbf{N})})' = (e^{i\beta\mathbf{N}})' = A' \ni 1$, so $e^{-i\alpha j} \in A'$, and this implies $B \subseteq A'$, then $B' \subseteq A'$, and finally $\mathbf{U} = A' \cup B' = A'$. $\square$

Example 2B. Let K_1 be the function from Example 1A, K_2 the function from Example 2A, $\mu \ge 0$, $\nu \ge 0$, and take $K = RK_1^\mu K_2^\nu$.

Then

$$\log\frac{K(tx)}{K(x)} = \log t^{\rho} + o(1) + \mu\log t\sin\left(\frac{\pi}{4} + \log\log x\right) + o(1) + \nu\log\frac{K_2(tx)}{K_2(x)}$$

as $x \to \infty$. Applying the operation $\overline{\lim}_{x\to\infty}$, we obtain $\log k(t) = \log t^{\rho} + f(t)$, where

$$\begin{aligned} f(t) &:= \overline{\lim_{x\to\infty}}\left(\mu\log t\sin\left(\frac{\pi}{4} + \log\log x\right) + \nu\log\frac{K_2(tx)}{K_2(x)}\right) \\ &\leq \mu|\log t| + \nu\log k_2(t) = \log k_1^{\mu}(t)k_2^{\nu}(t) \end{aligned}$$

(according to the subadditivity of $\overline{\lim}_{x\to\infty}$). On the other hand, for $n \in \mathbf{N}^*$ and $x = e^{cn}$ we have $\log x = cn$ and $\log\bigl(K_2(tx)/K_2(x)\bigr) = \log K_2(t) = \log k_2(t)$, so

$$f(t) \geq \mu\overline{\lim_{n\to\infty}}\log t\sin\left(\frac{\pi}{4} + \log cn\right) + \log k_2^{\nu}(t) = \mu|\log t| + \log k_2^{\nu}(t);$$

by Lemma 3 the set $\sin(\pi/4 + \log c\mathbf{N}^*) \supseteq \sin(\pi/4 + \log c + (\log j)\mathbf{N}^*)$ for $j = 2, 3$ is dense in the interval $[-1, 1]$. The final conclusion is that $k = rk_1^{\mu}k_2^{\nu}$ and this equality implies

$$\begin{aligned} \underset{\sim}{\kappa} &= \rho - \mu + \nu h(c-), & \underline{\kappa} &= \rho - \mu + \nu c^{-1}H(c), \\ \tilde{\kappa} &= \rho + \mu + \nu h(0+), & \bar{\kappa} &= \rho + \mu + \nu c^{-1}H(c). \end{aligned}$$

Remark. Let $\alpha \leq \beta \leq \gamma \leq \delta$ be an increasing sequence in $\mathbf{R}$. To the question "Is there a K with the property index $K = (\alpha, \beta, \gamma, \delta)$?" we have a positive answer in cases: $\alpha = \beta = \gamma = \delta$ (Example 0), $\alpha = \beta < \gamma = \delta$ (Example 1A), $\alpha < \beta = \gamma < \delta$ (Example 2A), $\alpha < \beta < \gamma < \delta$ (Example 2B with $\rho = (\gamma + \beta)/2$, $\mu = (\gamma - \beta)/2$, $\nu = 1$, $c = 1$

$$h(y) = \begin{cases} \delta - \gamma & \text{for } 0 \leq y < (\delta - \gamma)/(\beta - \alpha + \delta - \gamma) \\ \alpha - \beta & \text{for } (\delta - \gamma)/(\beta - \alpha + \delta - \gamma) \leq y < 1). \end{cases}$$

Other cases are: $\alpha = \beta = \gamma < \delta$ or $\alpha < \beta = \gamma = \delta$ and $\alpha = \beta < \gamma < \delta$ or $\alpha < \beta < \gamma = \delta$.

Example 3. The function k is continuous.

PROPOSITION 3. *Suppose k is continuous. Then*

$$\underset{\sim}{\kappa} = \lim_{t\to 1-}\frac{\log k(t)}{t} = k'_{-}(1) \quad \textit{and} \quad \tilde{\kappa} = \lim_{t\to 1+}\frac{\log k(t)}{\log t} = k'_{+}(1).$$

Proof. We shall prove the second statement (the first one is then obtained by taking $1/K$ instead of K).

Assume $t > 1$ and $M_n = \sup\{k(s) \mid 1 < s \leq t^{1/n}\}$ for $n = 1, 2, \ldots$. By our assumption then $\lim_{n\to\infty} M_n = 1$. Let us assign a natural number $j \geq n$ to each

$1 < s \le t^{1/n}$ so that $s^j < t \le s^{j+1}$. Then we have $1 < \theta := ts^{-j} \le s \le t^{1/n}$, and hence

$$k(t) = k(s^j\theta) \le k^j(s)k(\theta) \le k^j(s)M_n,$$
$$\log k(t) \le j\log k(s) + \log M_n, \qquad \log t \ge j\log s.$$

Therefore,

$$\frac{\log k(t)}{\log t} \le \frac{\log k(s)}{\log s} + \frac{\log M_n}{\log t},$$

so using operations $\inf_{1<s\le t^{1/n}}$ and $\lim_{n\to\infty}$ we obtain $\tilde{\kappa} \le \underline{\lim}_{s\to 1+} \frac{\log k(s)}{\log s}$. On the other hand, we have $\tilde{\kappa} = \overline{\lim}_{s\to 1+} \frac{\log k(s)}{\log s}$, by the Proposition 1(c). Since $\log t \sim t-1$ as $t \to 1$, we have

$$\lim_{t\to 1+} \frac{\log k(t)}{\log t} = \lim_{t\to 1+} \frac{\log k(t)}{t-1} = (\log k)'_+(1) = k'_+(1) \quad \text{as } k(1) = 1. \quad \square$$

Some examples of function K with continuous k can be obtained from Example 2A, as we now exhibit.

If $c = 1$ and $h(y) = 1/\sqrt{y}$ for $0 < y < 1$, then $h(0+) = \infty$, $h(1-) = 1$, $H(y) = \int_0^y h = 2\sqrt{y}$ for $0 \le y \le 1$, $H(1) = 2$ and $H(y) = -2[y] + 2\sqrt{y-[y]}$ for $y \in \mathbf{R}$, therefore the function $K_2(x) = \exp H(\log x)$ has the properties $k_2 = K_2$, $\underset{\sim}{\kappa}_2 = 1$, $\underline{\kappa}_2 = 2 = \bar{\kappa}_2$ and $\tilde{\kappa}_2 = \infty$. For the function $K = RK_1^\mu K_2^\nu$ with real $\mu \ge 0$ and $\nu > 0$ we have $k = rk_1^\mu k_2^\nu$, $\underset{\sim}{\kappa} = \rho - \mu + \nu$, $\underline{\kappa} = \rho - \mu + 2\nu$, $\bar{\kappa} = \rho + \mu + 2\nu$, $\tilde{\kappa} = \infty$, and for real numbers $\alpha < \beta \le \gamma$ the statements "index $K = (\alpha, \beta, \gamma, \infty)$", "$\rho = (4\alpha - 3\beta + \gamma)/2$, $\mu = (-\beta + \gamma)/2$ and $\nu = -\alpha + \beta$" and "index$(1/K) = (-\infty, -\gamma, -\beta, -\alpha)$" are all equivalent.

If $c = 1$ and $h(y) = 1/\sqrt{y} - 1/\sqrt{y-1}$ for $0 < y < 1$, then $h(0+) = \infty$, $h(1-) = -\infty$, $H(y) = 2\sqrt{y} + 2\sqrt{1-y} - 2$ for $0 \le y \le 1$, $H(1) = 0$, $H(y) = 2\Big(\sqrt{y-[y]} + \sqrt{1-y+[y]} - 1\Big)$ for $y \in \mathbf{R}$. Therefore the function $K_2(x) = \exp H(\log x)$ has the properties: $k_2 = K_2$, $\underset{\sim}{\kappa}_2 = -\infty$, $\underline{\kappa}_2 = 0 = \bar{\kappa}_2$ and $\tilde{\kappa}_2 = \infty$, while for the function $K = RK_1^\mu K_2$ with real $\mu \ge 0$ one has $k = rk_1^\mu k_2$, $\underset{\sim}{\kappa} = -\infty$, $\underline{\kappa} = \rho - \mu$, $\bar{\kappa} = \rho + \mu$ and $\tilde{\kappa} = \infty$. Hence, for real $\beta \le \gamma$, index $K = (-\infty, \beta, \gamma, \infty)$ if and only if $\rho = (\beta + \gamma)/2$ and $\mu = (-\beta + \gamma)/2$.

One more example of this kind is the function $K(x) = 1 + |\sin\log x|$ for which $k = K$ and index $K(-\infty, 0, 0, \infty)$. This can be easily seen using Rule of computing 2.

Remark. The O-regular variation story could be finished with the case "k is continuous". However, this restriction would exclude the function $K(x) = \exp(\log x - [\log x])$ for which, by Rule of computing 2, $k = K$ and so index $K = (-\infty, 0, 0, 1)$, $\varphi(u) = e$ for $u \le 0$, $\psi(u) = \max(e^{1-u}, 1)$ for $u \ge 0$.

Other interesting types of O-regular variation would be also excluded. Some examples are "the function K is increasing" [Matuszewska 1964] and "the function k

is bounded on I_+" [Seneta 1976]. The reader can find other examples in [Bingham, Goldie, Teugels 1987] and [Geluk, de Haan 1987].

Example 4. The function k is nonmeasurable.

Example 4A. Let $A \subseteq \mathbf{R}$ be a measurable set which has the following properties: $-A = A$, $A \cap (A+2) = \varnothing$, $A+4 = A$ and $A - A := \{x - y \mid x \in A \text{ and } y \in A\}$ is nonmeasurable [Rubel 1963], and let $G(y) = \varphi_A(y) - \varphi_{A+2}(y)$ for all $y \in \mathbf{R}$. The function G is periodic and takes only finitely many values, $G(\mathbf{R}) = \{-1, 0, 1\}$, so

$$g(u) := \varlimsup_{y\to\infty} (G(u+y) - G(y)) = \max_{y\in\mathbf{R}} (G(u+y) - G(y)).$$

The function g is nonmeasurable since the set $\{u \in \mathbf{R} \mid g(u) = 2\}$ is nonmeasurable (observe that for $u \in \mathbf{R}$ the following conditions are equivalent: "$g(u) = 2$", "$G(u+y) = 1$ and $G(y) = -1$ for some y". "$u + y \in A$ and $y \in A + 2$ for some y". "$u \in A - A + 2$"). Besides that, the set $g(\mathbf{R}) \subseteq G(\mathbf{R}) - G(\mathbf{R})$ is finite.

The function $K_4(x) := \exp G(\log x)$ is O-regularly varying. The function $k_4(t) = \exp g(\log t)$ is nonmeasurable; it is bounded, hence $\underline{\kappa}_4 = 0 = \bar{\kappa}_4$. The function φ_4 is continuous on the interval $]-\infty, 0]$ and it takes on it finitely many values, so $\varphi_4(u) = \varphi_4(0) = e^2$ for all $u \leq 0$, therefore $\underline{\kappa}_4 = -\infty$. Since $\check{k} = k$, it follows that $\psi_4(u) = \varphi_4(-u) = e^2$ for $u \geq 0$, thus $\tilde{\kappa}_4 = \infty$.

The uniform convergence

Let $T \subseteq I$ be a nonempty compact set. Then

$$\infty > \varlimsup_{x\to\infty} \sup_{t\in T} \frac{K(tx)}{K(x)} = \lim_{u\to\infty} \sup_{x\geq u} \sup_{t\in T} \frac{K(tx)}{K(x)} = \lim_{u\to\infty} \sup_{t\in T} \sup_{x\geq u} \frac{K(tx)}{K(x)}$$

(The Uniform Convergence Theorem). On the other side, $k(t) = \lim_{u\to\infty} \sup_{x\geq u} \frac{K(tx)}{K(x)}$ (by the definition of k), thus one can ask if there is a finer form of the Uniform Convergence Theorem:

$$\lim_{u\to\infty} \sup_{t\in T} \left(\sup_{x\geq u} \frac{K(tx)}{K(x)} - k(t) \right) = 0.$$

The fineness in the question can be better recognized if we write the quoted statements in the form "$\sup_{x\geq u} \frac{K(tx)}{K(x)} = o(1)$ as $u \to \infty$ uniformly on $t \in T$", "$\sup_{x\geq u} \frac{K(tx)}{K(x)} = k(t) + o(1)$ as $u \to \infty$ uniformly on $t \in T$".

We shall prove the finer form under the assumption "k is continuous". Such a proof can be also found in [Hille, Phillips 1957], [Trautner 1987], [Geluk, de Haan 1987].

THEOREM 1 (on uniform convergence). *Let $T \subseteq I$ be a nonempty compact set, and suppose k is continuous. Then*

$$\lim_{u\to\infty} \sup_{t\in T} \left(\sup_{x\geq u} \frac{K(tx)}{K(x)} - k(t) \right) = 0.$$

Proof. We shall prove the theorem for the case $(+,+)$. So let $c \in \mathbf{R}$, $F : [c,\infty) \to \mathbf{R}$ is measurable,

$$f(u) := \overline{\lim_{y\to\infty}} (F(u+y) - F(u)) < \infty \qquad \text{for all } u \in \mathbf{R},$$
$$\Phi(u,y) := F(u+y) - F(y) - f(u),$$

$\delta \in I$ and $U = [-\delta, \delta]$, and let f be continuous. We are going to prove $\lim_{n\to\infty} \sup_{u\in U} \sup_{y\geq n} \Phi(u,y) = 0$, which is equivalent to $(\forall\varepsilon > 0)(\exists n \in \mathbf{N})(\forall u \in U)$ $(\forall y \geq n)[\Phi(u,y) < 2\varepsilon]$, since under the operator $\lim_{n\to\infty}$ is a positive decreasing sequence. Assume the contrary:

$$(\exists\varepsilon > 0)(\forall n \in \mathbf{N})(\exists u \in U)(\exists y \geq n)[\Phi(u,y) \geq 2\varepsilon].$$

Choose $\varepsilon > 0$, a sequence $(u_n)_{n\in\mathbf{N}}$ in U and a sequence $(y_n)_{n\in\mathbf{N}}$ in $\mathbf{R}$ with the properties: $y_n \geq n$ (observe that this implies $\lim_{n\to\infty} y_n = \infty$) and $\Phi(u_n, y_n) \geq 2\varepsilon$ for all $n \in \mathbf{N}$. The bounded sequence $(u_n)_{n\in\mathbf{N}}$ has a convergent subsequence, say $(u_{\alpha(n)})_{n\in\mathbf{N}}$. For $n \in \mathbf{N}$ put $v_n := u_{\alpha(n)}$, $z_n := y_{\alpha(n)}$, and $\bar{v} = \lim_{n\to\infty} v_n$. Observe that $\lim_{n\to\infty} z_n = \infty$ and $\Phi(v_n, z_n) \geq 2\varepsilon$ for all $n \in \mathbf{N}$. The function F is measurable by assumption, and f is continuous, thus measurable too, therefore Φ is measurable in its first variable. Hence, the following sets are measurable:

$$A_n := \{w \in U \mid \Phi(w, z_n) \geq \varepsilon\}, \qquad C_n := v_n - B_n.$$
$$B_n := \{w \in U \mid \Phi(v_n, z_n) - \Phi(w, z_n) \geq \varepsilon\},$$

Since $\Phi(w, z_n) + (\Phi(v_n, z_n) - \Phi(w, z_n)) = \Phi(v_n, z_n)$, it follows that $A_n \cup B_n = U$ for all $n \in \mathbf{N}$, and so

$$U = \overline{\lim_{n\to\infty}} A_n \cup \underline{\lim_{n\to\infty}} B_n.$$

Finally, we have 1° $\overline{\lim_{n\to\infty}} A_n \neq \varnothing$ or 2° $\underline{\lim_{n\to\infty}} B_n = U$.

1° If $w \in \overline{\lim_{n\to\infty}} A_n$, then the set $N := \{n \in \mathbf{N} \mid w \in A_n\}$ is infinite, so $\lim_{\mathbf{N}\ni n\to\infty} z_n = \infty$. Applying the operation $\overline{\lim}_{\mathbf{N}\ni n\to\infty}$ to "$\varepsilon \leq \Phi(w, z_n)$ for all $n \in \mathbf{N}$", we obtain

$$\varepsilon \leq \overline{\lim_{N\ni n\to\infty}} (F(w+z_n) - F(z_n)) - f(w) \leq f(w) - f(w) = 0.$$

2° Let $\underline{\lim_{n\to\infty}} B_n = U$. Then $\lim_{n\to\infty} B_n = U$ (since $B_n \subseteq U$ for all n in $\mathbf{N}$), thus $\lim_{n\to\infty} \lambda(B_n) = \lambda(U)$. From

$$(\forall n \in \mathbf{N}) \bigcup_{i\geq n} C_i \subseteq \bigcup_{i\geq n} [v_i - \delta, v_i + \delta] \subseteq \left[\inf_{i\geq n} v_i - \delta, \sup_{i\geq n} v_i + \delta\right]$$

it follows that

$$\bigcap_{n\geq 0} \bigcup_{i\geq n} C_i \subseteq \bigcap_{n\geq 0} \left[\inf_{i\geq n} v_i - \delta, \sup_{i\geq n} v_i + \delta\right]$$

$$= \Big[\sup_{n\geq 0}\inf_{i\geq n} v_i - \delta,\ \inf_{n\geq 0}\sup_{i\geq n} v_i + \delta\Big] = \Big[\varliminf_{n\to\infty} v_n - \delta,\ \varlimsup_{n\to\infty} v_n + \delta\Big]$$

$$= \bar v + U, \qquad \text{i.e.} \quad \varlimsup_{n\to\infty} C_n \subseteq \bar v + U.$$

Since

$$\lambda(U) = \varlimsup_{n\to\infty} \lambda(B_n) = \varlimsup_{n\to\infty} \lambda(C_n) \leq \lambda\Big(\varlimsup_{n\to\infty} C_n\Big) \leq \lambda(\bar v + U) = \lambda(U),$$

we have $\lambda\Big(\varlimsup_{n\to\infty} C_n\Big) = \lambda(\bar v + U)$. We conclude that the set $\varlimsup_{n\to\infty} C_n$ is dense in $\bar v + U$.

If $w \in \varlimsup_{n\to\infty} C_n$, then the set $P := \{n \in \mathbf{N} \mid w \in C_n\}$ is infinite. Further, $v_n - w \in v_n - (v_n - B_n) = B_n$ for all $n \in P$. Applying the operation $\varlimsup_{P\ni n\to\infty}$ to "$\varepsilon \leq \Phi(v_n, z_n) - \Phi(v_n - w, z_n) = F(v_n + z_n) - F(v_n - w + z_n) - f(v_n) + f(v_n - w)$ for all $n \in P$", we obtain $\varepsilon \leq f(w) - f(\bar v) + f(\bar v - w)$. This inequality holds for all $w \in \bar v + U$ since the set $\varlimsup_{n\to\infty} C_n$ is dense in $\bar v + U$ and the function f is continuous. Take $w = \bar v$. Then $\varepsilon \leq f(\bar v) - f(\bar v) + f(0) = 0$.

We conclude that $\varepsilon \leq 0$. This contradicts the assumption $\varepsilon > 0$. □

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