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A CLASSIFICATION OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS BY MEANS OF REGULARLY VARYING FUNCTIONS

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Abstract. Necessary and sufficient conditions are obtained such that all (positive) decreasing solutions of the equation (E) are slowly or regularly or rapidly varying functions in the sense of Karamata.

1. Introduction. The asymptotic integration of the equation

$$y'' - f(x)y = 0 \tag{E}$$

has been studied in a very large number of papers (cf. e.g. [**7**, Chs. X and XI] or [**5**, Ch. IV]). In that, very different hypotheses on $f(x)$ are used to obtain results which vary not only in methods of proof but also in the statements pertinent to the form and the rate of increase (decrease) of the relevant asymptotic solutions. These hypotheses (in terms of analysis) run from certain smoothness properties of $f(x)$ (e.g. concerning $f''(x)$) to integrability. As an illustration we quote a few known results (formulated here for positive $f(x)$ and for decreasing solutions only).

1. [**5**, Ch. IV, Th. 14]. If

$$\int_a^\infty |f^{-3/2}f''|\,dt < \infty$$

then (E) has a solution $y(x)$ such that for $x \to \infty$

$$y(x) \sim f^{-1/4}(x)\exp\Big(-\int_a^x f^{1/2}\,dt\Big), \quad -y'(x) \sim f^{1/4}(x)\exp\Big(-\int_a^x f^{1/2}\,dt\Big).$$

2. [**7**, Ch. XI, Ex. 9.9.b]

If for some $p \in [1,2]$

$$\int_a^\infty t^{2p-1}f^p\,dt < \infty$$

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then (E) has a solution $y(x)$ such that for $x \to \infty$

$$y(x) \sim \exp\left(-\int_a^x tf\,dt\right), \qquad y'(x)/y(x) = o(1/x).$$

3. [**10**, Satz 23]
If for some $\alpha > 0$

$$\int_a^\infty t|f - \alpha/t^2|\,dt < \infty$$

then (E) has a solution $y(x)$ such that for $x \to \infty$

$$y(x) \sim x^\beta, \qquad y' \sim \beta x^{\beta-1}$$

where β is the negative root of the equation $r(r-1) = \alpha$.

4. [**3**, Ch. 6, Th. 12]. Let $f(x) = 1 + \varphi(x)$, $\varphi(x) \to 0$ as $x \to \infty$ and

$$\int_a^\infty \varphi^2\,dt < \infty$$

then (E) has a solution $y(x)$ such that for $x \to \infty$

$$y(x) \sim \exp\left(-x - \frac{1}{2}\int_a^x \varphi\,dt\right).$$

In the present paper we assume that $f(x)$ is positive and continuous on the positive half-axis and give a classification of (e.g. positive, decreasing) solutions of (E) by showing that the set of all such solutions can be split into three disjoint subsets whose elements are functions belonging to classes that strongly differ among each other. This indicates, on one hand, why one has such a variety of hypotheses on $f(x)$ and of statements concerning asymptotics of solutions, but on the other hand, shows that there are only three essentially different groups of these which are mutually exclusive (which is not quite clear from the results quoted above).

Necessary and sufficient conditions for such a trichotomy are expressed in terms of limits, for $x \to \infty$, of

$$x\int_x^\infty f(t)\,dt \qquad \text{or} \qquad x\int_x^{kx} f(t)\,dt, \quad k > 1.$$

The classes to which solutions belong consist of various forms of Karamata's functions as specified below. A good insight into the nature of solutions and many of their properties are then a direct consequence of the general theory of such functions, rather than of their specific asymptotic form. Moreover, this opens a possibility, in two of three mentioned cases, to obtain the asymptotic behaviour of solutions for $f(x)$ not covered by the known results. This will be done in a subsequent paper.

We shall first present definitions of Karamata's functions and the needed properties of these.

Definition 1. [**8**]. A positive measurable function $L(x)$ defined on (a, ∞) is said to be slowly varying (s.v.) at infinity if for all $\lambda > 0$

$$L(\lambda x)/L(x) \to 1, \qquad \text{as } x \to \infty. \tag{1}$$

Definition 2. [**8**]. The function $f(x) = x^\alpha L(x)$, α real, is said to be regularly varying at infinity of index α.

Definition 3. [**2**]. A positive measurable function g defined on (a, ∞) is said to be rapidly varying if for all $\lambda > 1$

$$g(\lambda x)/g(x) \to 0 \quad (\text{or } \infty) \qquad \text{as } x \to \infty. \tag{2}$$

We summarize the properties of the above functions which will be needed in the sequel. (Cf. [4] or [6] or [**12**]):

1° *The representation theorem for slowly varying functions.* A function $L(x)$ is s.v. (at ∞) if and only if it can be written in the form

$$L(x) = c(x) \exp\left(- \int_a^x (\varepsilon(t)/t)\, dt \right) \tag{3}$$

for some $a > 0$, where, as $x \to \infty$, $c(x) \to c > 0$, $\varepsilon(t)$ is measurable and tends to zero. If in particular, in (3), $c(x) = c$, then $L(x)$ is called a normalized s.v.f. and is denoted by $L_0(x)$. For these $\varepsilon(x) = xL'(x)/L(x) \to 0$, as $x \to \infty$. Conversely, given a continuously differentiable function $L(x)$ such that $xL'(x)/L(x) \to 0$, as $x \to \infty$, then $L(x)$ is a normalized s.v.f., i.e. $L(x) = L_0(x)$. This class will show itself in the sequel to play a particular role for solutions of equation (E).

2° For any $\varepsilon > 0$ one has for $x \to \infty$

$$x^\varepsilon L(x) \to \infty, \qquad x^{-\varepsilon} L(x) \to 0.$$

LEMMA 1 [**1**]. *If $L'(x)$ decreases then $xL'(x)/L(x) \to 0$, as $x \to \infty$. In other words: any convex decreasing s.v.f. is a normalized one.*

LEMMA 2 [**6**, Prop. 1.7] *If $f(x) = x^{-\alpha}L(x)$, $\alpha > 0$ and $f(x) = \int_x^\infty \psi(t)\, dt$ with ψ nonincreasing, then $-xf'(x)/f(x) \to -\alpha$, as $x \to \infty$.*

LEMMA 3. *For any rapidly varying function $g(x)$ which tends to zero for $x \to \infty$ and $g'(x)$ increases one has*

$$-xg'(x)/g(x) \to \infty, \qquad \text{as } x \to \infty. \tag{4}$$

Conversely, if a continuously differentiable function $g(x)$ satisfies (4), then it is rapidly varying.

Remark. It follows from the proof of the direct statement that if $g(x)$ is rapidly varying only for a sequence $\{x_i\}$ tending to infinity with i, i.e. if for all $\lambda > 1$, $g(\lambda x_i)/g(x_i) \to 0$, then (4) remains valid for $x = x_i$.

Proof. Since $-g'(x)$ decreases one has for all $\lambda > 1$ and some $\xi \in (x, \lambda x)$

$$-g(\lambda x) + g(x) = -(\lambda - 1)xg'(\xi) \le -(\lambda - 1)xg'(x)$$

and so

$$-xg'(x)/g(x) \ge \{1 - g(\lambda x)/g(x)\}/(\lambda - 1).$$

But $\lambda > 1$ is independent of x and can be chosen arbitrarily close to 1. Hence, due to (2) the statement (4) follows. The converse is obvious: By integrating (4) over (a, x) one gets

$$g(x) = g(a)\exp\left(-\int_a^x (\eta(t)/t)\,dt\right), \qquad \text{where } \eta(x) \to \infty.$$

Hence, for any $\lambda > 1$

$$g(\lambda x)/g(x) = \exp\left(-\int_x^{\lambda x} (\eta(t)/t)\,dt\right) = \exp(-\eta(\xi)\ln\lambda), \qquad x \le \xi \le \lambda x.$$

Consequently $g(\lambda x)/g(x) \to 0$, as $x \to \infty$, and $g(x)$ is rapidly varying according to Definition 3.

We refer to slowly, regularly and rapidly varying functions as to *Karamata's functions*.

2. Results. We prove the following

THEOREM. i) *All (positive) decreasing solutions $y(x)$ of the equation* (E) *are:* a) *slowly varying functions,* b) *regularly varying functions of index $-\alpha = (1 - (1 + 4c)^{1/2})/2$, if and only if, for $x \to \infty$,*

$$\text{(5)} \qquad \text{a) } x\int_x^\infty f(t)\,dt \to 0, \qquad \text{b) } x\int_x^\infty f(t)\,dt \to c, \quad c \in (0, \infty)$$

and the solutions $y(x)$ are of the form $y(x) = L_0(x)$, in case a) *and $y(x) = x^{-\alpha}L_0(x)$ in case* b).

ii) *All (positive) decreasing solutions $y(x)$ of* (E) *are rapidly varying functions if and only if, for all $k > 1$ and $x \to \infty$,*

$$\text{(5')} \qquad x\int_x^{kx} f(t)\,dt \to \infty,$$

where the occurring integral may converge or diverge.

Remark. Conditions (5) and (5′) are in particular fulfilled if $x^2 f(x) \to c$ as $x \to \infty$, $0 \le c \le \infty$. For that case, the sufficiency part of Theorem 1 is proved in [9] (Cf also [**11**]).

Proof. i) a) *Necessity*. Let $y(x) = L(x)$ where $L(x)$ is s.v. Then, since $y(x)$ is convex, $L(x)$ is, due to Lemma 1, the normalized s.v.f., i.e. $L_0(x)$, and hence there holds for $x \to \infty$,

$$\text{(6)} \qquad xL_0'/L_0 \to 0.$$

Also, since $L_0''/L_0 \equiv (L_0'/L_0)' + (L_0'/L_0)^2$, the equation (E) becomes

$$(L_0'/L_0)' + (L_0'/L_0)^2 = f(x),$$

or by integrating over (x, ∞) and multiplying throughout by x,

$$-xL_0'/L_0 + x\int_x^\infty (tL_0'/L_0)^2 t^{-2}\,dt = x\int_x^\infty f(t)\,dt.$$

Now, because of (6), both integrals converge. Moreover the left-hand side and hence the right-hand one of the above identity tends to zero as $x \to \infty$, q.e.d.

Sufficiency. By integrating both sides of (E) over (x, ∞) and since $y(x)$ is decreasing and such that $y'(x) \to 0$, as $x \to \infty$, one has, observing the condition (5), a),

$$-xy'/y = x\int_x^\xi f(t)\,dt = \varepsilon(x)$$

with $\varepsilon(x) \to 0$, as $x \to \infty$. Hence $y(x) = k\exp\left(-\int_a^x (\varepsilon(t)/t)\,dt\right)$ and so, by Representation theorem, $y(x) = L_0(x)$.

i) b) *Necessity*. Assume $y(x) = x^{-\alpha}L(x)$; this implies

$$y'/y + \alpha/x = L'/L, \tag{7}$$

and by Lemma 2 one has $xL'/L \to 0$, as $x \to \infty$, so that $L(x) = L_0(x)$. By using the identity $y''/y \equiv (y'/y)' + (y'/y)^2$ to replace y''/y in (E) and by integrating both sides of the obtained equation over (x, ∞), one has

$$-xy'/y + x\int_x^\infty (ty'/y)^2 t^{-2}\,dt = x\int_x^\infty f(t)\,dt. \tag{8}$$

The convergence of the integrals follows from (7).

Finally, by substituting y'/y from (7) into (8), one obtains for $x \to \infty$, because of (6),

$$\alpha + \alpha^2 + o(1) = x\int_x^\infty f(t)\,dt,$$

which gives condition (5), b) with $\alpha + \alpha^2 = c$.

Sufficiency. We shall show that if (5), b) holds, then for $x \to \infty$

$$-xy'/y \to \alpha, \qquad \text{where } \alpha = ((1+4c)^{1/2} - 1)/2; \tag{9}$$

in other words, $y(x) = x^{-\alpha}L_0(x)$. To that end put into (E)

$$y(x) = k\exp\left(-\int_a^x (\alpha + \eta(t))t^{-1}\,dt\right) \tag{10}$$

where $k > 0$ and the positive real number α is chosen to satisfy

$$\alpha^2 + \alpha = c. \tag{11}$$

By the substitution (10) one concludes that the function $\eta(x) = -xy'/y-\alpha$ satisfies the equation

$$(\alpha+\eta)^2/x^2 - (x\eta' - (\alpha+\eta))/x^2 = f(x). \tag{12}$$

Furthermore, from (E) it follows that

$$0 < -xy'/y \le x \int_x^\infty f(t)\,dt,$$

so that because of (5), b) $\eta(t)$ is also bounded and by definition of $\eta(x)$

$$\alpha + \eta(x) > 0, \tag{13}$$

$y(x)$ being decreasing.

By integrating both sides of (12) over (x,∞), multiplying by x and using the condition (5), b), one obtains

$$x\int_x^\infty (\alpha+\eta)^2 t^{-2}\,dt - x\int_x^\infty (t\eta' - (\alpha+\eta))t^{-2}\,dt = c + \varepsilon(x) \tag{14}$$

where $\varepsilon(x) \to 0$, as $x \to \infty$. All occurring integrals are convergent due to boundedness of $\eta(t)$. By partial integration of the term of (14) containing $\eta'(t)$ and observing (11), the equality (14) is reduced to

$$x\int_x^\infty (2\alpha+\eta)\eta t^{-2}\,dt + \eta(x) = \varepsilon(x). \tag{15}$$

Due to (13), the factor $2\alpha+\eta$ is ultimately positive so that the sign of the left-hand side of (15) depends on the sign of η only. Consequently, one concludes from (15), using the mean-value theorem, that

$$\eta(\xi)(2\alpha+\eta(\xi)) + \eta(x) = \varepsilon(x), \qquad \text{where } \xi \ge x.$$

Therefore, if $\eta(x)$ is ultimately of constant sign, it tends to zero as $x \to \infty$, since $\varepsilon(x)$ does. If, on the other hand, $\eta(x)$ has infinitely many zeros x_i^0, denote by x_i the closest point preceeding x_i^0, $i = 1, 2, \ldots$, in which $\eta(x)$ possesses a maximum, integrate both sides of (12) over (x_i, x_i^0) and then multiply throughout by x_i. This leads to

$$x_i\int_{x_i}^{x_i^0} (\alpha+\eta)^2 t^{-2}\,dt - x_i\int_{x_i}^{x_i^0} (t\eta' - (\alpha+\eta))t^{-2}\,dt = x_i\int_{x_i}^{x_i^0} f(t)\,dt. \tag{16}$$

Now, in the left-hand side of (16) integrate partially the term $\int(\eta'/t)\,dt$ and use the mean value theorem as in the previous case. In addition, write the right-hand side as

$$x_i\int_{x_i}^{x_i^0} f(t)\,dt = c(1/x_i - 1/x_i^0) - \varepsilon(x_i^0)x_i/x_i^0 + \varepsilon(x_i),$$

which is a direct consequence of (5) written as $\int_{x_i}^\infty f(t)\,dt = (c+\varepsilon(x_i))x_i^{-1}$. Thus, observing also (11), the equation (16) is reduced to

$$(2\alpha+\eta(\xi))\eta(\xi)\{1 - x_i/x_i^0\} + \eta(x_i) = \varepsilon(x_i) - \varepsilon(x_i^0)x_i/x_i^0, \qquad x_i \le \xi \le x_i^0.$$

Since the right-hand side of the above equality tends to zero, the same is true for the left-hand one. But all its terms are positive so that the sequence of maxima $\eta(x_i)$ tends to zero (and similarly the sequence of minima). Consequently $\eta(x) \to 0$, as $x \to \infty$. Hence (10) becomes $y(x) = x^{-\alpha}L_0(x)$ with $L_0(x) = k\exp\left(-\int_a^x \eta(t)t^{-1}\,dt\right)$, and $y(x)$ is regularly varying of index $-\alpha$, q.e.d.

ii) *Necessity*. Let $y(x)$ be rapidly varying. First observe that there is no sequence $\{x_i\}$ tending to infinity with i and such that for $x = x_i$ and $x_i \to \infty$,

$$y'(x) \sim y'(\lambda x) \qquad \text{for all } \lambda > 1. \tag{17}$$

Otherwise, because of the continuity and monotonicity of $y'(x)$ and since (17) holds for any fixed λ, it holds also for any $\xi_i \in (x_i, \lambda x_i)$. Then, one can extend $y'(x)$ into a continuous monotone function $\bar{y}'(x)$ defined for all, sufficiently large, values of x such that there holds $\bar{y}'(x) \sim y'(\lambda x)$, as $x \to \infty$ and $\bar{y}'(x) = y'(x)$ for $x \in (x_i, \lambda x_i)$. Hence by Definition 1, $\bar{y}'(x)$ is s.v. and in intervals $(x_i, \lambda x_i)$ coincides with $y'(x)$. But, since $y(x) \to 0$, as $x \to \infty$ and since $-y'(x)$ decreases, one has for $x = x_i$,

$$\begin{aligned} -y'(\lambda x)\lambda x(\lambda - 1)/\lambda \le -(\lambda - 1)y'(\xi)x = -\int_x^{\lambda x} y'(t)\,dt \\ = y(x) - y(\lambda x) = o(1), \quad x \to \infty. \end{aligned}$$

Hence $-y'(\lambda x)\lambda x \to 0$ as $x \to \infty$ which is impossible since $y'(x)$ coincides on $(x_i, \lambda x_i)$ with the slowly varying function $\bar{y}'(x)$ for which there holds $x\bar{y}'(x) \to \infty$, as $x \to \infty$, by virtue of the property 2° of the Introduction.

Further by (E) and since $y(x)$ decreases

$$-y'(x)\{1 - y'(\lambda x)/y'(x)\} \le y(x)\int_x^{\lambda x} f(t)\,dt.$$

Therefore, since (17) cannot hold, there exists a constant $k > 0$ such that

$$-kxy'(x)/y(x) \le x\int_x^{\lambda x} f(t)\,dt.$$

Hence, by the direct part of Lemma 3 the condition (5′) follows.

Sufficiency. We shall show for any sequence $\{x_i\}$ tending to infinity with i, for which (5′) holds, one has for $x_i \to \infty$, $y(x_i)/y(\lambda x_i) \to \infty$ so that $y(x)$ is rapidly varying according to Definition 3.

Consider the function

$$\varphi(x) = -xy'(x)/y(x)$$

which is continuous, positive and defined for all x. Let $\{x_i\}$ be any sequence for which (5′) holds; then none of these is such that $\varphi(x_i) \to 0$, $x_i \to \infty$. If, on the contrary, for $x_i \to \infty$

$$\varphi(x_i) = x_i y'(x_i)/y(x_i) \to 0, \quad \text{or} \quad x_i y'(\lambda x_i)/y(\lambda x_i) \to 0, \qquad \lambda \ge 1 \tag{18}$$

one has for any $\delta > 0$

$$0 < \{y(x_i) - y(x_i + \delta x_i)\}/y(x_i) \to 0$$

and so as $x_i \to \infty$,

$$y(x_i + \delta x_i)/y(x_i) = y(\lambda x_i)/y(x_i) \to 1, \tag{19}$$

But, from (19) and since

$$y'(\lambda x_i) - y'(x_i) \geq y(\lambda x_i) \int_x^{\lambda x_i} f(t)\,dt$$

one obtains, for some $k > 0$

$$-x_i y'(x_i)/y(x_i) \geq k x_i \int_{x_i}^{\lambda x_i} f(t)\,dt.$$

Hence (18) cannot hold since the right-hand side of the above inequality tends to infinity because of (5′). The sequence $\varphi(x_i)$ is defined for all x_i satisfying (5′). Since, by the above consideration, those x_i for which $\varphi(x_i) \to 0$, are excluded, we are left with the two remaining possibilities: $\underline{\lim}\,\varphi_i = m > 0$ and $\overline{\lim}\,\varphi_i \leq \infty$; here $\varphi_i = \varphi(x_i)$.

We next consider the following two separate cases:

(a) For a sequence $\{x_i\}$ there holds for any fixed $\lambda \geq 1$

$$\varphi(x_i) = -x_i y'(\lambda x_i)/y(\lambda x_i) \to \infty$$

(b) $\varphi(x_i) = -x_i y'(\lambda x_i)/y(\lambda x_i) \leq k < \infty$ for some $k > 0$.

In the case a) one has for some $\delta > 0$ and some ξ_i such that $x_i < \xi_i < x_i + \delta x_i$,

$$\begin{aligned}\{y(x_i) - y(x_i + \delta x_i)\}/y(x_i + \delta x_i) &= -\delta x_i y'(\xi_i)/y(x_i + \delta x_i) \\ &\geq -\delta x_i y'(x_i + \delta x_i)/y(x_i + \delta x_i);\end{aligned}$$

hence, because of a),

$$y(x_i)/y(x_i + \delta x_i) - 1 \geq -\delta x_i y'(x_i + \delta x_i)/y(x_i + \delta x_i) \to \infty,$$

and so

$$y(x_i)/y(\lambda x_i) \to \infty \quad \text{as } x_i \to \infty \quad \text{and } \delta + 1 = \lambda$$

In the case b), using also the fact that (17) cannot hold one obtains

$$\begin{aligned}y(\lambda x_i) \int_{x_i}^{\lambda x_i} f(t)\,dt &\leq \int_{x_i}^{\lambda x_i} y(t) f(t)\,dt = \int_{x_i}^{\lambda x_i} y''(t)\,dt \\ &= y'(\lambda x_i) - y'(x_i) \leq -k_1 y'(x_i) \leq k_2 y(x_i)/x_i,\end{aligned}$$

$0 < k_1 \leq 1$, $0 < k_2 < \infty$. Consequently, due to (5′)

$$y(x_i)/y(\lambda x_i) \geq k_3 x_i \int_{x_i}^{\lambda x_i} f(t)\,dt \to \infty, \quad \text{as } x_i \to \infty.$$

To complete the proof suppose that there exists a sequence $\{x_i\}$, $x_i \to \infty$, such that $y(x_i)/y(\lambda x_i)$ does not tend to infinity with x_i. Then there exists a subsequence (denoted again by $\{x_i\}$ for convenience) such that for some $\bar{k} > 0$

$$1 < y(x_i)/y(\lambda x_i) < \bar{k} < \infty. \tag{20}$$

Now, for $\varphi(x_i)$ where the sequence $\{x_i\}$ satisfies (20), one has either, for some $\bar{k} > 0$, $\overline{\lim}\,\varphi(x_i) < \bar{k} < \infty$, or $\overline{\lim}\,\varphi(x_i) = \infty$. Hence, one can find a subsequence $\{x_i'\}$ of the sequence $\{x_i\}$ such that $0 < k_1 < \varphi(x_i') < k_2 < \infty$ or such that $\varphi(x_i') \to \infty$. As shown above, in both cases $y(x_i')/y(\lambda x_i')$ as $x_i \to \infty$, in the first one because of b) and in the second one because of a). But this is impossible because of (20), $\{x_i'\}$ being a subsequence of $\{x_i\}$. Hence (20) cannot hold and so $y(x_i)/y(\lambda x_i) \to \infty$ as $x_i \to \infty$ for any sequence $\{x_i\}$.

REFERENCES

[1] S. Aljančić, R. Bojanić, M. Tomić, *Sur le valeur asymptotique d'une classe d'integrales définies*, Publ. Inst. Math. (Beograd) **7** (1954), 81–94.

[2] A. Békéssy, *Eine Verallgemeinerung der Laplaceschen Methode*, Publ. Math. Hungar. Acad. Sci. **2** (1957), 105–120.

[3] R. Bellman, *Stability Theory of Differential Equations*, Mc. Graw-Hill, New York-Toronto-London, 1953.

[4] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular Variation*, Cambridge Univ. Press, 1987.

[5] W. A. Coppel, *Stability and Asymptotic Behavior of Differential Equations*, D. C. Heath, Boston, 1965.

[6] J. L. Geluk, L. de Haan, *Regular variation, extensions and Tauberian theorems*, VWI Tract 40, Amsterdam, 1987.

[7] P. Hartman, *Ordinary Differential Equations*, J. Wiley & Sons, New York-London-Sidney, 1964.

[8] J. Karamata, *Sur une mode de croissance régulière des functions*, Mathematica (Cluj) **4** (1930), 38–53.

[9] V. Marić, M. Tomić, *A trichotomy of solutions of second order linear differential equations*, Univ. u Novom Sadu, Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **14** (2) (1984), 1–11.

[10] J. Mařik, M. Ráb, *Asymptotische Eigenschaften von Lösungen der Differentialgleichung $y'' = A(x)y$ in nichtoscillatorischen Fall*, Czech. J. Math. **10** (85) (1960), 501–520.

[11] E. Omey, *Regular variation and its applications to second order linear differential equations*. Bull. Soc. Math. Belg. **32** (1981), 207–229.

[12] E. Seneta, *Regularly Varying Functions*, Lecture Notes in Mathematics 508, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

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