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SOME RESULTS ON SUBEXPONENTIAL DISTRIBUTIONS

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Abstract. Subexponential distributions form a subclass within the family of distributions whose tails are functions of regular variation. They are important in queueing, renewal, and extreme value theory as well as in other areas of applied probability. Some applications of the subexponential class $\mathcal{S}$ are discussed and a subadditivity property necessary for class membership is considered. A lemma giving new necessary and sufficient conditions for membership in $\mathcal{S}$ is proved. Some sufficient conditions resulting from the lemma are compared to previously known conditions.

1. Introduction. Let $F(x)$ be a proper distribution on $(0, \infty)$ such that $F(x) < 1$ for $x < \infty$ and

$$\lim_{x\to\infty} \frac{1 - F^{(2)}(x)}{1 - F(x)} = 2.$$

F is said to belong to the subexponential class $\mathcal{S}$. Chistyakov [1964] introduced the class in a paper on branching processes and showed that a necessary condition for $F \in \mathcal{S}$ is that

$$\frac{1 - F(x - y)}{1 - F(x)} \to 1 \quad \text{as } x \to \infty$$

for every fixed y. In other words, the tail of F must be a functon of moderate growth. He also showed that a sufficient condition for $F \in \mathcal{S}$ is that the tail of F be a function of regular variation:

$$1 - F(x) = x^{-\alpha} L(x), \qquad \alpha \geq 0$$

and $L(x)$ is a function of slow growth. Thus one may regard the class $\mathcal{S}$ as an expansion of the class of tail functions of regular variation. The class has proved fruitful in various areas of applied probability. See, for example, Embrechts' [1985] review paper outlining applications of subexponential distributions to ruin, random walk, branching process, and standard renewal theory problems.

In section 2 we consider some problems where the subexponential class arises. In section 3 we discuss some necessary conditions for membership in $\mathcal{S}$ and prove a

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lemma giving new necessary and sufficient conditions for membership. This leads to some new sufficient conditions which are simpler than those of Goldie and Resnick (1988) and of Klüppelberg (1987).

2. Examples involving the class $\mathcal{S}$. The subexponential class $\mathcal{S}$ plays an important role in the theory of transient renewal processes. It is not surprising, therefore, that it also arises in various queueing, risk, and branching theory problems, since these can often be regarded as transient renewal problems.

Let $\{X_i\}_{i\geq 1}$ be a sequence of iid random variables having defective distribution F on $(0,\infty)$; let $F(\infty)=\omega<1$. Define $S_n=\sum_{i=1}^n X_i$ and let $N(t)$ be the integer k such that $S_k\leq t<S_{k+1}$. The renewal function associated with F is $H(t)=\sum_{n=1}^\infty F^{(n)}(t)$. There will be some smallest index J such that $X_J=\infty$, and $\{X_i\}_{i=1}^J$ is called a transient renewal process. Note that $H(\infty)=\omega(1-\omega)^{-1}$. Teugels [1975] proved that

$$\begin{aligned}&F^{-1}(\infty)F(x)\in\mathcal{S}\iff H^{-1}H(x)\in\mathcal{S}\\ &\iff \lim_{x\to\infty}[1-H^{-1}(\infty)H(x)]/[1-F^{-1}(\infty)F(x)]=(1-\omega)^{-1}\end{aligned}\tag{2.1}$$

Teugels' result generalizes a result of Callaert and Cohen [1972] which relates the tails of F and H when the tails are functions of regular variation.

The equivalence relation can be applied, for example, to the GI/G/1 queue. Consider the queue based on independent sequences $\{T_n\}$, $\{V_n\}$ of iid inter-arrival and service times. Let A and B be the distribution functions of the T's and V's. Let $\alpha=E(T_n)$, $\beta=E(V_n)$, and assume that $\alpha>\beta$ so that the queue is stable. Let $X_n=V_n-T_n$ and suppose $P\{X_n>0\}>0$. One is interested in $Q(x)$, the stationary d.f. of a customer's queueing time. The stationary probability a customer must queue is $\pi=1-Q(0+)$. Let $\{Z_n\}_{n=1}^N$ be the positive ladder variables built from the X's. N is the finite number of ladder r.v's observed; $P\{N=n\}=(1-\pi)\pi^n$. It is well known that

$$Q(x)=1-\pi+\sum_{r=1}^{\infty}(1-\pi)\pi^r L^{(r)}(x)\tag{2.2}$$

where $L(x)$ is the d.f. of a Z.

Define the defective distributon $F(x)=\pi L(x)$. Then (2.2) can be rewritten as

$$Q(x)=(1-\pi)[1+H(x)]\tag{2.3}$$

and thus

$$1-Q(x)=\pi[1-H^{-1}(\infty)H(x)].\tag{2.4}$$

Teugels' result implies that

$$1-Q(x)\sim\pi(1-\pi)^{-1}[1-L(x)]\iff L\in\mathcal{S}\iff Q\in\mathcal{S}.\tag{2.5}$$

This strengthens a result of Smith [1972]. Smith [1972] and Pakes [1975] also relate the tail of Q to the tail of $\bar{B}$, where $\bar{B}$ is the so-called integrated tail distribution

$$\bar{B}(x) = \beta^{-1} \int_0^x [1 - \bar{B}(y)]\, dy. \tag{2.6}$$

Pakes shows that $Q \in \mathcal{S}$ if and only if $\bar{B} \in \mathcal{S}$ and that if either is the case, then

$$1 - Q(x) \sim \tau(1-\tau)^{-1}[1 - \bar{B}(x)] \tag{2.7}$$

where $\tau = \beta\alpha^{-1}$ is the traffic intensity.

This result thus encourages the search for conditions on the service time distribution B which ensure that $\bar{B} \in \mathcal{S}$. Klüppelberg [1988] defines a subclass of $\mathcal{S}$ called $\mathcal{S}^*$; a distribution F in $\mathcal{S}^*$ must have a finite mean μ and satisfy

$$\lim_{x\to\infty} \int_0^x \frac{1 - F(x-y)}{1 - F(x)} [1 - F(y)]\, dy = 2\mu. \tag{2.8}$$

$\Bigg($This will happen if

$$\lim_{x\to\infty} \int_0^x \int_0^u \frac{1 - F(x-y)}{1 - F(x)}\, dy\, F(du) = \lim_{x\to\infty} \int_0^x u\, F(du).\Bigg)$$

Klüppelberg shows that if $F \in \mathcal{S}^*$, then $\bar{F} \in \mathcal{S}$.

One can also use Teugels' equivalence result to study transient renewal processes themselves. Given the process $\{X_i\}_{i=1}^J$, $S_n < \infty$ for $n < J$ and $S_J = \infty$. Define the forward delay $\zeta_t = S_{N(t)+1} - t$. For sufficiently large t, $\zeta_t = \infty$ and $N(t) = J - 1$. (The renewals cease.) The properties of ζ_t and $N(t)$ can be studied by conditioning on the event that all lifetimes X_i begun by time t are finite. We say the renewal process is "alive" at time t in this case and use the shorthand $\{A(t) = 1\} = \{S_{N(t)+1} < \infty\}$. Observe that

$$\begin{aligned} P\{A(t) = 1\} \equiv q(t) &= \omega - F(t) + \sum_{n=1}^{\infty} \int_0^t [\omega - F(t-\tau)] F^{(n)}(d\tau) \\ &= (1-\omega)[H(\infty) - H(t)]. \end{aligned} \tag{2.9}$$

Teugels' result relating the tails of H and F is therefore useful in deriving the conditional properties of ζ_t and $N(t)$. For example, Murphree [1987] shows that if $\omega^{-1}F(x) \in \mathcal{S}$ and $u(t) > 0$, then

$$\begin{aligned} &\lim_{t\to\infty} P\{\zeta_t \le u(t) \mid A(t) = 1\} = \lim_{t\to\infty} \int_t^{t+u(t)} \frac{1 - F(t + u(t) - \tau)}{(1-\omega)[H(\infty) - H(t)]}\, H(d\tau) \\ &= \lim_{t\to\infty} \frac{H(t+u(t)) - H(t)}{H(\infty) - H(t)} = 1 - \lim_{t\to\infty} \frac{F(\infty) - F(t+u(t))}{F(\infty) - F(t)}. \end{aligned} \tag{2.10}$$

As a particular example, if $\omega - F(t) \sim \exp(-t^{\beta})$, $0 < \beta < 1$, then

$$\lim_{t\to\infty} P\{\zeta_t \le \lambda t^{1-\beta} \mid A(t) = 1\} = 1 - \exp(-\beta\lambda).$$

She also derives the k^{th} conditional and unconditional moments of $N(t)$ for k a positive integer and shows that

$$\lim_{t\to\infty} E[N(t)^k \mid A(t)=1] = \lim_{t\to\infty} E[N(t)^k]$$

if and only if $\omega^{-1}F(x) \in \mathcal{S}$. The common limit is a k^{th} degree polynomial in $H(\omega)$.

Goldie and Resnick [1988] study the connection between subexponential and extreme value distributions. They actually consider a wider class than $\mathcal{S}$ in which

$$\lim_{x\to\infty} \frac{1-F^{(2)}(x)}{1-F(x)} = d < \infty$$

and do not always restrict themselves to distributions on $(0,\infty)$, but we will concentrate on their results about $\mathcal{S}$. Suppose there exist sequences $a_n > 0$ and $b_n \in \mathbf{R}$ such that

$$\lim_{n\to\infty} P\Big\{\max_{i\le n} X_i \le a_n x + b_n\Big\} = \exp\{-e^{-x}\} \equiv L(x).$$

Then $F \in D(L)$, the domain of attraction of an extreme-value distribution. A representation of such an F is

$$1-F(x) \sim \exp\Big\{-\int_0^x \frac{1}{f(u)}\,du\Big\} \tag{2.11}$$

where $f \ge 0$ is absolutely continuous with density f' satisfying

$$f'(x) \to 0 \quad \text{as } x \to \infty.$$

This representation suggests a connection between $D(L)$ and $\mathcal{S}$ because a necessary condition for $F \in \mathcal{S}$ is that

$$1-F(x) \sim \exp\Big\{-\int_0^x \alpha(u)\,du\Big\} \tag{2.12}$$

where $\alpha(u)$ is a bounded nonnegative function and $\alpha(u) \to 0$ as $u \to \infty$. ((2.14) follows from the fact that $1-F$ is a function of moderate growth.)

Goldie and Resnick consider conditions which imply that $F \in \mathcal{S} \cap D(L)$.

3. Some conditions on $\Lambda(x)$. Define $\Lambda(x) = -\ln[1-F(x)]$. In this section we investigate some implications for Λ if $F \in \mathcal{S}$ as well as some of the common conditions imposed on Λ to ensure $F \in \mathcal{S}$.

Choose $g(x)$ to be any function such that both $g(x)$ and $x - g(x)$ diverge as $x \to \infty$. Then one can easily show that $F \in \mathcal{S}$ is equivalent to

$$\lim_{x\to\infty} \int_0^{g(x)} \frac{1-F(x-y)}{1-F(x)}\,F(dy) = 1; \tag{A}$$

$$\lim_{x\to\infty} \frac{[1-F(x-g(x))]\,[1-F(g(x))]}{1-F(x)} = 0; \quad \text{and} \tag{B}$$

$$\lim_{x\to\infty}\int_0^{x-g(x)} \frac{F(x)-F(x-y)}{1-F(x)}\,F(dy)=0. \tag{C}$$

Taking $g(x) = x/2$ gives a characterization of $\mathcal{S}$ which is used frequently (see Goldie [1978] or Pitman [1980]), for then (C) follows automatically from (A) and there are just two conditions to check. However, it is worth noting that (A), (B), and (C) must hold for any g as described. Of course, once they hold for one such g, they hold for all such g.

If we translate our conditions on F into conditions on Λ, we find that $F \in \mathcal{S}$ if and only if

$$\lim_{x\to\infty}\int_0^{x/2} \exp\{\Lambda(x)-\Lambda(x-y)-\Lambda(y)\}\Lambda(dy)=1; \tag{3.1}$$

and

$$\lim_{x\to\infty}\big(\Lambda(x)-2\Lambda(x/2)\big)=-\infty. \tag{3.2}$$

Necessary conditions are:

$$\Lambda(x)-\Lambda(x-y)\to 0 \quad \text{as } x\to\infty \text{ for each fixed } y \tag{3.3}$$

and

$$\Lambda(x)-\Lambda(x-g(x))-\Lambda(g(x))\to -\infty \quad \text{as } x\to\infty \tag{3.4}$$

for $g(x)$ as described.

Typically, sufficient conditions for $F \in \mathcal{S}$ are derived by assuming (3.3) holds and by strengthening (3.4) considerably. For example, Teugels [1975] assumes that Λ is asymptotically concave; Pitman [1980] supposes that Λ has a derivative $\Lambda'(x)$ which is a decreasing function, at least for sufficiently large x. Well known examples of functions Λ corresponding to subexponential distributions have concave forms:

$$\begin{array}{lll}\Lambda(x)\sim x(\ln x)^{-\beta}, & \beta>0 & \text{(Pitman [1980])}\\ \Lambda(x)\sim cx^{\alpha}, & 0<\alpha<1 & \text{(Teugels [1975])}\\ \Lambda(x)\sim \alpha\ln x, & \alpha>0 & \text{(Chistyakov [1964]).}\end{array}$$

It is not necessary for $\Lambda(x)$ to be concave or even for $\Lambda(x)/x$ to be decreasing. However, if one drops assumptions about concavity, it is important to impose other conditons ensuring the asymptotic subadditivity condition (3.4). In general, conditions slowing the growth rate of $\Lambda(x)$ are not enough to do this. For example, Pakes [1975] claims that if $1-F$ is a function of moderate growth and if

$$\limsup_{x\to\infty}\big(\Lambda(2x)/\Lambda(x)\big)<2, \tag{3.5}$$

then $F \in \mathcal{S}$. This, however, is incorrect, as is shown in Murphree [1989]. Murphree constructs a function $\Lambda(x)$ having both properties assumed by Pakes yet satisfying

$$\Lambda(4^n)+\Lambda(3\cdot 4^n)=\Lambda(4^{n+1}), \qquad n=1,2,\ldots \tag{3.6}$$

Since this violates the subadditivity condition, $F \notin \mathcal{S}$. The function $\Lambda(x)$ in this example is constant throughout the intervals $[4^n, 1.5(4^n)]$ and $[2 \cdot 4^n, 3 \cdot 4^n]$; these long intervals of flatness make Λ insufficiently smooth.

What minimal smoothness conditions on Λ follow from its asymptotic subadditivity? In order to show $F \in \mathcal{S}$, one wants to control the size which differences $\Lambda(x) - \Lambda(x-y)$ can take when $x \geq 2y$. For if

$$\int_0^{x/2} \exp\{\Lambda(x) - \Lambda(x-y) - \Lambda(y)\}\, \Lambda(dy) < C,$$

then dominated convergence will imply the integral has limit

$$\int_0^{\infty} \exp\{-\Lambda(y)\}\, \Lambda(dy) = 1.$$

In searching for conditions on Λ, one prefers that they be only asymptotic conditions since subexponentiality is concerned only with the limiting behavior of tails. One might ask whether the subadditivity requirement (3.4) implies that

$$(g(x)/x)\Lambda(x) - \Lambda(g(x)) \to -\infty. \tag{3.7}$$

If this were true, we could claim that $\dfrac{\Lambda(x+y)}{x+y} < \dfrac{\Lambda(x)}{x}$ whenever x and y are sufficiently large. Unfortunately, this is not the case. Consider the following example:

$$\Lambda(x) = \begin{cases} x^{1/2}, & 4^n \leq x \leq 2 \cdot 4^n \\ 2^{n+1/2} + (1.5 - \sqrt{2})2^{-n}(x - 2 \cdot 4^n), & 2 \cdot 4^n \leq x \leq 3 \cdot 4^n \\ 2^{-(n+1)}x, & 3 \cdot 4^n \leq x \leq 4^{n+1} \end{cases}$$

Notice that $\Lambda(x)$ is increasing, continuous, and that

$$\frac{\Lambda(x+y)}{x+y} \leq \frac{\Lambda(x)}{x} \quad \text{for all } y \geq 0. \tag{3.8}$$

In addition, one can show that

$$\Lambda(2x) - 2\Lambda(x) \leq -(1.5 - \sqrt{2})\sqrt{x}. \tag{3.9}$$

(3.9) is certainly enough to imply that

$$\int_0^{\infty} \exp\left\{\frac{1}{2}[\Lambda(2x) - 2\Lambda(x)]\right\} \Lambda(dx) < \infty. \tag{3.10}$$

Murphree [1989] shows that (3.8), (3.10), and the moderate growth of $1 - F$ are sufficient to imply that $F \in \mathcal{S}$. Hence the function Λ fulfills the subadditivity conditon (3.4) and yet

$$\alpha\Lambda(4^{n+1}) = \Lambda(\alpha \cdot 4^{n+1}) \quad \text{for any } 3/4 \leq \alpha \leq 1. \tag{3.11}$$

Thus (3.7) fails to hold even for $g(x) = \alpha x$, $0 < \alpha < 1$.

We now prove a lemma giving new necessary and sufficient conditions for $F \in \mathcal{S}$. In this lemma, the usual integrability conditon (3.1) is replaced by a summability condition. We begin by noting that if $F \in \mathcal{S}$, we may construct a continuous version of F which also belongs to $\mathcal{S}$. This follows from Teugels' [1975] result that if $F \in \mathcal{S}$ and $1 - F(x) \sim 1 - G(x)$, then $G \in \mathcal{S}$. The continuous version, for example, could agree with F at the integers and be linear in between. Thus with no loss of generality, we assume F is continuous in what follows.

Define $w_j = \Lambda^{-1}(j)$; that is,

$$w_j = \min\{\alpha : \Lambda(\alpha) = j\}, \qquad j = 0, 1, \ldots . \tag{3.12}$$

Since $\Lambda(x)/x \to 0$ as $x \to \infty$, we know $w_j/j \to \infty$ as $j \to \infty$ through the integers. Also let $N_x = [\Lambda(x/2)]$.

LEMMA. *$F \in \mathcal{S}$ if and only if $1 - F(x)$ is a function of moderate growth, $\Lambda(x) - 2\Lambda(x/2) \to -\infty$ as $x \to \infty$, and*

$$\sum_{j=0}^{N_x} \exp\{\Lambda(x) - \Lambda(x - w_j) - j\} \tag{3.13}$$

is bounded.

Proof. We will show that whenever $\Lambda(x) - 2\Lambda(x/2) \to -\infty$, the sum (3.13) is bounded if and only if

$$\int_0^{x/2} \exp\{\Lambda(x) - \Lambda(x-y) - \Lambda(y)\}\, \Lambda(dy)$$

is bounded. Note that $\int_0^{x/2} = \int_0^{w_{N_x}} + \int_{w_{N_x}}^{x/2}$.

But

$$\int_{w_{N_x}}^{x/2} \exp\{\Lambda(x) - \Lambda(x-y) - \Lambda(y)\}\, \Lambda(dy) \leq \exp\{\Lambda(x) - \Lambda(x/2) - N_x\}$$
$$\leq \exp\{\Lambda(x) - 2\Lambda(x/2) + 1\} \to 0.$$

Also,

$$\begin{aligned} \int_0^{w_{N_x}} &= \sum_{j=0}^{N_x - 1} \int_{w_j}^{w_{j+1}} \exp\{\Lambda(x) - \Lambda(x-y) - \Lambda(y)\}\, \Lambda(dy) \\ &\leq \sum_{j=0}^{N_x-1} \exp\{\Lambda(x) - \Lambda(x - w_{j+1}) - \Lambda(w_j)\} \\ &= \sum_{j=0}^{N_x-1} \exp\{\Lambda(x) - \Lambda(x - w_{j+1}) - (j+1) + 1\} \end{aligned}$$

$$\leq e \sum_{j=0}^{N_x} \exp\{\Lambda(x) - \Lambda(x - w_{j+1}) - j\} \tag{3.14}$$

$$\int_0^{w_{N_x}} \geq \sum_{j=0}^{N_x - 1} \exp\{\Lambda(x) - \Lambda(x - w_j) - (j+1)\}. \tag{3.15}$$

Thus if $\Lambda(x) - 2\Lambda(x/2) \to -\infty$, $\int_0^{x/2} \exp\{\Lambda(x) - \Lambda(x-y) - \Lambda(y)\}\, \Lambda(dy)$ is bounded if and only if $\sum_{j=0}^{N_x} \exp\{\Lambda(x) - \Lambda(x - w_j) - j\}$ is. Of course, if the integral is bounded, its limit is one.

The advantage of the summability condition (3.13) is that it depends only on the growth of Λ between $x - w_j$ and x and not with its growth over fixed finite intervals.

Comments. 1) The sum (3.13) is bounded if there exist a finite constant C and a $\delta > 0$ such that

$$\Lambda(x) - \Lambda(x - w_j) \leq j + C - (1 + \delta) \ln j \tag{3.16}$$

whenever $x \geq 2w_j$.

Condition (3.16) holds, for example, when $F \in \mathcal{D}$, the class of dominated-variation distributions. In this case

$$\overline{\lim_{x \to \infty}} \{\Lambda(2x) - \Lambda(x)\} < \infty.$$

Goldie [1978] has shown that whenever $1 - F$ is a function of moderate growth and $F \in \mathcal{D}$, then $F \in \mathcal{S}$. Of course, (3.16) is a much weaker condition than $F \in \mathcal{D}$.

2) Pitman [1980] shows that if $\Lambda(x) = \int_0^x \alpha(u)\, du$ where $\alpha(u)$ is eventually decreasing to 0, then $F \in \mathcal{S}$ if and only if

$$\lim_{x \to \infty} \int_0^x \exp\{y\alpha(x) - \Lambda(y)\}\, \Lambda(dy) = 1. \tag{3.17}$$

If $\alpha(u)$ is eventually decreasing to 0, then with no loss of generality one may assume that $\alpha(u)$ is nondecreasing for all $u \geq 0$. Hence for $w_j \leq w_{N_x} \leq x/2$,

$$\alpha(x) w_j \leq \Lambda(x) - \Lambda(x - w_j) \leq \alpha(x - w_j) w_j \leq \alpha(w_j) w_j$$

and thus

$$\sum_{j=0}^{N_x} \exp\left[\alpha(x) w_j - j\right] \leq (3.13) \leq \sum_{j=0}^{N_x} \exp\left[\alpha(w_j) w_j - j\right]. \tag{3.18}$$

But the left-most sum is bounded if and only if $\int_0^{x/2} \exp[y\alpha(x) - \Lambda(y)]\, \Lambda(dy)$ is, and if this integral over $(0, x/2]$ is bounded, the integral over $(0, x]$ is also. This follows from the fact that

$$\begin{aligned}
&\int_{x/2}^{x} \exp[y\alpha(x) - \Lambda(y)]\,\Lambda(dy) \\
&= \int_0^{x/2} \exp[(u + x/2)\alpha(x) - \Lambda(u + x/2)]\alpha(u + x/2)\,du \\
&= \int_0^{x/2} \exp[u\alpha(x) - \Lambda(u)]\exp[(x/2)\alpha(x) + \Lambda(u) - \Lambda(u + x/2)]\alpha(u + x/2)\,du \\
&\le \int_0^{x/2} \exp[u\alpha(x) - \Lambda(u)]\alpha(u)\,du. \qquad (3.19)
\end{aligned}$$

Similarly, the right-most sum in (3.18) is bounded if and only if

$$\int_0^{x/2} \exp[y\alpha(y) - \Lambda(y)]\,\Lambda(dy)$$

is, and if this integral over $(0, x/2]$ is bounded, the integral over $(0, x]$ is also. Pitman shows that given his conditions on $\alpha(u)$, a sufficient condition for $F \in \mathcal{S}$ is

$$\int_0^{\infty} \exp[y\alpha(y) - \Lambda(y)]\,\Lambda(dy) < \infty. \qquad (3.20)$$

Thus, when $\alpha(u)$ is eventually decreasing, (3.18) shows that the summability of (3.13) is a stronger condition than Pitman's necessary and sufficient condition but weaker than his sufficient condition. An advantage of the lemma and (3.13), of course, is that they do not require $\alpha(u)$ to be eventually decreasing.

3) Goldie and Resnick [1988] and Klüppelberg [1987] have shown that if $1 - F$ is a function of moderate growth and if

$$\overline{\lim_{x\to\infty}}\, \frac{x\Lambda'(x)}{\Lambda(x)} = \rho < 1, \qquad (3.21)$$

then $F \in \mathcal{S}$. Equation (3.21) implies that if y is sufficiently large, then

$$\Lambda(z)/\Lambda(y) \le [z/y]^{\rho}, \qquad z > y. \qquad (3.22)$$

(3.22) ensures that $\Lambda(2x) - 2\Lambda(x) \to -\infty$ and also implies that for sufficiently large j,

$$\begin{aligned}
\Lambda(x) - \Lambda(x - w_j) &\le \Lambda(x - w_j)\left[\left(\frac{x}{x - w_j}\right)^{\rho} - 1\right] \le \frac{\rho\Lambda(x - w_j)w_j}{x - w_j} \\
&= \frac{\rho j w_j}{x - w_j}\cdot\frac{\Lambda(x - w_j)}{\Lambda(w_j)} \le \rho j\left(\frac{w_j}{x - w_j}\right)^{1-\rho} \le \rho j, \qquad x \ge 2w_j. \qquad (3.23)
\end{aligned}$$

Hence condition (3.16) is implied by (3.21).

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