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REGULAR VARIATION IN PROBABILITY THEORY

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Our aim here is to present a survey of the role of Karamata's theory of regular variation in probability theory. We refer for fuller detail to Chapter 8 of Bingham, Goldie & Teugels (1987/89) (hereafter, 'BGT').

1. Weak limit theorems. We begin by discussing the classical core of limit theory in probability, the theory of addition of independent random variables. As we shall see, regular variation plays a crucial role in weak (distributional) limit theorems, though not in strong (almost-sure) results.

We confine attention to the simplest case: let $X, X_1, X_2, \dots$ be independent, identically distributed random variables, with law F and characteristic function φ; write $S_n := \sum_1^n X_k$. The two prototypical results are:

I (Weak Law of Large Numbers: WLLN). If X has mean μ, then

$$S_n/n \to \mu \quad (n \to \infty) \quad \text{in probability.}$$

Recall the standard proof, indicated by the calculation

$$E[\exp\{itS_n/n\}] = \varphi(t/n)^n = \big(1 + i\mu t/n + o(1/n)\big)^n \to e^{i\mu t} \quad (n \to \infty)$$

and an appeal to Lévy's continuity theorem.

II (Central Limit Theorem: CLT). If X has mean μ and variance σ^2, then

$$(S_n - n\mu)/(\sigma\sqrt{n}) \to \Phi \quad (n \to \infty) \quad \text{in distribution,}$$

where Φ denotes the standard Gaussian law.

As above, writing φ_0 for the characteristic function of $X-\mu$, $(S_n - n\mu)/(\sigma\sqrt{n})$ has characteristic function

$$\varphi_0\left(\frac{t}{\sigma\sqrt{n}}\right) = \left(1 - \frac{t^2/2}{n} + o\left(\frac{1}{n}\right)\right)^n \to e^{-t^2/2},$$

AMS Subject Classification (1985): Primary 60-02 the characteristic function of Φ, and we use the continuity theorem again.

One may ask how far these basic results may be generalised. To take I first, one may ask for necessary and sufficient conditions for convergence in probability of S_n/n to μ. Each of the following is such a condition:

(a) $\varphi(t)$ is differentiable at the origin and $\varphi'(0) = i\mu$,

(b) $xP(|X| > x) \to 0$ and $\int_{-x}^{x} y\,dF(y) \to \mu$ $(x \to \infty)$;

see Ehrenfeucht & Fisz (1960), Feller (1971), XVII.2a, VII.7.

One may generalise even further, and ask for necessary and sufficient conditions for convergence in probability to a constant limit when S_n is normalised by a more general sequence a_n. It is convenient for our purposes to postpone discussion of this question to §4 below. For further discussion of the weak law of large numbers, and its relationship to the strong law, we refer to §2 of Bingham (1989).

We turn now to II, and ask in a similar vein for necessary and sufficient conditions for existence of centring constants a_n and norming constants b_n with $(S_n - a_n)/b_n \to \Phi$ $(n \to \infty)$ in distribution; one writes $F \in D(\Phi)$ and says that F belongs to the *domain of attraction* of Φ ('F is attracted to Φ'). Write R_ρ for the class of functions regularly varying in Karamata's sense with index ρ; thus R_0 is the class of slowly varying functions. The first clear glimpse of the crucial role of regular variation in probability theory is provided by the striking answer to the question above: *the truncated variance should be slowly varying*:

$$F \in D(\Phi) \iff \int_{-x}^{x} y^2\,dF(y) \in R_0$$

(BGT, Theorem 8.3.1(i); briefly, 'T8.3.1(i)').

This remarkable result was discovered independently by Khinchin (1935), Feller (1935) and Lévy (1935); the first textbook treatment is Lévy (1937). Unfortunately, its simplicity was masked by a failure to use the language of regular variation, although this had already been provided by Karamata (1930).

Before discussing further the history of $D(\Phi)$, let us generalise the problem by omitting all reference to variances and normality. Suppose that a non-degenerate law G and constants a_n, b_n exist with $(S_n - a_n)/b_n \to G$ $(n \to \infty)$ in distribution. We again say that F is attracted to G, $F \in D(G)$, and call the law G *stable*. Clearly if G is stable, so is any law G^* obtained from it by a change of location and scale: that is, our problem is invariant under affine transformations, and we may work to within 'type'. It turns out that the stable types form a two-parameter family $\{G_{\alpha\beta} : 0 < \alpha \leq 2, -1 \leq \beta \leq 1\}$, where α is called the *index* ($\alpha = 2$ gives normality) and β the skewness parameter. Then (T8.3.1(ii))

$$F \in D(G_{\alpha,\beta}) \quad \text{with } 0 < \alpha < 2 \iff$$

(a) $P(|X| > x) \in R_{-\alpha}$, and

(b) $P(X > x)/P(|X| > x) \to p$, $P(X < -x)/P(|X| > x) \to q$ $(x \to \infty)$, where $(p + q = 1$ and$)$ $\beta = p - q$.

Thus (a) is again a regular variation condition, and (b) is a *tail-balance* condition.

This result was found by Gnedenko (1939) and Doeblin (1940); a text-book synthesis of the cases $\alpha = 2$ and $\alpha < 2$ was given by Gnedenko & Kolmogorov (1949, 1954). Again, no use was made of the language of regular variation, essential to a simple statement of the result.

The realisation of the essential role of regular variation in this context is due to Sakovich (1956). Unfortunately, Sakovich's work seems to have been overlooked; his results were rediscovered by Feller (1966, 1971), where they entered the text-book literature. Feller's influential book did much to popularise the use of regular variation among probabilists.

The above results extend also to the multidimensional case; see Rvacheva (1954), Sakovich (1956).

2. Self-similarity. One may consider, instead of individual random variables S_n, stochastic processes $\{S_{[nt]} : t \geq 0\}$, and processes obtained from them by centring and norming. Such processes have independent increments, and their finite-dimensional distributions converge under the conditions of §1 to those of certain limit processes, the stable processes of P. Lévy; these have independent increments and their one-dimensional distributions are the stable laws above. Convergence to stable processes may be strengthened to weak convergence under a suitable (Skorohod) topology; see Skorohod (1957), or for a modern treatment in a more general setting, Jacod & Shiryayev (1987).

This prompts the search for analogous results in settings more general than that of §1. Suppose we have processes $X = (X_t)_{t>0}$, $Y = (Y_t)_{t>0}$ and centring and norming functions $g(\cdot)$, $f(\cdot)$ such that

$$\left((X_{ut} - g(u))/f(u)\right)_{t>0} \to (Y_t)_{t>0} \qquad (u \to \infty)$$

in the sense of convergence of finite-dimensional distributions. Let us impose suitable mild regularity conditions on Y: it suffices that Y_1 be non-degenerate and that $t \mapsto Y_t$ be right-continuous in law. Then it turns out that the limit process Y replicates itself under a change of scale, to within an affine transformation: there exist a_t, b_t such that

$$(Y_{ut})_{t>0} = (a_u Y_t + b_u)_{t>0} \qquad \forall u > 0,$$

in the sense of equality of finite-dimensional distributions.

This property is called *self-similarity*, and has been much studied in recent years in connection with fractals (Mandelbrot (1977)). The concept was introduced earlier by Lamperti (1962a) under the name 'semi-stability', chosen to reflect the self-replicating property of Lévy's stable processes.

It turns out that regular variation is present implicitly here, just as it was in the more special situation of §1. There exists $\rho \in \mathbf{R}$, called the *index* of Y, with (a) $a_t \equiv t^\rho$, (b) $f \in R_\rho$ (Lamperti (1962a), Vervaat (1981); T8.5.2).

This result accounts for the ubiquitous appearance of regular variation in weak (distributional) limit theorems in probability theory; self-similarity requires that norming functions are regularly varying.

So far as the class of self-similar processes is concerned, the strength of the above result is its great generality: all interesting limit processes are self-similar. The weakness is that the class of such processes is vast — too vast to admit a useful description in this degree of generality. To study self-similarity further, one proceeds by appropriate specialisation — for instance, to the Markov case (Lamperti (1972)). For recent work, see O'Brien & Vervaat (1985), Vervaat (1985), (1986), Taqqu (1986), Eberlein & Taqqu (1986), Kasahara, Maejima & Vervaat (1988).

3. Arc-sine laws. We saw in §2 that limit processes are self-similar, but the class of such processes is too big to allow a parametric description in full generality. But by imposing suitable structure, one may seek to reduce the class of limits to more manageable size. In §1 we considered the classical case of stability: independent increments structure, and a two-parameter limit family. Here we turn to another classical instance with special structure, giving a one-parameter family.

Consider the familiar setting of renewal theory: the X_n are (independent and) positive; X_n is the lifetime of the nth lightbulb used. At time 0, a new bulb is installed and switched on. The light is kept on, bulbs being replaced immediately on failure. At time t, let Y_t denote the length of time the current bulb has been in use. Thus the age-process $Y = ({}_t)_{t>0}$ is a random saw-tooth with slope 1, starting afresh at 0 at each renewal epoch (Lamperti (1962a), Horowitz (1971), (1972)). What can be said about its limiting behaviour?

For $0 < \alpha < 1$, consider the law Q on $[0, 1]$ with density

$$q_\alpha := \begin{cases} \pi^{-1}\sin(\pi\alpha)\, x^{-\alpha}(1-x)^{\alpha-1}, & (0 < x < 1) \\ 0, & \text{else.} \end{cases}$$

As $\alpha \downarrow 0$, Q_α converges weakly to δ_1, and as $\alpha \uparrow 1$, $Q_\alpha \to \delta_0$; we extend to $\alpha \in [0,1]$ by writing Q_0 for δ_1, Q_1 for δ_0. Then Q_α has mean $1-\alpha$, and we call Q_α the *generalised arc-sine law* with parameter α. The case $\alpha = 1/2$ — density $q_{1/2}(x) = 1/\big(\pi x^{1/2}(1-x)^{1/2}\big)$, law $Q_{1/2}(x) = 2\pi^{-1}\arcsin\sqrt{x}$ — arises in the classical arc-sine for coin-tossing; see Feller (1957, 1968), III.4.

The limit theory of Y_t is due to Dynkin (1961), Lamperti (1962b) ('Dynkin-Lamperti theorem': T8.6.3, 8.6.5). It turns out that only normings Y_t/t give non-degenerate limits — thus the fraction of time elapsed in which our current lightbulb has been in use is significant, indicating that our lifetime law F is 'heavy-tailed'. It turns out also that convergence in the weakest reasonable sense (of means) is equivalent to convergence in the strongest sense (weak convergence), and to a condition of regular variation. The following are equivalent:

(i) $Y_t/t \to Q_\alpha \quad (t \to \infty)$ in distribution,

(ii) $E[Y_t/t] \to 1-\alpha$,

(iii) $E[(Y_t/t)^k] \to (-1)^k \binom{\alpha-1}{k}$ $(k = 0, 1, 2, \dots)$,

(iv) $\begin{cases} 1 - F \in R_{-\alpha} & (0 \le \alpha < 1) \\ \int_0^x y\,dF(y) \in R_0 & (\alpha = 1), \end{cases}$,

and the Q_α are the only possible limits. Further, the age-process Y and its limit have a Markov structure, and one may extend the list of equivalences to include

(v) convergence of Markov transition kernels, and

(vi) weak convergence under the Skorohod (J_1) topology; see Lamperti (1962a), Bingham (1973).

We mention in passing an interesting open problem in this area, the 'Dynkin-Lamperti problem' (BGT, §8.6.3). If $1 - F \in R_\alpha$, the renewal function $U(x) := \sum_0^\infty F^{n*}(x)$ satisfies

$$(1 - F(x))U(x) \to \frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)} = \frac{\sin \pi\alpha}{\pi\alpha} \qquad (x \to \infty).$$

Conversely, does convergence of $(1 - F(x))U(x)$ to a (positive, finite) limit c imply regular variation of $1 - F$ as above?

4. Relative stability. Recall that in §1 we dealt with the CLT, II, in full generality, but left our treatment of the weak law, I, less complete.

Suppose first that the X_n are positive. We ask for the condition that there exist a_n with S_n/a_n convergent in probability to a constant limit. We take this limit non-zero (to avoid the possibility that a_n is merely chosen too large), and we may take the limit 1 on absorbing a scale-factor. We say that F is *relatively stable* if there exists (a_n) with $S_n/a_n \to 1$ $(n \to \infty)$ in probability. It turns out that relative stability may be studied conveniently as the case $\alpha = 1$ of the work above: F is relatively stable if and only if $\int_0^x y\,dF(y) \in R_0$, and then $a_n \in R_1$ (Rogozin (1971); T8.8.1).

We note the classical instance of the St. Petersburg game. For each play we toss a fair coin repeatedly; the player wins 2^n (£, say) if the coin first falls heads at the nth trial. Then S_n is the accumulated gain in n plays. One has $S_n/(n \log n/\log 2) \to 1$ $(n \to \infty)$ in probability (see e.g. Feller (1957, 1968), X.4; for almost-sure behaviour see Aaronson (1978)). If we ask what is the fair price of a ticket to play the St. Petersburg game, we must conclude that the price c_n of a ticket for the nth play should vary with n (since $\sum_1^n c_k \sim n \log n/\log 2$). This seems paradoxical, as each play is probabilistically a replica of every other; however, this is an inescapable consequence of the fact that the expectation EX is infinite. This 'St. Petersburg paradox' goes back to the work of Daniel Bernoulli (1700–1782), and illustrates the difficulties confronting the early probabilists in their efforts to formulate a concept of 'moral expectation' (what *should* one pay for a ticket?). For historical background, see e.g. Todhunter (1965), 220–222, Sheynin (1972), Daston (1980), Shoesmith (1983).

One may generalise further by relaxing the condition that F be concentrated on $(0, \infty)$. With relative stability defined as above, the necessary and sufficient condition for it may be shown to be $xP(|X| > x)/\int_{-x}^{x} y\,dF(y) \to 0$ $(x \to \infty)$; see Rogozin (1976), Maller (1978), (1979). (The difficulty associated with the denominator vanishing — e.g. with F symmetric — is only apparent: we cannot have $S_n/a_n \to +1$ in probability for such F.) This may be compared with the condition $F \in D(\Phi)$ in CLT, which may be written, using Karamata's theorem, in the form

$$x^2 P(|X| > x) \Big/ \int_{-x}^{x} y\,dF(y) \to 0 \qquad (x \to \infty)$$

(Feller (1966, 1971), IX.8, XVII.5).

5. Fluctuation theory. The result of §3 ('arc-sine law for renewal theory') may be applied in another apparently unrelated context, that of random walks. With S_n $(= \sum_1^n X_k)$ a random walk, consider the time $N_n := \sum_0^n I(S_k > 0)$ that the walk has spent in the positive half-line by time n. What can be said about its limit behaviour? One has ('arc-sine law for random walks'; Spitzer (1956), T8.8.1)

$$N_n/n \to Q_{1-\rho} \quad (n \to \infty) \qquad \text{in distribution}$$

if and only if 'Spitzer's condition' holds:

$$\frac{1}{n} \sum_{0}^{n} P(S_k > 0) \to \rho \qquad (n \to \infty),$$

and these are the only possible limit laws. For proof, consider L_n, defined as the first time that the maximum value of the random walk for times $k \leq n$ is attained. The link with renewal theory of §3 is that, if 'renewal' is occurrence of a strict, ascending ladder point (strict maximum to date) in the walk, then $L_n = n - Y_n$, as is easy to check. Also, N_n and L_n have the same law ('equivalence principle': Sparre Andersen (1953), T8.9.5), so

$$E(N_n/n) = E(L_n/n) = 1 - E(Y_n/n),$$

and the result follows by §3.

There are other, related links between regular variation and fluctuation-theoretic results of Spitzer type. For instance, if

$$u_n := P(n \text{ is a strict ascending ladder point})$$

((u_n) is the renewal sequence here), one has

$$\sum_{0}^{\infty} u_n s^n = \exp\left\{ \sum_{1}^{\infty} \frac{s^n}{n} P(S_n > 0) \right\}$$

(BGT, Proposition 8.9.8). The right visibly simplifies on logarithmic differentiation, an operation commonly used for picking out an index of regular variation. These ideas lead to an easy proof of another result of Rogozín (1971) (T8.9.12).

6. Reliability theory. We return to the renewal-theoretic setting of §3. Suppose we know that at time t the lightbulb in use is the original one (installed new at time $t = 0$). What can we now say about limiting behaviour at $t \to \infty$? Since now $Y_t = t$, we look forwards in time and consider the 'residual lifetime at great age', to use the terminology of Balkema & de Haan (1974). We note in passing that in the Dynkin-Lamperti theorem of §3, one may consider the 'residual lifetime' Z_t as well as (or instead of) the 'spent lifetime', Y_t, or even the bivariate process (Y_t, Z_t).

One of the striking features of §3 is the equivalence of convergence in law (of Y_t/t, or Z_t/t) with convergence of a single moment, the mean. A similar feature is present here. Writing X for the lifetime of our bulb, one has (Balkema & de Haan (1974), Th. 8a) that the conditional distribution of X/t given $X > t$ converges in law as $t \to \infty$ if and only if a suitable conditional moment $E[(X/t)^\xi \mid X > t]$ converges. The possible limit laws form a one-parameter family $\{\Gamma_\alpha : \alpha > 0\}$ of laws on $(1, \infty)$ given by $\Gamma_\alpha(x) := 1 - x^{-\alpha}$ $(x \geq 0)$, and we need to take $\xi \in (0, \alpha)$. For the proof, observe that the conditional moment is

$$\int_t^\infty \left(\frac{x}{t}\right)^\xi \frac{dF(x)}{1 - F(t)} = 1 + \xi \frac{\int_t^\infty x^{\xi-1}(1 - F(x))\, dx}{x^\xi(1 - F(x))}.$$

By Karamata's theorem, the condition for convergence of the right is regular variation of $1 - F$: $1 - F \in R_{-\alpha}$, say. This is the condition in extreme-value theory for $\max(X_1, \ldots, X_n)$ to have limit law Φ_α (Gnedenko (1943); T8.13.2), and this last is the condition for the required convergence in law (Balkema & de Haan (1974), Th. 4).

It is interesting to compare this with the situation in §3 (with Z_t in place of Y_t). Here there is no renewal-theoretic content: we are conditioning on absence of renewal epochs (no failures). Rather, this is the context of reliability theory, where we condition on no failure to date and examine the limiting behaviour (see e.g. Barlow & Proschan (1975)). In other words, in reliability theory we envisage death on first failure, and in renewal theory, regeneration. This difference explains the more difficult proofs in renewal theory (T8.6.5; P8.6.4): we do not know how many renewals have taken place by time t, and have to sum over all possibilities. It is this summation which brings in the renewal function $U = \sum_0^\infty F^{n^*}$ characteristic of the subject.

Other interesting questions in this area involve comparison of the maximum and the sum of $X_1, \ldots, X_n$; see e.g. BGT, §8.15, Omey & Willekens (1986), Willekens & Resnick (1989).

7. Occupation times. Suppose that X is a Markov process, A a compact set, and one considers the occupation time $T(t) := \int_0^t I(X_u \in A)\, du$ of X in A over $[0, t]$. What can be said about the limiting behaviour of $T(t)$ as $t \to \infty$? Subject to mild regularity conditions, $T(t)/f(t)$ converges in law as $t \to \infty$ if and only if $f(t)$ (and $ET(t)$) are regularly varying. If $ET(\cdot) \in R_\alpha$ $(0 \leq \alpha \leq 1)$, the limit law is the Mittag-Leffler law of index α: the law with Laplace-Stieltjes transform the

Mittag-Leffler function $\sum_0^\infty (-s)^n/\Gamma(1+n\alpha)$. This result is due to Darling & Kac (1957) (T8.11.3); see also Athreya (1986), Aaronson (1986).

The Darling-Kac theorem provides a third instance of a class of self-similar limit processes describable parametrically. The limit process $Y = (Y_t)_{t>0}$ in this case can be identified as the inverse of the stable subordinator X of index α: if $X_t := \inf\{u : Y_u > t\}$, then $(X_t)_{t>0}$ has stationary independent stable increments, non-decreasing paths, and Laplace-Stieltjes transform $E\exp(-sX_t) = \exp(-ts^\alpha)$. For details see Bingham (1971); the link between the Mittag-Leffler and one-sided stable laws is in Feller (1966, 1971), XIII.8.

It is interesting to compare this situation (A compact, Mittag-Leffler limit laws) with that of §5 (A a half-line, arc-sine limit laws). It turns out that for random walks the two theorems can be linked; see Kesten (1968), Bingham & Hawkes (1983) (T8.11.6, 8.11.7).

8. Other areas. The selection of topics here is a subjective one. We mention in addition two important areas we have omitted: branching processes (see particularly the works of Eugene Seneta; BGT §8.12) and extremes (see particularly the works of Laurens de Haan; BGT §8.13).

9. Recent progress. An indication of recent progress in the field may be obtained from the following references. The list below complements the Additional References in the 1989 edition of BGT, and for convenience we label topics by the section numbers there.

§8.3: Stability and domains of attraction: Zolotarev (1986), Hall & Seneta (1988).

§8.6: Renewal theory: Anderson & Athreya (1988a), (1988b).

§8.6.3: The Dynkin-Lamperti condition: Isaac (1988).

§8.6.6: Generalisations: Omey & Willekens (1987).

§8.6.7: Murphree (1987a), (1987b).

§8.9: Fluctuation theory: Doney (1987a), (1987b), (1989).

§8.11: Occupation times: Lee (1989).

§8.12: Branching processes: Vatutin (1983), Yakimiv (1988).

§8.13: Extremes: Leadbetter & Rootzén (1988), Galambos & Obretenov (1987), de Haan & Weissman (1988).

§8.13.6: Rates of convergence: Cohen (1988).

§8.13.7: Statistical estimation of tails: de Haan & Resnick (1980), Lo (1986), Dekkers & de Haan (1989), Dekkers, Einmahl & de Haan (1989).

§8.16.1: Strong convergence: Haeusler & Mason (1987), Révész & Willekens (1987), Mikosch (1988).

A1.5: Multivariate extremes: Hüsler (1986), Takahashi (1987), Norberg (1987).

A4: Subexponentiality: Willekens (1988a), (1988b).

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