PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 48 (62), 1990, 133–168

SOLVABILITY OF OPERATOR EQUATIONS AND PERIODIC SOLUTIONS OF SEMILINEAR HYPERBOLIC EQUATIONS

P. S. Milojević

Abstract. Let A be a closed densely defined linear map in a Banach space with infinite dimensional null space and N be a nonlinear map such that $A - N$ is pseudo A-proper. We present a solvability theory for the equation $Ax - Nx = f$ using only the Brouwer degree theory and the finite dimensional Morse theory. Applications to the problem of existence of weak periodic solutions of (systems of) hyperbolic equations in one and several space variables are given. Both nonresonance as well as resonance problems are considered.

I. Introduction

Let H be a separable real Hilbert space, X be a Banach space density and continuously embedded in H, $A : D(A) \subset H \to H$ be a densely defined closed linear map and N be a nonlinear map from $D(A) \cap X$ into H. We are interested in studying operator equations of the form

$$(1.1) \qquad Ax - Nx = f$$

where $f \in H$ is given.

Equations of this form appear in a variety of situations, and in particular in the theory of ordinary and partial differential equations. For example, they can describe nonlinear elliptic boundary value problems, or problems concerning periodic solutions of semilinear hyperbolic equations, or Hamiltonian systems of ordinary differential equations, etc.

Eq. (1.1) has been studied extensively by various topological as well as variational methods. When A is a Fredholm map of nonnegative index, depending on the nature of a nonlinearity, various degree theories (e.g., for the compact or condensing perturbations of the identity, coincidence degree, etc.) have been used in conjunction with the Liapunov-Schmidt technique (cf. the references). When the null space of A is infinite dimensional, Eq. (1.1) is much harder to study and

AMS Subject Classification (1980): Primary 47H15, 35L70, 35L75

the most often used topological approach is a combination of the Leray-Schauder and coincidence degrees and the monotone operator theory (cf. Brezis-Nirenberg **[Br-Ni-1-2]**, Mawhin **[Ma-1-3]** and the references in there). Essential to this approach is the existence of a compact partial inverse $A^{-1} : R(A) \to H$ of A, where the range $R(A)$ of A is closed. However, if A contains also nonzero eigenvalues of infinity multiplicity, then A^{-1} is not compact and this approach is not suitable. This is the situation that occurs when studying the existence of periodic weak solutions of semilinear wave equations in more than one space variables. We note that, using the monotone operator theory and certain approximation procedure, Amann **[Am-2]** was able to obtain rather general unique solvability results for Eq. (1.1) without requiring the compactness of A^{-1}.

In **[Mi-4]**, we initiated a new approach for the study of Eq. (1.1) (with $\dim\ker A = \infty$) based on a Galerkin type method. This approach requires that $A - N$ is pseudo A-proper w.r.t. a scheme $\Gamma = \{H_n, H_n, P_n\}$ for (X, H), i.e. that the corresponding Galerkin procedure leads to a solution of (1.1). Here $\{H_n\}$ is a sequence of finite dimensional subspaces of $D(A) \cap X$, whose union is dense in both X and H, and $P_n : H \to H_n$ are the orthogonal projections. The only topological tool the method requires is the Brouwer degree theory and, as shown in **[Mi-4-8]**, it is applicable to the situations studied by the above authors as well as to many new ones when neither A^{-1} is compact nor N is of monotone type. Moreover, in studying the existence of nontrivial solutions, we also utilize the finite-dimensional Morse theory.

In this paper we shall present a solvability theory for Eq. (1.1) using the pseudo A-proper mapping approach and give applications to the problem of existence of periodic weak solutions of (systems of) hyperbolic equations in one and more space variables. Both nonresonance as well as resonance cases are considered, i.e. when the nonlinearity N stays away from the spectrum $\sigma(A)$ of A, or interacts with it in some way.

Section II is devoted to the study of nonresonance problems for Eq. (1.1) with nonlinearities N that are asymptotically close to linear maps or are asymptotically $\{B_1, B_2\}$-quasilinear. We prove a number of solvability results for Eq. (1.1) for each f as well as the existence of nontrivial solutions of $Ax + Nx = 0$ when $A + N$ is of gradient type. Resonance problems for Eq. (1.1) are studied in Section IV, using both direct and perturbation methods. The abstract theory is then used in Sections III and V to study the existence of weak periodic solutions of various classes of semilinear hyperbolic equations without resonance as well as in resonance.

II. Semilinear equations without resonance

In this part we shall study nonresonance problems for Eq. (1.1) assuming that $A - N$ is a pseudo A-proper map. We begin by studying such problems involving not necessarily monotone quasibounded nonlinear pertubations of both selfadjoint and nonselfadjoint linear maps with possibly infinite dimensional null space. We continue our study of Eq. (1.1) involving asymptotically $\{B_1, B_2\}$-

quasilinear nonlinearities under the nonresonance conditions introduced by Amann [**Am-2**]. Finally, the existence of nontrivial solutions is studied using Morse theory.

2.1 Semilinear Equations with Quasibounded Nonlinearities. We begin by defining precisely the class of (pseudo) A-proper maps. Let X and Y be separable Banach spaces, $\{X_n\}$ and $\{Y_n\}$ be finite dimensional subspaces of X and Y with $\dim X_n = \dim Y_n$ and $\mathrm{dist}(x, X_n) \to 0$ as $n \to \infty$ for each $x \in X$. If $Q_n : Y \to Y_n$ are linear projections such that $\delta = \max\|Q_n\| < \infty$, then $\Gamma = \{X_n, Y_n, Q_n\}$ is a projection scheme for (X, Y).

Let $D \subset X$ and $T : D \to Y$. Recall [**Pet**]

Definition 2.1 A map $T : D \subset X \to Y$ is A-proper (resp., pseudo A-proper) w.r.t. Γ if $Q_nT : D \cap X_n \to Y_n$ is continuous for each n and, whenever $\{x_{n_k} | x_{n_k} \in D \cap X_{n_k}\}$ is bounded and $Q_{n_k}Tx_{n_k} - Q_{n_k}f \to 0$ as $k \to \infty$ for some $f \in Y$, then some subsequence $x_{n_{k(i)}} \to x$ (resp., there is an $x \in D$) and $Tx = f$.

The pseudo A-properness of $T = A - N$ has been established under various assumptions on A and N in [**Mi-4-7**] and more details with some new examples are given in [**Mi-4,8**]. Throughout the paper we shall assume that $(X, \|\cdot\|_0)$ is a Banach space continuously and densely embedded in a Hilbert space H.

Without the monotonicity assumption on N, we have

PROPOSITION 2.1 *Let X be a reflexive Banach space compactly embedded in a Hilbert space H, $A : D(A) \subset H \to H$ be a closed and densely defined linear map and $N : H \to H$ be a nonlinear continuous map. Then $A - N : D(A) \cap X \subset X \to H$ is pseudo A-proper w.r.t. $\Gamma = \{H_n, H_n, P_n\}$ for (X, H) with $P_nAx = Ax$ on H_n.*

Proof. Let $\{x_{n_k} \in X_{n_k}\}$ be bounded in X and $y_k \equiv P_{n_k}(A - N)x_{n_k} \to f$ in H. Then we may assume that $x_{n_k} \rightharpoonup x$ (weakly) in X, $x_{n_k} \to x$ in H, and $Nx_{n_k} \to Nx$. Hence, $Ax_{n_k} = y_k + P_{n_k}Nx_{n_k} \to f + Nx$. Since A is a closed map, $x \in D(A)$ and $Ax - Nx = f$. □

Let us first study Eq. (1.1) when there is no resonance at infinity. We have

THEOREM 2.1 (cf. [**5, 8**]) *Let $A : D(A) \subset H \to H$ be a linear selfadjoint map and $\Gamma = \{H_n, H_N, P_n\}$ be a projection scheme for (X, H) with $P_nAx = Ax$ on H_n. Let $N : X \to H$ be a nonlinear map such that for some positive constants a, b, c and r and selfadjoint $N_\infty : H \to H$*

(2.1) *If $(A - N_\infty)x = y$ for $y \in H$, then $x \in X$ and $\|x\|_0 \le c\|y\|$.*

(2.2) $\|Nx - N_\infty x\| \le a\|x\| + b$ *for* $\|x\|_0 \ge r$.

(2.3) $0 < a < \min\{|\mu| \mid \mu \in \sigma(A - N_\infty)\}$.

Suppose that either one of the following conditions holds·

(2.4) *$A - N : D(A) \cap X \subset X \to H$ is pseudo A-proper w.r.t. Γ,* (2.5) *X is a reflexive space compactly embedded in H and N is continuous in H*

Then Eq. (1.1) *is solvable for each* $f \in H$.

Proof. In view of Proposition 2.1, $A-N : D(A)\cap X \subset X \to H$ is pseudo A-proper w.r.t. Γ if (2.5) holds. Therefore, it remains to prove the theorem assuming (2.4).

Now, since $A - N_\infty$ is self-adjoint in H, we have that $\min\{|\mu| \mid \mu \in \sigma(A-N_\infty)\} < \|(A-N_\infty)^{-1}\|^{-1}$. Let $f \in H$ be fixed and $H(t,x) = (A-N_\infty)x - t(N-N_\infty)x$ on $[0,1]\times D(A)\cap X$. Then there are $\gamma > 0$, $R \geq r$ and $n \geq n_0$ such that

$$\|P_nH(t,x) - tP_nf\| \geq \gamma \quad \text{for} \quad t \in [0,1],\ x \in \partial B_X(0,R),\ n \geq n_0. \tag{2.6}$$

If not, then there would exist $t_n \in [0,1], t_n \to t_0$, and $x_{n_k} \in H_{n_k}$ such that $\|x_{n_k}\|_0 \to \infty$ and $P_{n_k}H(t_k,x_{n_k}) - t_kP_{n_k}f \to 0$ as $k \to \infty$. Then $z_k \equiv y_{n_k} - t_kP_{n_k}(N-N_\infty)(A-N_\infty)^{-1}y_{n_k} - t_kP_{n_k}f \to 0$, where $y_{n_k} = (A-N_\infty)x_{n_k}$. By (2.1) and (2.2) $\|y_{n_k}\| \geq \|x_{n_k}\|_0/c \to \infty$ and

$$\begin{aligned}\|y_{n_k}\| &\leq \delta\|(N-N_\infty)(A-N_\infty)^{-1}y_{n_k}\| + \delta\|f\| + \|z_k\| \\ &\leq \delta a\|(A-N_\infty)^{-1}\|\,\|y_{n_k}\| + \delta(b+\|f\|) + \|z_k\|.\end{aligned} \tag{2.7}$$

Dividing (2.7) by $\|y_{n_k}\|$ and taking the limit we get $1 \leq \delta a\|(A-N_\infty)^{-1}\|$ in contradiction to (2.3). Hence, (2.6) holds and the Brouwer degree

$$\deg(P_n(A-N), B_X(0,R)\cap X_n, P_nf) = \deg(P_n(A-N_\infty), B_X(0,R)\cap X_n, 0) \neq 0$$

for each $n \geq n_0$. Hence, there exists an $x_n \in B_X(0,R) \cap X_n$, such that $P_n(A-N)x_n = P_nf$ for $n \geq n_0$, and by the pseudo A-properness of $A-N$, there is an $x \in D(A)$ such that $Ax - Nx = f$. □

Remark 2.1 If there are real numbers $\alpha < \beta$ such that $\sigma(A)\cap(\alpha,\beta)$ consists of at most a finite number of eigenvalues, and if $\lambda_k < \lambda_{k+1}$ are some consecutive eigenvalues in (α,β) and $\lambda = (\lambda_{k+1}-\lambda_k)/2$, then (2.3) holds with $N_\infty = \lambda I$ if $\alpha = \gamma = (\lambda_{k+1}-\lambda_k)/2$. Indeed, the spectral gap for $A-\lambda I$ induced by the gap $(\lambda_k,\lambda_{k+1})$ is $(-\gamma,\gamma)$ and therefore $(A-\lambda I)^{-1} : H \to H$ is a bounded selfadjoint map whose spectrum lies in $(-1/\gamma, 1/\gamma)$. Hence, $\|(A-\lambda I)^{-1}\| = 1/\gamma$.

Remark 2.2 If $X = H$ and $N_\infty = \lambda I$ with $\lambda \notin \sigma(A)$, then (2.3) implies (2.1) since $\min\{|\mu| : |\mu \in \sigma(A-\lambda I)\} = \|(A-\lambda I)^{-1}\|^{-1}$ and $\|(A-\lambda I)x\| \geq \|(A-\lambda I)^{-1}\|^{-1}\|x\|$ for all $x \in H$. Moreover, if $R(A)$ is the orthogonal complement of $\ker A$ and X is compactly embedded in H, then it can be shown that (2.1) with $N_\infty = \lambda I$ implies that $\dim\ker A$ is finite.

Regarding condition (2.1), we have the following result useful in applications.

LEMMA 2.1 *Let* $A : D(A) \subset H \to H$ *be selfadjoint with the spectum* $\sigma(A)$ *consisting only of eigenvalues* $\{\lambda_i \mid i \in I\}$ *having no accumulation points and* *each λ_i have a finite multiplicity j_i. Suppose that the corresponding eigenvectors $\{e_{ij} | i \in I, 1 \le j \le j_i\}$ form a complete basis for X and H. Suppose that there is a constant $c_0 > 0$ such that if $Ax = y$ for some $y \in \ker A^{\perp}$, then $x \in X$ and $\|x\|_0 \le c_0 \|y\|$. Then condition (2.1) holds.*

Proof. For $x \in D(A)$ we have

$$x = \sum_i \sum_{j=1}^{j_i} x_{ij} e_{ij} \quad \text{and} \quad Ax = \sum_i \lambda_i \sum_{j=1}^{j_i} x_{ij} e_{ij},$$

where $x_{ij} = (x, e_{ij})$. If $\lambda \notin \sigma(A)$ and $(A - \lambda I)x = y$ for $y \in H$ then

$$(A - \lambda I)x = \sum_i (\lambda_i - \lambda) \sum_{j=1}^{j_i} x_{ij} e_{ij} = \sum_i \sum_{j=1}^{j_i} y_{ij} e_{ij}$$

and therefore

$$x = (A - \lambda I)^{-1} y = \sum_i \frac{1}{(\lambda_i - \lambda)} \sum_{j=1}^{j_i} y_{ij} e_{ij}.$$

Since $\ker(A - \lambda I) = \{0\}$, there is a constant $\alpha > 0$ such that $|\lambda_i - \lambda| \ge \alpha$ for all $i \in I$. If not, then a subsequence $\lambda_{i_k} \to \lambda$ in contradiction to the fact that $\{\lambda_i\}$ has no accumulation points. Since $x_{ij} = y_{ij}/(\lambda_i - \lambda)$, we have that

$$|x_{ij}| \le \alpha^{-1} |y_{ij}| \quad \text{and} \quad \|x\| \le c_1 \|y\|$$

for some $c_1 > 0$ independent of y. Moreover, $x \in X$ and $\|x\|_0 \le c_0 \|y + \lambda x\|$ if $Ax = y + \lambda x \ne 0$. If $y + \lambda x = 0$, then $x = y/\lambda \in \ker A$ and $\|x\|_0 = \|y\|_0/\lambda \le c\|y\|/\lambda$ since $\dim \ker A < \infty$. Hence $\|x\|_0 \le c(\lambda)\|y\|$ with some $c(\lambda) > 0$. □

If A is not selfadjoint, analyzing the proof of Theorem 2.1 we see that the following more general version of it holds.

THEOREM 2.2 *Let $C : X \to H$ be a linear map such that $A - C : D(A) \cap X \to H$ is a bijection and for some positive constants a, b, c, $\delta = \max \|P_n\|$ and r with $ac\delta < 1$:*

$$\|(A - C)^{-1} y\|_0 \le c\|y\| \quad \text{for each} \quad y \in H, \tag{2.8}$$

$$\|Nx - Cx\| \le a\|x\|_0 + b \quad \text{for each} \quad \|x\|_0 \ge r. \tag{2.9}$$

Then the conclusions of Theorem 2.1 are valid.

2.2 Semilinear Equations with $\{B_1, B_2\}$-quasilinear Perturbations. We continue our study of Eq. (1.1) with the so-called asymptotically $\{B_1, B_2\}$-quasilinear nonlinearities N. Such maps have been introduced by Perov [**Per**] and Krasnoselskii-Zabreiko [**Kr-Za**] in their study of the existence of fixed points of compact maps. Let $B_1, B_2 : H \to H$ be selfadjoint maps with $B_1 \le B_2$, i.e. $(B_1 x, x) \le (B_2 x, x)$ for $x \in H$.

Definition 2.2 a) A nonlinear map $K : H \to H$ is $\{B_1, B_2\}$-quasilinear on a set $S \subset H$ if for each $x \in S$ there exists a linear selfadjoint map $B : H \to H$ such that $B_1 \le B \le B_2$ and $Bx = Kx$;

b) A map $N : H \to H$ is said to be asymptotically $\{B_1, B_2\}$-quasilinear if there is a $\{B_1, B_2\}$-quasilinear outside some ball map K such that

$$|N - K| = \limsup_{\|x\| \to \infty} \frac{\|Nx - Kx\|}{\|x\|} < \infty.$$

We do not require that $\{B_1, B_2\}$ is a regular pair as in [**Kr-Za, Per**]. This class of maps is rather large. For example, let $K : H \to H$ have a selfadjoint weak Gateaux derivative N' i.e. $N'(x)$ is a selfadjoint map on H for each x and

$$\lim_{t \to 0} t^{-1}\big(N(x + th) - N(x), y\big) = \big(N'(x)h, y\big)$$

for all $x, y, h \in H$. Assume that $B_1 \le N'(x) \le B_2$ for each x and some selfadjoint maps B_1 and B_2. Then N is asymptotically $\{B_1, B_2\}$-quasilinear with $|N - K| = 0$. Indeed, for every $x, y, z \in H$, the mean value theorem implies the existence of a number $t \in (0, 1)$ such that

$$(Nx - Ny, z) = \left(N'(y + t(x - y))(x - y), z\right).$$

When $y = 0$, this gives

$$\left(Nx - N'(tx)x, z\right) = (N(0), z)$$

and therefore, if we set $Kx = N'(tx)x$, then

$$\|Nx - Kx\| = \sup_{\|z\| \le 1} |(N(0), z)| \le \|N(0)\|.$$

Hence, $|N - K| = 0$ and N is asymptotically $\{B_1, B_2\}$-quasilinear. In the nondifferentiable case, if $Nx = B(x)x + Mx$ for some nonlinear map M with the quasinorm $|M| < \infty$ and selfadjoint maps $B(x) : H \to H$ with $B_1 \le B(x) \le B_2$ for each $x \in H$, then N is asymptotically $\{B_1, B_2\}$-quasilinear.

We need the following preliminary result (cf. [**Mi-8**]).

Lemma 2.2 *Let $A : D(A) \subset H \to H$ and $B^{\pm} \in L(H)$ be selfadjoint with $B^- \le B^+$ and $H^{\pm}$ be subspaces with $H = H^- \oplus H^+$ and such that for some $\gamma_1 > 0$ and $\gamma_2 > 0$*

$$((A - B^-)x, x) \le -\gamma_1 \|x\|^2 \quad \textit{for all} \quad x \in H^- \cap D(A), \tag{2.10}$$

$$((A - B^+)x, x) \ge \gamma_2 \|x\|^2 \quad \textit{for all} \quad x \in H^+ \cap D(A). \tag{2.11}$$

Then there are $\epsilon > 0$ and $c > 0$ such that for any selfadjoint maps $B_1, B_2, C \in L(H)$ with $B_1 \le B^-$ and $B^+ \le B_2$ and

$$B_1 - \epsilon I \le C \le B_2 + \epsilon I \tag{2.12}$$

we have that

$$\|Ax - Cx\| \geq c\|x\| \quad \textit{for} \;\; x \in D(A). \tag{2.13}$$

Proof. If (2.13) does not hold, then there would exist selfadjoint maps B_{1n}, B_{2n} and $C_n \in L(H)$ and $x_n \in D(A)$ with $\|x_n\| = 1$ such that $B_{1n} \leq B^-$, $B^+ \leq B_{2n}$ and

$$B_{1n} - I/n \leq C_n \leq B_{2n} + I/n \tag{2.14}$$

and

$$\|Ax - C_n x_n\| \leq \|x_n\|/n.$$

Set $y_n = Ax_n - C_n x_n$. Then, we have $x_n = x_{1n} + x_{2n} \in H^- \oplus H^+$ and

$$(Ax_{2n} - C_n x_{2n}, x_{2n}) - (Ax_{1n} - C_n x_{1n}, x_{1n}) = (y_n, x_{2n} - x_{1n}).$$

Since $B_{1n} \leq B^-$ and $B^+ \leq B_{2n}$, (2.14) and (2.12)–(2.13) imply that

$$(Ax_{1n} - C_n x_{1n}, x_{1n}) \leq (Ax_{1n} - B^- x_{1n}, x_{1n}) + \|x_{1n}\|^2/n \leq (1/n - \gamma_1)\|x_{1n}\|^2.$$

Subtracting the first equation from the second one, we get

$$\begin{aligned}\gamma_1\|x_{1n}\|^2 + \gamma_2\|x_{2n}\|^2 - \left(\|x_{1n}\|^2 + \|x_{2n}\|^2\right)/n \\ &\leq (Ax_{2n} - C_n x_{2n}, x_{2n}) - (Ax_{1n} - C_n x_{1n}, x_{1n}) \\ &= (y_n, x_{2n} - x_{1n} \leq \|x_{2n} - x_{1n}\|/n.\end{aligned}$$

Hence, if $\gamma = \min\{\gamma_1, \gamma_2\}$, by the parallelogram law we get

$$(\gamma - 1/n)\left(\|x_{2n}\|^2 + \|x_{1n}\|^2\right) \leq 2/n\left(\|x_{2n}\|^2 + \|x_{1n}\|^2\right)$$

and therefore $\gamma \leq 1/n \to 0$ as $n \to \infty$, in contradiction to $\gamma > 0$. Hence, (2.13) holds. □

Now we are ready to prove our basic solvability result for Eq. (1.1) involving asymptotically $\{B_1, B_2\}$-quasilinear nonlinearities N without resonance at infinity. It is based on the following continuation theorem.

THEOREM 2.3 **[Mi-3]** *Let V be dense subspace of a Hilbert space H, $D \subset H$ be open and bounded subset and a homotopy $H : [0,1] \times (\bar{D} \cap V) \to H$ be such that*

(i) *H is an A-proper homotopy w.r.t. $\Gamma = \{H_n, P_n\}$ on $[0, \epsilon] \times (\partial D \cap V)$ for each $\epsilon \in (0,1)$ and H_1 is pseudo A-proper w.r.t. Γ,*

(ii) *$H(t,x)$ is continuous at 1 uniformly for $x \in \partial D \cap V$,*

(iii) *$H(t,x) \neq f$ and $H(0,x) \neq tf$ for $t \in [0,1]$, $x \in \partial D \cap V$.*

(iv) *$\deg(P_n H_0, D \cap H_n, 0) \neq 0$ for all large n.*

Then the equation $H(1,x) = f$ is solvable in $\bar{D} \cap V$.

Now, we have (cf. **[Mi-8]**)

THEOREM 2.4 *Let $A : D(A) \subset H \to H$ satisfy (2.10)–(2.11) and $N : H \to H$ be bounded asymptotically $\{B_1, B_2\}$-quasilinear with $|N - K|$ sufficiently small. Suppose that a selfadjoint map $C_0 \in L(H)$ satisfies (2.12) with sufficiently small ϵ and that $H(t,.) = A - (1-t)C_0 - tN$ is A-proper w.r.t. $\Gamma = \{H_n, P_n\}$ with $P_n Ax = Ax$ on H_n for each $t \in [0,1)$ and H_1 is pseudo A-proper w.r.t. Γ. Then Eq. (1.1) is solvable for each $f \in H$.*

Proof. Since $N_f x = Nx - f$, $f \in H$, has the same properties as N, it suffices to solve the equation $Ax - Nx = 0$. Let $\epsilon_0 > 0$ be such that $|N - K| + \epsilon_0 < c$, where c is from Lemma 2.2. Then there is an $r > 0$ such that

$$\|Nx - Kx\| \leq (|N - K| + \epsilon_0)\|x\|, \quad \text{for each } \|x\| \geq r.$$

Moreover,

$$H(t,x) = Ax - (1-t)C_0 x - tNx \neq 0 \quad \text{for} \quad x \in \partial B_r \cap D(A), t \in [0,1]. \tag{2.15}$$

If not, then $H(t,x) = 0$ for some $\|x\| = r$ and $t \in [0,1]$. Hence, subtracting tKx from both sides, we get

$$\|Ax - tKx - (1-t)C_0 x\| = t\|Nx - Kx\| < c\|x\|.$$

Since K is $\{B_1, B_2\}$-quasilinear, there is a selfadjoint map $C^* \in L(H)$ such that $Kx = C^* x$, $B_1 \leq C^* \leq B_2$ and therefore

$$\|Ax - tC^* x - (1-t)C_0 x\| < c\|x\|. \tag{2.16}$$

But, $C = tC^* + (1-t)C_0$ is selfadjoint, satisfies (2.12) in Lemma 2.1 and therefore (2.13) holds. This contradicts (2.16) and so (2.15) is valid.

Next, since C_0 and N are bounded maps, $H(t,x)$ is an A-proper homotopy on $[0,\epsilon] \times (\bar{B}_r \cap D(A))$ w.r.t. Γ for each ϵ in $(0,1)$ and is continuous at 1 uniformly for $x \in \bar{B}_r \cap D(A)$. Hence, the solvability of $Ax - Nx = 0$ follows from Theorem 2.3. □

Let us now discuss some conditions on $B^{\pm}$ which imply (2.10)–(2.11). Assume, as in Amann [**Am-2**],

(2.17) a) $A : D(A) \subset H \to H$ is self adjoint

b) $B^{\pm} = \sum_{i=1}^{m} \lambda_i^{\pm} P_i^{\pm}$ commute with A, where $P_i^{\pm} : H \to \ker(B^{\pm} - \lambda_i)$ are orthogonal projections, $\lambda_1^{\pm} \leq \cdots \leq \lambda_m^{\pm}$ and $\lambda_i^{\pm}$ are pairwise distinct.

c) $\bigcup_{i=1}^{m} [\lambda_i^-, \lambda_i^+] \subset \rho(A)$-the resolvent set of A.

Being selfadjoint, A possesses a spectral resolution

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda$$

where $\{E_\lambda \mid \lambda \in R\}$ is a right continuous spectral family. Since $B^{\pm}$ commute with A, it is known that $P_i^{\pm}$ commute with the resolution of the identity $\{E_\lambda \mid \lambda \in R\}$. Hence, the selfadjoint maps $A - B^{\pm}$ have the spectral resolution

$$A - B^{\pm} = \sum_{i=1}^{m} \int_{-\infty}^{\infty} (\lambda - \lambda_i^{\pm}) dE_\lambda P_i^{\pm}. \tag{2.18}$$

Define the orthogonal projections $P^\pm$ by

$$P^- = \sum_{i=1}^m E(-\infty, \lambda_i^-)P_i^- \quad \text{and} \quad P^+ = \sum_{i=1}^m E(\lambda_i^+, \infty)P_i^+,$$

where

$$E(\alpha, \beta) = \int_\alpha^\beta dE_\lambda$$

for all $\alpha, \beta \in \rho(A) \cup \{\pm\infty\}$ with $\alpha < \beta$. Define $H^\pm = P^\pm(H)$ and note that by (2.17)-c,

$$P^+ = \sum_{i=1}^m E(\lambda_i^-, \infty)P_i^+,$$

and

$$\gamma = \operatorname{dist}\left(\bigcup_{i=1}^m [\lambda_i^-, \lambda_i^+], \sigma(A) \right) > 0.$$

Moreover, by (2.18), we have that

$$\begin{aligned} \big((A - B^-)x, x\big) &\le -\gamma\|x\|^2 \quad \text{for } x \in D(A) \cap H^-, \\ \big((A - B^-)x, x\big) &\ge \gamma\|x\|^2 \quad \text{for } x \in D(A) \cap H^+. \end{aligned}$$

Hence, we have

LEMMA 2.3 *If* (2.17) *holds and* $P_i^- = P_i^+$ *for* $1 \le i \le m$, *then there are orthogonal subspaces* $H^\pm$ *such that* $H = H^- \oplus H^+$ *and conditions* (2.10)–(2.11) *hold with* $\gamma_1 = \gamma_2 = \gamma > 0$.

Proof. It remains only to show that H^+ and H^- are orthogonal. Since $P_i^- = P_i^+$ for $i = 1, \ldots, m$, and $E(-\infty, \lambda_i^-) = I - E(\lambda_i^-, \infty)$, we get that $P^+ = I - P^-$' and therefore, $H^+ = (H^-)^\perp$. □

When $B^\pm$ are not of the form (2.17)-b, we need to assume more on the linear part A.

(2.19) Suppose A is selfadjoint possessing a countable spectrum $\sigma(A)$ consisting of eigenvalues and whose eigenvectors form a complete orthonormal system in H.

(2.20) There are selfadjoint maps $C_1, C_2 \in L(H)$ and two consecutive finite multiplicity eigenvalues $\lambda_k < \lambda_{k+1}$ of A such that

$$\lambda_k\|x\|^2 < (C_1x, x) \le (C_2x, x) < \lambda_{k+1}\|x\|^2 \quad \text{for } x \in H \setminus \{0\}.$$

Let H^- (resp. H^+) be the subspaces of H spanned by the eigenvectors of A corresponding to the eigenvalues $\lambda_i \le \lambda_k$ (resp. $\lambda_i \ge \lambda_{k+1}$).

LEMMA 2.4 (cf. [Mi-8]) *Let* (2.19)–(2.20) *hold. Then there are* $\gamma_1 > 0$ *and* $\gamma_2 > 0$ *such that for any selfadjoint maps* $B^\pm \in L(H)$ *satisfying* $C_1 \le B^-$ *and* $B^+ \le C_2$ *on* H, *we have that* (2.10)–(2.11) *hold.*

Proof. It is enough to show that (2.10) holds since the same arguments also give (2.11). Since $\lambda_k < C_1 \le B^-$, it is enough to show that there is a $\gamma_1 > 0$ such that

$$(Ax - C_1x, x) \le -\gamma_1\|x\|^2 \quad \text{for all} \quad x \in D(A) \cap H^-.$$

If such a $\gamma_1 > 0$ did not exist, then there would exist $\{x_n\} \subset D(A) \cap H^-$ with $\|x_n\| = 1$ and such that

$$-1/n \le (Ax_n - C_1x_n, x_n) \le (Ax_n - \lambda_k x_n, x_n), \quad n = 1, 2, \dots .$$

Decompose $H^- = \tilde{H} \oplus \bar{H}$, where $\tilde{H}$ is spanned by the eigenvaectors $\{e_i\}$ corresponding to $\lambda_i \le \lambda_{k-1}$ and $\bar{H}$ is the finite dimensional space spanned by the eigenvectors corresponding to λ_k. Then $x_n = \tilde{x}_n + \bar{x}_n \in \tilde{H} \oplus \bar{H}$ and

$$\begin{aligned} -1/n &\le (Ax_n - \lambda_k x_n, x_n) = (A\tilde{x}_n - \lambda_k \tilde{x}_n, \tilde{x}_n) \\ &= \sum_{i \le k-1} \lambda_i (\tilde{x}_n, e_i)^2 - \lambda_k \|\tilde{x}_n\|^2 \le (\lambda_{k-1} - \lambda_k)\|\tilde{x}_n\|^2 \end{aligned}$$

Hence, $\tilde{x}_n \to 0$ as $n \to \infty$ and $x_n \to \bar{x} \in \bar{H}$ with $\|\bar{x}\| = 1$ since $\|x_n\|^2 = 1 - \|\tilde{x}_n\|$ and $\bar{H}$ is finite dimensional. Thus,

$$\begin{aligned} -1/n &\le (\lambda_k x_n - C_1 x_n, x_n) + (Ax_n - \lambda_k x_n, x_n) \\ &\le (\lambda_k x_n - C_1 x_n, x_n) + (\lambda_{k-1} - \lambda_k)\|\tilde{x}_n\|^2 \end{aligned}$$

and passing to the limit as $n \to \infty$, we get

$$0 \le ((\lambda_k - C_1)\bar{x}, \bar{x}), \bar{x} \ne 0$$

in contradiction to (2.20). Hence, (2.10) holds. □

Remark 2.3 If λ_k (resp., λ_{k+1}) is of infinite multiplicty, then Lemma 2.3 is still valid if we assume in (2.20)

$$(\lambda_k + \epsilon)\|x\|^2 \le (C_1x, x) \quad \text{resp.} \quad (C_2x, x) \le (\lambda_{k+1} - \epsilon\|x\|^2) \quad \text{for} \quad 0 \ne x \in H.$$

This is easy to check by analyzing the proof of this lemma.

Next, let us look at the case when $H^- \oplus H^+ \ne H$, which is also useful in applications.

Recall that a closed subspace $X \subset H$ is said to *reduce* A if A commutes with the orthogonal projection P of H onto X, i.e., if $PA = AP$. As before, let $P^\pm$ be the orthogonal projections of H onto $H^\pm$. Regarding A and H we require that the following conditions hold for a scheme $\Gamma = \{H_n, P_n\}$ (cf. **[Am-2]**):

(2.21) (i) $H_1 \subset H_2 \subset \cdots$ with each H_n closed in H, $\dim H_n = \infty$ and $\bigcup H_n$ is dense in H;

(ii) each H_n reduces A;

(iii) the orthogonal projections $P_n : H \to H_n$ commute with $P^\pm$;

(iv) $H_n = H_n^- \oplus H_n^+$, where $H_n^\pm = H^\pm \cap H_n$;

(v) $Q^{\pm}(D(A) \cap H_n) \subset D(A)$ for each n, where $Q^{\pm} : H^- \oplus H^+ \to H^{\pm}$ are orthogonal projections.

We have the following extension of Theorem 2.4.

THEOREM 2.5 **[Mi-8]** *Let $H^{\pm}$ be closed subspaces of H such that $H^- \cap H^+ = \{0\}$, $A : D(A) \subset H \to H$ satisfy conditions* (2.10)–(2.11) *and* (2.21) *hold. Suppose that $N : H \to H$ is bounded asymptotically $\{B_1, B_2\}$-quasilinear with $|N - K|$ sufficiently small and $A - N : D(A) \subset H \to H$ is pseudo A-proper w.r.t. $\Gamma = \{H_n, P_n\}$. Suppose that a selfadjoint map $C_0 \in L(H)$ satisfies* (2.12) *and for each large n the map $H_n(t,\cdot) = P_n(A - (1-t)C_0 - tN) : D(A) \cap H_n \to H_n$ is A-proper w.r.t. $\Gamma_n = \{H_{n,k}, Q_k\}$ for H_n, with $Q_k Ax = Ax$ on $H_{n,k}$, for each $t \in [0,1)$ and $H_n(1,\cdot)$ be pseudo A-proper w.r.t. Γ_n. Then Eq.* (1.1) *is solvable for each $f \in H$.*

When $P_i^- \neq P_i^+$ for $1 \leq i \leq m$ and $H \neq H^- \oplus H^+$, regarding conditions (2.10)–(2.11) we have (cf. **[Am-2]**).

LEMMA 2.5 *Let* (2.17) *hold and there exist a unitary map $U \in L(H)$ such that A commutes with U and*

$$P_i^- = U P_i^+ U^{-1}, \quad i = 1, \ldots, m.$$

Suppose that A has a pure point spectrum in $(\lambda_1^-, \lambda_m^+)$. Then there are closed subspaces $H^{\pm}$ of H with $H^- \cap H^+ = \{0\}$ such that conditions (2.10)–(2.11) *hold with $\gamma_1 = \gamma_2 = \gamma > 0$, and a scheme $\Gamma = \{H_n, P_n\}$ satisfying* (2.21).

2.3 Asymptotically linear equations and Morse theory. In this section we shall study the existence of nontrivial solutions in a Hilbert space H of equations of the form

$$Ax + Nx = 0 \tag{2.22}$$

of potential type using a combination of the finite dimensional Morse theory and the A-proper mappping approach. We note that Galerkin methods in conjunction with Morse theory have been used earlier by many authors in the study of equations of the form (2.22) with N compact.

The novelty in our approach is that we require that $A + N$ is A-proper with respect to a projectionally complete scheme, which is so in particular when N is compact, or ball condensing or monotone (of Section 2 in **[Mi-8]** and Proposition 2.3).

Throughout the section, we assume that $A : D(A) \in H \to H$ is a linear densely defined, selfadjoint map with $\dim \ker A \leq \infty$ and $N : H \to H$ is a nonlinear map such that $A + N : D(A) \subset H \to H$ is potenial and A-proper with respect to a projectinally complete scheme $\Gamma = \{H_n, P_n\}$ with $P_n Ax = Ax$ for $x \in H_n$.

Let $f : H \to R$ be such that $f'(x) = Ax + Nx$. We say that f is asymptotically quadratic at infinity if there is a selfadjoint bounded linear map $N_\infty : H \to H$ such that

$$|N - N_\infty| = \limsup_{\|x\| \to \infty} \frac{\|Nx - N_\infty x\|}{\|x\|} < \infty. \tag{2.23}$$

Definition 2.3 A C^1 function $f : H \to R$ is said to satisfy the Palais-Smale condition (PS) if any sequence $\{x_n\} \in H$ with $f'(x_n) \to 0$ and $\{f(x_n)\}$ bounded posesses a convergent subsequence.

Let f_n be the restriction of f to H_n.

Definition 2.4 f is said to satisfy the $(PS)_n$ condition if any sequence $\{x_n | x_n \in H_n\}$ with $\|f'(x_n)\| \to 0$ and $\{f(x_n)\}$ bounded posesses a convergent subsequence.

The next result provides conditions on the A-proper map $f' = A + N$ which guarantee condition $(PS)_n$ for f.

PROPOSITION 2.2 *Let $f : H \to R$ with $f_n \in C^1(H_n, R)$, (2.23) hold with $|N - N_\infty|$ sufficiently small, $A + N_\infty$ and $A + N$ be A-proper with respect to $\Gamma = \{H_n, P_n\}$ and $A + N_\infty$ be injective. Then*

(i) *f_n is asymptotically quadratic and*

$$\limsup_{\|x\| \to \infty, x \in H_n} \frac{\|f'(x) - (A + P_n N_\infty)x\|}{\|x\|} \leq |N - N_\infty|.$$

(ii) *f_n satisfies condition (PS) for each large n.*

(iii) *f satisfies condition $(PS)_n$.*

(iv) *The equation $Ax + Nx = f$ is approximation solvable for each $f \in H$.*

Proof. (i) Let $\epsilon > 0$ and $R > 0$ such that $\|Nx - N_\infty x\| \leq (|N - N\infty| + \epsilon)\|x\|$ for all $\|x\| \geq R$. Since $f_n'(x) = Ax + P_n Nx$ for each $x \in H_n$, we get that

$$\|f_n'(x) - (A + P_n N_\infty)x\| = \|P_n(N - N_\infty)x\| \leq (|N - N_\infty| + \epsilon)\|x\| \tag{2.24}$$

for all $x \in H_n$, $\|x\| \geq R$ which proves (i).

(ii) Since $A + N_\infty$ is injective and A-proper, arguing by contradiction we see that there are $c > 0$ and $n_0 \geq 1$ such that

$$\|P_n(A + N_\infty)x\| \geq c\|x\| \quad \text{for all } x \in H_n, \; n \geq n_0. \tag{2.25}$$

Let $\{x_k\} \subset H_n$ with $\|f_n'(x_k)\| \to 0$ as $k \to \infty$ and $\{f_n(x_k) | k \geq 1\}$ bounded. Then, by (2.24) and (2.25),

$$\begin{aligned} c\|x_k\| &\leq \|P_n(A + N_\infty)x_k\| = \|P_n(A + N)x_k - P_n(N - N_\infty)x_k\| \\ &= \|f_n'(x_k) - P_n(N - N_\infty)x_k\| \leq \|f_n'(x_k)\| + \|f_n'(x_k) - (A + P_n N_\infty)x_k\| \\ &\leq \epsilon + (|N - N_\infty| + \epsilon)\|x_k\|, \quad \text{for all } n \geq n_1 \geq n_0. \end{aligned}$$

Taking $\epsilon > 0$ such that $|N - N_\infty| + \epsilon < c$, this yields the boundedness of $\{x_k\}$ in H_n. Hence a subsequence $\{x_{k_i}\}$ converges in H_n.

(iii) Let $\{x_n \in H_n\}$ satisfy $\|f_n'(x_n)\| \to 0$. As above, by (2.24) and (2.25), the sequence $\{x_n\}$ is bounded. Since $f_n'(x_n) = Ax_n + P_n Nx_n \to 0$, by the A-properness of $A+N$ a subsequence $x_{n_k} \to x$ and $f'(x) = Ax + Nx = 0$, i.e. x is a critical point of f.

(iv) This is a special case of Theorem 2.1. □

Remark 2.4 The boundedness of $\{f_n(x_k) \mid k \geq 1\}$ and $\{f(x_n)\}$ was not used in the proofs of (ii) and (iii).

Next, we give special classes of A-proper maps appearring in Proposition 2.2.

PROPOSITION 2.3 *Let $A : D(A) \subset H \to H$ be a densely defined Fredholm map of index zero and $N : H \to H$ be a continuous nonlinear and bounded map. Suppose that either one of the following conditions holds:*

(i) *N is k-ball contractive with k sufficiently small and $|N - N_\infty| = 0$ for some continuous linear map N_∞;*

(ii) *The partial inverse A^{-1} of A is compact.*

Then $A+N : D(A) \subset H \to H$ and $A + N_\infty$ are A-proper with respect to $\Gamma = \{H_n, P_n\}$ with $P_n Ax = Ax$ for $x \in H_n$.

Proof (i) Since the index $i(A) = 0$, we have that $H = H_0 \oplus H_1$, where $H_0 = \ker A$. Let $\{x_{n_k} \in H_{n_k}\}$ be bounded and $y_k \equiv P_{n_k}(A+N)x_{n_k} \to f$. Then $x_{n_k} = x_{0n_k} + x_{1n_k}$ with $x_{in_k} \in H_i$, $i = 0, 1$, and we may assume that $x_{0n_k} \to x_0 \in H_0$ and

$$x_{1n_k} + A^{-1}(I-P)P_{n_k}Nx_{n_k} = A^{-1}(I-P)y_k \to A^{-1}(I-P)f,$$

where $P : H \to H_0$ is the orthogonal projection. Hence, the ball-measure of noncompactness

$$\chi(\{x_{1n_k}\}) = \chi(\{A^{-1}(I-P)P_{n_k}Nx_{n_k}\}) \leq \|A^{-1}\| k \chi(\{x_{n_k}\}) = k\|A^{-1}\|\chi(\{x_{1n_k}\}).$$

Since k is sufficiently small, we get that $\chi(\{x_{1n_k}\}) = 0$ and therefore $x_{1n_k} \to x_1 \in H_1$. Thus, $x_{n_k} \to x = x_0 + x_1$ and $Ax + Nx = f$. Since it is well known that N_∞ is also k-ball contractive, the same arguments show that $A + N_\infty$ is also A-proper with respect to Γ.

(ii) Let $\{x_{n_k}\}$ be as in i). Then $x_{0n_k} \to x_0 \in H_0$ and $P_{n_k}Nx_{n_k} \rightharpoonup u$ by the boundedness of N. Hence, $x_{1n_k} = A^{-1}(I-P)(y_k - P_{n_k}Nx_{n_k}) \to A^{-1}(I-P)(f-u) = x_1$ so that $x_{n_k} \to x = x_0 + x_1$. Since $Nx_{n_k} \to Nx$, it follows that $Ax_{n_k} = y_k - P_{n_k}Nx_{n_k} \to f - Nx$ and so $Ax + Nx = f$ by the clossedness of A. □

Remark 2.5 When $\dim \ker A = \infty$, various conditions on A and N that imply the A-properness of $A+N$ have been discussed in [**Mi-4-8**].

To obtain nontrivial solutions of Eq. (2.22), we need to impose some additional conditions on A and N. To that end we shall use some finite dimensional results on the existence of nontrivial critical points. For a symmetric matrix S, denote by $m^-(S)$, $m^0(S)$ and $m^+(S)$ the negative, zero and positive Morse index of S. We have (cf. [**A-Z, Ch, L-L**])

THEOREM 2.6 *Let $f : R^n \to R$ be a C^1-function and A_0 and A_∞ be symmetric matrices such that*

$$(2.26)\qquad \frac{\|f'(x) - A_\infty x\|}{\|x\|} \to 0 \quad as \ \|x\| \to \infty$$

and

$$(2.27)\qquad \frac{\|f'(x) - A_0 x\|}{\|x\|} \to 0 \quad as \|x\| \to 0.$$

If $m^0(A_0) = m^0(A_\infty) = 0$ and $m^-(A_0) \neq m^-(A_\infty)$, then f has at least one nontrivial critical point. Moreover, if for some $r > 0$

$$(2.28)\qquad \|f'(x) - A_0 x\| < \|x\|/(2\|A^{-1}\|) \quad for \ \|x\| < r.$$

then the nontrivial critical points of f lie outside the ball $B(0, r)$.

Suppose that there is a selfadjoint map $N_0 : H \to H$ such that

$$(2.29)\qquad \|Nx - N_0 x\|/\|x\| \to 0 \quad \text{as} \ \|x\| \to 0.$$

For each n, define the numbers $I^-(f_n, \infty) = m^-(P_n(A + N_\infty)P_n)$, $I^-(f_n, 0) = m^-(P_n(A + N_0)P_n)$ and $I^0(f_n, 0) = m^0(P_n(A + N_0)P_n)$.

The Morse indices of $A + N_\infty$ and $A + N_0$ may be infinite and the Morse indices of the approximate operators $I^-(f_n, \infty)$ and $I^-(f_n, 0)$ converge to infinity as $n \to \infty$. We have the following extension of Theorem 2.6 to the infinite dimensional case.

THEOREM 2.7 *Let $f : H \to R$ with $f_n \in C^1(H_n, R)$, (2.23) and (2.29) hold with $|N - N_\infty| = 0$ and $A + N_0$ and $A + N_\infty$ be both injective A-proper maps with respect to $\Gamma = \{H_n, P_n\}$. Suppose that $I^-(f_n, 0) \neq I^-(f_n, \infty)$ for all large n. Then f has at least one nontrivial critical point.*

Proof. Since $A + N_\infty$ and $A + N_0$ are injective A-proper maps, there are $c > 0$ and $n_0 \geq 1$ such that for each $n \geq n_0$

$$(2.30)\quad \|(A + P_n N_0)x\| \geq c\|x\| \quad \text{and} \quad \|(A + P_n N_\infty)x\| \geq c\|x\| \quad \text{for all} \ x \in H_n.$$

Let $r > 0$ be so small that

$$\|f'(x) - (A + N_0)x\| \leq \frac{c}{2}\|x\| \quad \text{for} \ \|x\| < r.$$

Then, for each $n \geq n_0$ and $x \in H_n$ with $\|x\| < r$,

$$\begin{aligned}\|f_n'(x) - (A_n + P_n N_0 P_n)x\| &\leq \|f'(x) - (A + N_0)x\| \\ &\leq \frac{c}{2}\|x\| \leq \frac{1}{2}\|(A_n + P_n N_0 P_n)^{-1}\|^{-1}\|x\|\end{aligned}$$

by (2.30), where A_n is the restriction of A to H_n. By Theorem (2.6), f_n has a nontrivial critical point $x_n \in H_n$ with $\|x_n\| \geq r$ for all $n \geq n_0$. Since f satisfies the $(PS)_n$ condition by Proposition 2.2, a subsequence $x_{n_k} \to x$ and x is a critical point of f since $f' = A + N$ is A-proper, with $\|x\| \geq r$. □

Let us now show that under the conditions of Theorem 2.7, the conditions $I^-(f_n, 0) \neq I^-(f_n, \infty)$ for all large n are implied by the condition $m^-(P_n(A + N_0)P_n) - m^-(A_n) \neq m^-(P_n(A + N_\infty)P_n) - m^-(A_n)$ for some sufficiently large n.

The following algebraic lemma is needed (see [**L-L**]).

LEMMA 2.6 *Let S and T be two symmetric $n \times n$ matrices. Then $m^-(S) = m^-(S+T)$ provided that for any $d > 0$ there is an $r = r(d) > 0$ such that either one of the following conditions holds*:

(i) *S is invertible, $\|S^{-1}\| < d$ and $\|T\| < r$;*

(ii) *$\|S_1^{-1}\| < d$, $\|T\| < r$ and $TP = 0$, where $P : R^n \to \ker(S)$ is the orthogonal projection onto $\ker(S)$ and S_1 is the restriction of S to $(I - P)R^n$. The condition $TP = 0$ means that $\ker(S) \subset \ker(S+T)$.*

LEMMA 2.7 *Let $A : D(A) \subset H \to H$ be a Fredholm map of index zero and $B : H \to H$ be a continuous linear map such that $A + B$ is A-proper with respect to $\Gamma = \{H_n, P_n\}$ with $P_n Ax = Ax$ on H_n. If either*

(i) *$A + B$ is one-to-one, or*

(ii) *$P_n P_0 = P_0$, where $P_0 : H \to \ker(A+B)$ is the orthogonal projection onto $\ker(A+B)$,*

then $m^-(A_n + P_n BP_n) - m^-(A_n)$ is a constant for all large n.

Proof. (cf. also [**L-L**]) Let $F = \operatorname{span}\{H_n, H_m\}, Q : H \to F$ be the orthogonal projection and $Q_n : H \to F_n$ be the orthogonal projection onto F_n, the complement of H_n in F. Then $Q = P_n + Q_n$ and Q and Q_n commute with A since so do P_n and P_m. Assume that $F_n \neq \{0\}$ for each n. We represent the map $A + B$ in F by the matrix $W = S + T$:

$$W = \begin{pmatrix} P_n(A+B)P_n & P_n(A+B)Q_n \\ Q_n(A+B)P_n & Q_n(A+B)Q_n \end{pmatrix},$$

$$S = \begin{pmatrix} A_n + P_nBP_n & 0 \\ 0 & Q_n(A+P)Q_n \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & P_nBQ_n \\ Q_nBP_n & Q_n(B-P)Q_n \end{pmatrix},$$

where $P : H \to \ker(A)$ is the orthogonal projection, and we used the fact that $Q_nA = AQ_n$ and $P_nQ_n = Q_nP_n = 0$. We have that P is compact, commutes with P_n and Q_n, and since $A + P$ is invertible there are $c_1 > 0$ and $n_0 \geq 1$ such that

$$\|(Q_n(A+P)Q_n)^{-1}\| < c_1 \quad \text{for all } n \geq n_0. \tag{2.31}$$

Assume first that $A + B$ is invertible. Since it is A-proper with respect to Γ, there are c_2 and $n_2 \geq 1$ such that

$$\|P_n(A+B)P_nx\| > c_2\|P_nx\| \quad \text{for all } x \in H,\ n \geq n_2$$

and therefore, in H_n

$$\|(P_n(A+B)P_n)^{-1}\| < 1/c_2 \quad \text{for all } n \geq n_2. \tag{2.32}$$

Since $\|Q_nB\| = \|BQ_n\| \to 0$ and $\|Q_nP\| \to 0$ as $n \to \infty$, we get by Lemma 2.6 that for each n large

$$m^-(W) = m^-(S) = m^-(A_n + P_nBP_n) + m^-(Q_n(A+P)Q_n). \tag{2.33}$$

Using Lemma 2.6-(ii), we can also prove that (2.33) holds when $P_nP_0 = P_0$. In particular, taking $B = 0$, (2.33) becomes

$$m^-(QAQ) = m^-(A_n) + m^-(Q_n(A+P)Q_n). \tag{2.34}$$

Combining (2.33) and (2.34), we get

$$\begin{aligned} m^-(A_n + P_nBP_n) - m^-(A_n) &= m^-(W) - m^-(QAQ) \\ &= m^-(A_m + P_mBP_m) - m^-(A_m). \quad \square \end{aligned}$$

Using Lemma 2.7, we have the following special case of Theorem 2.7.

COROLLARY 2.1 *Let $f : H \to R$ with $f_n \in C^1(H_n, R)$, A be Fredholm of index zero,* (2.23) *and* (2.29) *hold with $|N - N_\infty| = 0$ and $A + N_0$ and $A + N_\infty$ be both injective and A-proper with respect to $\Gamma = \{H_n, P_n\}$. Then, $m^-(A_n + P_nN_0P_n) - m^-(A_n)$ and $m^-(A_n + P_nN_\infty P_n) - m^-(A_n)$ are both constant for each large n. If these two constants are different, then f has at least one nontrivial critical point.*

Proof. By Lemma 2.7 both $m^-(A_n + P_nN_0P_n) - m^-(H_n)$ and $m^-(A_n + P_nN_\infty P_n) - m^-(A_n)$ are constant for all large n. Moreover, if they are different constants, then $I^-(f_n, 0) \neq I^-(f_n, \infty)$ for all large n, and the conclusion follows from Theorem 2.7. $\square$

Remark 2.6 When $f \in C^1(H, R)$ with A and N compact and $A = A^*$, Lemma 2.7 and Corollary 2.1 were proven by Li-Liu [**L-L**] using different type of arguments.

Next, we shall treat the case when 0 is a degenerate critical point of f. We need the following finite-dimensional result.

THEOREM 2.8 [**L-L**] *Let $f \in C^1(R^n, R) \cap C^2(B(0,d), R)$ and satisfy* (2.26) *and* (2.27). *Suppose that $m^0(A_\infty) = 0$ and $m^-(A_\infty) \notin [m^-(A_0), m^-(A_0) + m^0(A_0)]$. Then f has at least one nontrivial critical point. Moreover, if*

$$\|f'(x) - A_0\| \le \frac{1}{2}\|A_0^{-1}\|^{-1} \quad \textit{for} \ \ \|x\| < d, \tag{2.35}$$

where A_0^{-1} is the inverse of A_0 restricted to the range of A_0, then at least one critical point of f lies outside the ball $B(0, d/2)$.

We have the following extension to the infinite dimensional case.

THEOREM 2.9 *Let $f : H \to R$ with $f_n \in C^1(H_n, R) \cap C^2(H_n \cap B(0,d), R)$ and satisfy* (2.23) *and* (2.29) *with $|N - N_\infty| = 0$. Suppose that $A + N_\infty$ is injective* *and A-proper with respect to* $\Gamma = \{H_n, P_n\}$ *with* $P_nAx = Ax$ *on* H_n, *and* $A + N_0$ *is A-proper with respect to* Γ *with* $\ker(A + N_0) \subset H_n$ *for all large n. If*

$$I^-(f_n, \infty) \notin [I^-(f_n, 0), I^-(f_n, 0) + I^0(f_n, 0)] \tag{2.36}$$

for all large n, then f has at least one nontrivial critical point.

Proof. Since $H_0 = \ker(A + N_0) \subset H_n$, then $H = H_0 \oplus \tilde{H}$ for some closed subspace $\tilde{H}$ and $H_n = H_0 \oplus H_{1n}$ with $H_{1n} = H_n \cap \tilde{H}$. Since the restriction of $A + N_0$ to $\tilde{H}$ is injective and $A + N_0$ is A-proper, it is easy to see that there are $c > 0$ and $n_0 \geq 1$ such that

$$\|P_n(A + N_0)P_nx\| > c\|x\| \quad \text{for} \quad x \in H_{1n}, n \geq n_0.$$

Hence, the inverses of the restrictions of $P_n(A + N_0)P_n$ to H_{1n} satisfy

$$\|(P_n(A + N_0)P_n)^{-1}\| < 1/c \quad \text{for} \quad n \geq n_0.$$

Let $r < d/2$ be sufficiently small so that

$$\|f''(x) - (A + N_0)\| < c/2 \quad \text{for} \quad \|x\| \leq 2r,$$

and therefore,

$$\begin{aligned} &\|f_n''(x) - (A_n + P_nN_0P_n)\| \leq \|f''(x) - (A + N_0)\| \\ &\leq c/2 < \|(P_n(A + N_0)P_n)^{-1}\|/2 \quad \text{for} \quad \|x\| \leq 2r, \, n \geq n_0. \end{aligned} \tag{2.37}$$

Hence, from (2.36), (2.37) and Theorem 2.8, we get a nontrivial critical point $x_n \in H_n$ of f_n with $\|x_n\| \geq r$ for each $n \geq n_0$. Since f satisfies the $(PS)_n$ condition, a subsequence $x_{n_k} \to x$ with $\|x\| \geq r$ and $f'(x) = 0$ by the A-properness of $A + N$. □

COROLLARY 2.2 *Let* $f : H \to R$ *with* $f_n \in C^1(H_n, R) \cap C^2(H_n \cap B(0, d), R)$, *A be Fredholm of index zero and conditions* (2.23) *and* (2.29) *hold with* $|N - N_\infty| = 0$. *Suppose that* $A + N_\infty$ *is injective and A-proper with respect to* Γ *and* $A + N_0$ *is A-proper with respect to* Γ *with* $\ker(A + N_0) \subset H_n$ *for all large n. Then,* $m^-(A_n + P_nN_0P_n) - m^-(A_n)$ *and* $m^-(A_n + P_nN_\infty P_n) - m^-(A_n)$ *are both constant for each large n. Moreover, if*

$$\begin{aligned} &m^-(A_n + P_nN_\infty P_n) - m^-(A_n) \\ &\notin [m^-(A_n + P_nN_0P_n) - m^-(A_n), \\ &\quad m^-(A_n + P_nN_0P_n) - m^-(A_n) + m^0(A_n + P_nN_0P_n)] \end{aligned} \tag{2.38}$$

with n large, then f has at least one nontrivial critical point.

Proof. By Lemma 2.7 (ii), $m^-(A_n + P_nBP_n) - m^-(A_n)$ is constant for all large n with $B = N_0$ and $B = N_\infty$. Moreover, (2.38) implies that

$$I^-(f_n, \infty) \notin \left[I^-(f_n, 0), I^-(f_n, 0) + I^0(f_n, 0)\right]$$

for all large n, and the conclusion follows from Theorem 2.9. □

Finally, we shall consider the existence of nontrivial critical points of f when 0 is a degenerate critical point at which f has a local linking as defined below. Again we begin with the finite dimensional case first.

Definition 2.5 Let $f : X = R^n \to R$ be of class C^1. We say that f has a local linking at 0 if there are subspaces Y and Z of X with $X = Y + Z$ and positive constants a and r such that

$$\begin{aligned} &f(y) \geq a && \text{for } y \in \partial(B_r \cap Y) \\ &f(y) \geq 0 && \text{for } y \in B_r \cap Y \\ &f(z) \leq -a && \text{for } z \in \partial(B_r \cap Z) \\ &f(z) \leq 0 && \text{for } z \in B_r \cap Z. \end{aligned} \tag{2.39}$$

THEOREM 2.10 ([**L-L**]) *Let* $f \in C^1(R^n, R)$ *satisfy* (2.26), (2.27), (2.39) *and* $m^-(A_\infty) = 0$, $m^-(A_\infty) \neq \dim Z$. *Then* f *has at least one nontrivial critical point* x *with* $\|x\| \geq r$ *and* $|f(x)| \geq a$.

The following result involves the local linking condition in the infinite dimensional case.

THEOREM 2.11 *Let* $f : H \to R$ *with* $f_n \in C^1(H_n, R)$ *and satisfy* (2.23) *and* (2.29) *with* $|N - N_\infty| = 0$. *Suppose that* $A + N_\infty$ *is injective and A-proper with respect to* $\Gamma = \{H_n, P_n\}$ *with* $P_n Ax = Ax$ *on* H_n *and for each large* n *there is a decomposition* $H_n = Y_n + Z_n$ *and positive constants* a *and* r *such that*

$$\begin{aligned} &f(y) \geq a && \text{for } y \in \partial(B_r \cap Y_n) \\ &f(y) \geq 0 && \text{for } y \in B_r \cap Y_n \\ &f(z) \leq -a && \text{for } z \in \partial(B_r \cap Z_n) \\ &f(z) \leq 0 && \text{for } z \in B_r \cap Z_n. \end{aligned} \tag{2.40}$$

If $I^-(f_n, \infty) \neq \dim Z_n - m^-(A_n)$ *for infinitely many* n, *then* f *has at least one nontrivial critical point.*

Proof. By Proposition 2.2 and (2.40), Theorem 2.10 yields critical points $x_n \in H_n$ of f_n with $\|x_n\| \geq r$ and $|f_n(x_n)| \geq a$ for each large n. Since f satisfies condition $(PS)_n$ and $A + N$ is A-proper, a subsequence $x_{n_k} \to x$ with $f'(x) = Ax + Nx = 0$ and $x \neq 0$. □

COROLLARY 2.3 *Let* $f : H \to R$ *with* $f_n \in C^1(H_n, R)$ *and satisfy* (2.23) *and* (2.29) *with* $|N - N_\infty| = 0$. *Suppose that* $A + N_\infty$ *is injective and A-proper with respect to* $\Gamma = \{H_n, P_n\}$ *with* $P_n Ax = Ax$ *on* H_n *and that there is a decomposition* $H = Y + Z$ *such that* $P_n Y \subset Y$ *and* $P_n Z \subset Z$ *and for some positive constants* a *and* r *such that*

$$\begin{aligned} &f(y) \geq a && \text{for } y \in \partial(B_r \cap Y) \\ &f(y) \geq 0 && \text{for } y \in B_r \cap Y \\ &f(z) \leq -a && \text{for } z \in \partial(B_r \cap Z) \\ &f(z) \leq 0 && \text{for } z \in B_r \cap Z. \end{aligned}$$

If $I^-(f_n, \infty) \neq dim P_n Z - m^-(A_n)$ for infinitely many n, then f has at least one nontrivial critical point.

Remark 2.7 When $f \in C^1(H, R) \cap C^2(B(0,r), R)$, respectively $f \in C^1(H, R)$, and A and N are compact maps with $A = A^*$, Corollary 2.2, respectively Theorem 2.11 and Corollary 2.3, were proven by Li-Liu **[L-L]** using different type of arguments.

III. Periodic Solution of Semilinear Wave Equations Without Resonance

Let $Q \subset R^n$ be a bounded domain with smooth boundary and set $\Omega = (0,T) \times Q$ and let $H = L_2(\Omega, R^m)$ with the inner product defined by

$$(u,v) = \int_0^T \int_Q (u(t,x), v(t,x))\, dx dt$$

where $(u(t,x), v(t,x))$, $(t,x) \in \Omega$, is the inner product in R^m. Let L_1 be a linear selfadjoint elliptic operator in space variables $x \in R^n$ with coefficients independent of t such that the induced bilinear form $a(u,v)$ on the Sobolev space $W_2^1(Q, R^m)$ is continuous and symmetric. Suppose that V is a closed subspace of $W_2^1(Q, R^m)$, containing the test functions, such that $a(u,v)$ is semi-coercive on V, i.e. there are constants $a_1 > 0$ and $a_2 \geq 0$ such that

$$a(u,v) \geq a_1 \|u\|_{2,1}^2 - a_2 \|u\|_{L_2}^2 \quad \text{for all } \ u \in V.$$

Define a linear map $L_0 : D(L_0) \subset L_2(Q, R^m) \to L_2(Q, R^m)$ by

$$(L_0 u, v) = a(u,v) \quad \text{for each } \ v \in V,$$

where

$$D(L_0) = \{u \in V \mid a(u, \cdot) \text{ is continuous on } V \text{ in the } L_2\text{-norm}\}.$$

It is well known that L_0 is selfadjoint and has a compact resolvent, since W_2^1 is compactly embedded in L_2. Next, define a selfadjoint map with compact resolvent $L : D(L) \subset H_1 = L_2(Q, R^m) \to H_1$ by:

$$D(L) = [D(L_0)]^m \quad \text{and} \quad L = \text{diag}(L_0, \ldots, L_0).$$

Since L has a compact resolvent, there is an orthonormal basis $\{\psi_j \mid j \in J\}$ in H_1 and a sequence of its eigenvalues $\{\mu_j \mid j \in T\}$ such that $|\mu_j| \to \infty$.

Let $F : R \times Q \times R^m \to R^m$ be a Caratheodory function and consider the semilinear system of wave equations:

$$(3.1) \qquad \begin{cases} u_{tt} - L_1 u - F(t,x,u) = f(t,x) \\ u(t,.) \in V^m \end{cases}$$

where $f \in L_2(\Omega, R^m)$ is a T-periodic function in t variable and $\tau = 2\pi/T$ is rational.

By a *T-periodic weak solution* of the variational boundary value problem (3.1) for the semilinear system of wave equations we mean a solution of the nonlinear operator equation

(3.2) $$Au - Nu = f, u \in D(A), f \in H$$

where $Au = \sum_{j,k} u_{j,k}(\mu_j - \tau^2 k^2)\psi_j(x)e^{i\tau kt}$ for

$$u \in D(A) = \left\{ u = \sum_{j,k} u_{j,k}\psi_j(x)e^{i\tau kt} \;\middle|\; \sum_{j,k} |u_{jk}(\mu_j - \tau^2 k^2)|^2 < \infty \right\}$$

and $Nu = F(t,x,u)$ for $u \in L_2(\Omega, R^m)$.

Regarding $F = F_1 + F_2$, we assume

(3.3) F_1 is a Caratheodory function such that for some $a_1 > 0$, $k \in (0,1)$ and $h_1 \in L_2(\Omega, R)$ it satisfies

$$|F_1(t,x,y)| \leq a_1|y|^k + h_1(t,x) \quad \text{for a.e. } (t,x) \in \Omega, \text{ all } y \in R^m.$$

(3.4) F_2 is a Caratheodory function and there are $h_2 \in L_2(\Omega, R)$ and some consecutive eigenvalues $\lambda_i < \lambda_{i+1}$ of A, and $0 < a_2 < \min\{\lambda - \lambda_i, \lambda_{i+1} - \lambda\}$ such that

$$|F_2(t,x,y) - \lambda y| \leq a_2|y| + h_2(t,x), \quad \text{for a.e. } (t,x) \in \Omega, \; y \in R^m.$$

Our first nonresonance result for (3.1) is for monotone nonlinearities F and is an application of Theorem 2.1 (cf. [**Mi-8**]).

THEOREM 3.1 *Let $Q = (0,\pi)$, $\mu_j \geq 0$ for each $j \in J$, and each nonzero eigenvalue of A is of finite multiplicity. Suppose that* (3.3)–(3.4) *hold and F is increasing in y. Then there is a T-periodic weak solution $u \in L_2$ of* (3.1) *for each $f \in L_2(\Omega, R^m)$.*

Proof. Let $\{H_n\}$ be an increasing sequence of finite dimensional subspaces spanned by the eigenfunctions $\{\psi_j(x)e^{i\tau kt}\}$ and $P_n : H \to H_n$ be the orthogonal projections onto H_n. Since the eigenfunctions are dense in $L_2(\Omega, R^m)$, $\Gamma = \{H_n, P_n\}$ is a projection scheme for L_2. Moreover, since $Nu = F(t,x,u) : L_2 \to L_2$ is monotone, the map $A - N : D(A) \subset L_2(\Omega, R^m) \to L_2(\Omega, R^m)$ is pseudo A-proper w.r.t. Γ. In view of Theorem 2.1 and Remark 2.2, it remains to verify conditions (2.2)–(2.3) with $X = H = L_2$.

First, note that the spectral gap of $A_\lambda = A - \lambda I$ induced by the gap $(\lambda_i, \lambda_{i+1})$ is $(\lambda_i - \lambda, \lambda_{i+1} - \lambda)$. Hence, A_λ has a bounded selfadjoint inverse in L_2 with the spectrum lying in $[(\lambda_i - \lambda)^{-1}, (\lambda_{i+1} - \lambda)^{-1}]$ and so $\|A_\lambda^{-1}\| \leq \max\{(\lambda - \lambda_i)^{-1}, (\lambda_{i+1} - \lambda)^{-1}\} \leq 1/a$. Hence, $0 < a < \|A_\lambda^{-1}\| = \min\{|\mu| \,|\mu \in \sigma(A - \lambda I)\}$. Next, set $N_i u = F_i(t,x,u)$ for $u \in L_2$, $i = 1,2$. Then, by the Minkowski and Hölder inequalities, (3.3)–(3.4) imply that for some constants c_1 and c_2

$$\|N_1 u\| \leq c_1\|u\|^k + \|h_1\| \quad \text{and} \quad \|N_2 u\| \leq c_2\|u\| + \|h_2\| \quad \text{for } u \in L_2.$$

Since $k < 1$, there are positive a, b and r such that $a < \min\{\lambda - \lambda_i, \lambda_{i+1} - \lambda\}$ and

$$||Nu - \lambda u|| \leq a||u|| + b \quad \text{for all} \quad ||u|| \geq r.$$

Hence, (2.2)–(2.3) hold and Theorem 2.1 is applicable by Remark (2.2). □

It is easy to see that condition (3.4) is implied by the following two conditions with $\lambda = (\lambda_i + \lambda_{i+1})/2$ and $a = (\lambda_{i+1} - \lambda_i)/2 - \epsilon$:

(3.5) There are constants $M > 0$ and $r > 0$ and $h \in L_2(\Omega, R)$ such that for each $1 \leq l \leq m$

$$|F_{2,l}(t,x,y)| \leq M|y_l| + h_2(t,x) \quad \text{for a.e. } (t,x) \in \Omega,\ y \in R^m.$$

(3.6) For a.e. $(t,x) \in \Omega$, $y = (y_1, \dots, y_m) \in R^m$ with $|y_l| \geq r$

$$\lambda_i + \epsilon \leq F_{2,l}(t,x,y)/y_l \leq \lambda_{i+1} - \epsilon.$$

Hence, when $m = 1$, we have

COROLLARY 3.1 *Let $F : R \times (0,\pi) \times R \to R$ be a 2π-periodic in t Caratheodory function satisfying conditions (3.5)–(3.6) and $F(t,x,\cdot)$ be increasing. Then, for each $f \in L_2$, there is a 2π-periodic weak solution $u \in L_2$ of*

$$\begin{aligned} &u_{tt} - u_{xx} - F(t,x,u) = f(t,x), && t \in R,\ x \in (0,\pi) \\ &u(t,0) = u(t,\pi) = 0, && t \in R \\ &u(t+2\pi, x) = u(t,x), && t \in R,\ x \in (0,\pi). \end{aligned} \tag{3.7}$$

Remark 3.1 Without the monotonicity of F, the solvability of (3.7) for a dense set of f's in L_2 was proved by Hofer [**Ho**] under a global Lipschitz condition on F (cf. also [**W**]) and by Tanaka [**Ta-1**] without this condition. When F is monotone, Corollary 3.1 is due to Mawhin [**Ma-2**].

We continue our study of (3.1) when a nonlinear perturbation F satisfies asymptotic nonuniform nonresonance conditions with respect to two consecutive eigenvalues of the associated linear problem. These conditions are more general than (3.4) and (3.5)–(3.6) and our method of study requires a different approach based on Theorem 2.4. When $m = n = 1$, this problem has been studied by Mawhin-Ward [**Ma-Wa**] for the wave equation (3.7).

Let A be the abstract realization of the linear problem associated with (3.1) and $\lambda_i < \lambda_{i+1}$ be two consecutive eigenvalues of A having finite multiplicities.

Let $F : \Omega \times R^m \to R^m$ be a Caratheodory function such that for each $r > 0$ and $1 \leq l \leq m$ there are functions a_l, $b_l \in L_\infty(\Omega)$ and $h_r \in L_2(\Omega)$ such that

$$F(t,x,y)| \leq h_r(t,x) \quad \text{for a.e. } (t,x) \in \Omega,\ |y| \leq r, \tag{3.8}$$

$$a_l(t,x) \leq \liminf_{|y_l| \to \infty} F_l(t,x,y) y_l^{-1} \leq \liminf_{|y_l| \to \infty} F_l(t,x,y) y_l^{-1} \leq \beta_l(t,x) \tag{3.9}$$

uniformly a.e. in $(t,x) \in \Omega$ and $(y_1, \dots, y_{l-1}, y_{l+1}, \dots, y_m) \in R^{m-1}$ and

$$\lambda_i \leq a_l(t,x) \leq \beta_l(t,x) \leq \lambda_{i+1} \quad \text{a.e. on } \Omega \tag{3.10}$$

with $\lambda_i < a_l(t,x)$ and $\beta_l(t,x) < \lambda_{i+1}$ on some sets of positive measure.

THEOREM 3.2 ([**Mi-8**]) *Let* (3.8)–(3.10) *hold and*

(a) $\operatorname{sign}\lambda_i F$ *be monotone if* $n = 1$, *i.e.,* $\operatorname{sign}\lambda_i (F(t,x,y) - F(t,x,z)) \cdot (y-z) \geq 0$ *for a.e.* $(t,x) \in \Omega$ *and all* $y, z \in R^m$;

(b) *If* $n > 1$, *then* $F = F_1 + F_2$ *and for some positive constants* c, k_1, k_2 *with* $(k_1 + k_2)\|A^{-1}\| < 1$, $k_2 < c$;

$$\operatorname{sign}\lambda_i (F_1(t,x,y) - F_1(t,x,z), y-z) \geq c|y-z|^2 \quad \text{for a.e. } (t,x) \in \Omega,\ y,z \in R^m, \tag{3.11}$$

$$|F_i(t,x,y) - F_i(t,x,z)| \leq k_i|y-z| \quad \text{for a.e. } (t,x) \in \Omega,\ y,z \in R^m \quad i = 1,2. \tag{3.12}$$

Then there is a T-periodic weak solution $u \in L_2(\Omega, R^m)$ *of* (3.1) *for each* $f \in L_2(\Omega, R^m)$.

Proof. We have that either $0 < \lambda_i < \lambda_{i+1}$ or $\lambda_i < \lambda_{i+1} < 0$. We may assume that $\lambda_i > 0$, for otherwise instead of the corresponding operator equation

$$Au - Nu = f,\ \ u \in D(A),\ f \in H = L_2(\Omega, R^m) \tag{3.13}$$

where $Nu = F(t,x,u)$, we can consider the equivalent equation

$$A_1 u - N_1 u = -f$$

with $A_1 = -A$, $N_1 = -N$, $\sigma(A_1) = \{\cdots < 0 < -\lambda_{i+1} < -\lambda_i < \cdots\}$. Then, setting

$$a_{1l} = -\beta_l, \beta_{1l} = -a_l\ \tilde{F} = -F,$$

we see that conditions (3.9), (3.10) and (3.12) hold with a_l, β_l, F, λ_i and λ_{i+1} replaced respectively by a_{1l}, β_{1l}, $\tilde{F}$, $-\lambda_i$ and $-\lambda_{i+1}$ and the function $\operatorname{sign}(-\lambda_i)\tilde{F}_1 = \operatorname{sign}\lambda_i F_1$ is monotone if $n = 1$, or satisfies (3.11) if $n > 1$. Hence, we can assume that $\lambda_i > 0$ and therefore $N : H \to H$ is monotone when $n = 1$ and N is c-strongly monotone and N_j are k_j-contractive, $j = 1,2$, when $n > 1$.

Next, we shall show that N is a bounded asymptotically $\{B_1, B_2\}$-quasilinear map with $B_1 = C_1 - \epsilon I$ and $B_2 = C_2 + \epsilon I$ for some $\epsilon > 0$, where C_1 and C_2 are $m \times m$ diagonal matrices with the diagonal entries $a_1(t,x), \cdots, a_m(t,x)$ and $\beta_1(t,x), \cdots, \beta_m(t,x)$ respectively. By (3.9), for $\epsilon > 0$ there is an $r > 0$ such that for each $1 \leq l \leq m$, for $a.e. (t,x) \in \Omega$ and all $y = (y_1, \dots, y_l, \dots, y_m) \in R^m$ with $|y_l| \geq r$

$$a_l(t,x) - \epsilon \leq F_l(t,x,y)y_l^{-1} \leq \beta_l(t,x) + \epsilon. \tag{3.14}$$

Hence, by (3.8),

$$|F_l(t,x,y)| \leq (\lambda_{i+1} + \epsilon)|y_l| + h_r(t,x) \quad \text{for a.e. } (t,x) \in \Omega,\ y \in R^m,$$

and therefore N is continuous and bounded in H. For each $1 \le l \le m$, define a function $G_l : \Omega \times R^m \to R$ by (cf. [**Ma-Wa**])

$$G_l(t,x,y) = \begin{cases} y_l^{-1} F_l(t,x,y), & \text{if } |y_l| \ge r \\ r^{-1} F_l(t,x,y_1,\dots,y_{l-1},r,y_{l+1},\dots,y_m)(y_l/r) & \\ \qquad + (1 - y_l/r) a_l(t,x), & \text{if } 0 \le y_l \le r \\ r^{-1} F_l(t,x,y_1,\dots,y_{l-1},-r,y_{l+1},\dots,y_m)(y_l/r) & \\ \qquad + (1 + y_l/r) a_l(t,x), & \text{if } -r \le y_l \le 0. \end{cases}$$

Using (3.14), it is easy to check that for $1 \le l \le m$

$$a_l(t,x) - \epsilon \le G_l(t,x,y) \le \beta_l(t,x) + \epsilon \quad \text{for a.e. } (t,x) \in \Omega,\ y \in R^m.$$

Moreover, $H_l(t,x,y) = F_l(t,x,y) - G_l(t,x,y)y_l$ is a Caratheodory function on $\Omega \times R^m$ for $1 \le l \le m$ and

$$|H_l(t,x,y)| \le 2h_r(t,x) \quad \text{for a.e. } (t,x) \in \Omega,\ y \in R^m.$$

Let $G(t,x,y)$ be the $m \times m$ diagonal matrix with the diagonal entries

$$G_1(t,x,y), \dots, G_m(t,x,y) \quad \text{for} \quad (t,x) \in \Omega,\ y \in R^m.$$

Similarly, let $N(t,x,y)$ be the $m \times m$ diagonal matrix with the diagonal entries $H_1(t,x,y), \dots, H_m(t,x,y)$.

Now, for each $u \in H$, define $B(u) : H \to H$ by $B(u)v = G(t,x,u)v$ and $Mu = H(t,x,u)$. Then, $Nu = B(u)u + Mu$ on H, and it is easy to check that $B(u)$ is selfadjoint and $B_1 \le B(u) \le B_2$ for each $u \in H$. Since $\|Mu\| \le 2\|h_r\|$ for each $u \in H$, it follows that N is asymptotically $\{B_1, B_2\}$-quasilinear. Moreover, by Lemma 2.4, conditions (2.10)–(2.11) hold for any selfadjoint maps $B^{\pm}$ with $C_1 \le B^-$ and $B^+ \le C_2$, where H^+ (resp., H^-) is a subspace of H spanned by the eigenfunctions of A corresponding to the eigenvalues $\lambda_k \le \lambda_i$ (resp., $\lambda_k \ge \lambda_{i+1}$).

Finally, define the selfadjoint map $C_0 : H \to H$ by $C_0 u : a(t,x)u$, where $a(t,x)$ is the $m \times m$ diagonal matrix with the diagonal entries $a_1(t,x), \dots, a_m(t,x)$, for $(t,x) \in \Omega$. Since each $a_l(t,x) \ge \lambda_i > 0$ *a.e.* on Ω, it follows that C_0 is continuous and a_i-strongly monotone. Hence, $H(t,x) = A - (1-t)C_0 - tN$ is A-proper w.r.t. $\Gamma = \{H_k, P_k\}$ with $P_k Ax = Ax$ on H_k for each $t \in [0,1)$ if $n \ge 1$, and H_1 is pseudo A-proper if $n = 1$ and is A-proper if $n > 1$ by Propositions 2.3 and 2.6 in [**Mi-8**]. Choosing $\epsilon > 0$ as in Lemma 2.4, we see that the conclusion of the theorem follows from Theorem 2.4. □

Remark 3.2 When $n = m = 1$, Theorem 3.2 with the Dirichlet boundary conditions for the semilinear wave equation (3.7) was proved by Mawhin-Ward [**Ma-Wa**] using rather different arguments based on the compactness of A^{-1} and the coincidence degree theory of Mawhin. Hence, their method is not applicable when $n > 1$ since A^{-1} is not compact due to the presence of nonzero eigenvalues of infinite multiplicity.

Remark 3.3 In view of Remark 2.3, Theorem 3.2 still remains valid if $\lambda_i = 0$ (resp., $\lambda_{i+1} = 0$) provided each a_l (resp., β_l) is a constant.

IV. Semilinear Equations at Resonance

In this section, we study Eq. (1.1) when $\dim\ker A = \infty$ and the nonlinearity interacts in some sense with the spectrum of A. Using a direct method, we first prove an extension of a theorem of Brezis- Nirenberg [**Br-Ni-1**] to pseudo A-proper maps $A - N$. Then, we use a perturbation method to study Eq. (1.1) assuming $A - N$ is a closed map. Moreover, when N is a gradient map, we also give a necessary and sufficient condition for the solvability of Eq. (1.1).

4.1 Direct Method. Assume that $A : D(A) \subset H \to H$ satisfies:

(4.1) There are positive constants $a_\pm$ and a_0 such that
 (i) $-a_+^{-1}\|Ax\|^2 \le (Ax, x) \le a_-^{-1}\|Ax\|^2$ for $x \in D(A)$,
 (ii) $\|x\| \le a_0\|Ax\|$ for $x \in \tilde{H} = \ker A^\perp$.

Let $\mathrm{Int}(D)$ denote the interior of D and $\mathrm{conv}\, D$ be the convex hull of D. We have [**Mi-7**]

THEOREM 4.1 *Let a linear closed map $A : D(A) \subset H \to H$ satisfy* (4.1) *and $N : H \to H$ be such that $\pm A + N : D(A) \subset H \to H$ is pseudo A-proper w.r.t. $\Gamma = \{H_n, P_n\}$ with $P_nAx = Ax$ on H_n. Suppose that*

(4.2) *There are $\gamma < a_\pm$ and $\tau < a_0^{-2}(\gamma^{-1} - a_\pm^{-1})$ such that for every $y \in H$ and every $\delta > 0$ there exist $c_i(y)$, $i = 1, 2$, and $k(\delta)$ such that for each $x \in H$*

$$(Nx - Ny, x) \ge \gamma^{-1}\|Nx\|^2 - c_1(y)\|Nx\| - \tau\|x_1\| - c_2(y)\|(\delta\|x_0\| + k(\delta)).$$

Then $\mathrm{Int}\,(R(A) + \mathrm{conv}\, R(N)) \subset R(\pm A + N)$. *Moreover, if N is onto, so is $\pm A + N$.*

Proof. We shall consider only $+A + N$, since the case $-A + N$ can be done in a similar way. Let $f \in \mathrm{Int}\,(R(A) + \mathrm{conv}\, R(N))$ be fixed and assume that

$$Ax_n + (1 - t_n)P_nPx_n + t_nP_nNx_n = t_nP_nf$$

for some $x_n \in H_n$, $t_n \in (0, 1]$ and each n, where $P : H \to \ker A$ is the orthogonal projection. We shall show that $\|x_n\| < r$ for some r and all n.

For each $h \in H$ with sufficiently small norm, there are $v \in D(A)$, $w_i \in H$ and $\lambda_i \ge 0$ with $\sum_{i=1}^k \lambda_i = 1$ such that

$$f + h = Av + \sum_{i=1}^{k} \lambda_i Nw_i.$$

Substituting this in the above equation, we get

$$(1 - t_n)P_nPx_n + t_nP_n\left(Nx_n - \sum_{i=1}^{k} \lambda_i Nw_i\right) + t_nP_nh = t_nP_nAv - Ax_n.$$

Taking the scalar product with x_n, we find

$$(1-t_n)\|Px_n\|^2 + t_n \sum_{i=1}^{k} \lambda_i \left(Nx_n - Nw_i, x_n\right) + t_n(h, x_n) = t_n(Av, x_n) - (Ax_n, x_n).$$

Hence, by (4.2), for each $\delta > 0$,

$$\begin{aligned}(1-t_n)\|Px_n\|^2 + t_n \sum_{i=1}^{k} \lambda_i(\gamma^{-1}\|Nx_n\|^2 - c_1(w_i)\|Nx_n\| - \tau\|x_{1n}\|^2 \\ - c_2(w_i)\left(\delta\|x_{0n}\| + k(\delta)\right)) + t_n(h, x_n) \\ \le t_n\|Av\|\,\|x_{1n}\| + a_+^{-1}\|Ax_n\|^2,\end{aligned}$$

or, after dropping the first term,

$$\begin{aligned}(h, x_n) + \gamma^{-1}\|Nx_n\|^2 \le c(h)\|Nx_n\| + \tau\|x_{1n}\|^2 + \|Av\|\,\|x_{1n}\| \\ + (t_n a_+)^{-1}\|Ax_n\|^2 + c(h)(\delta\|x_{0n}\| + k(\delta))\end{aligned}$$

for some constant $c(h)$. But, since

$$Ax_n + t_n(I-P)P_nNx_n = t_n(I-P)P_nf$$

we get that

$$\|Ax_n\| \le t_n\left(\|Nx_n\| + \|f\|\right) \quad \text{and} \quad \|x_{1n}\| \le a_0 t_n\left(\|Nx_n\| + \|f\|\right).$$

Hence,

$$\begin{aligned}(h, x_n) &\le a_+^{-1}\left(\|Nx_n\| + \|f\|\right)^2 - \gamma^{-1}\|Nx_n\|^2 + \tau a_0^2\left(\|Nx_n\| + \|f\|\right)^2 \\ &\quad + c\|Nx_n\| + a_0\|Av\|\left(\|Nx_n\| + \|f\|\right) + c(h)\left(\delta\|x_{0n}\| + k(\delta)\right) \\ &= (a_+^{-1} - \gamma^{-1} + \tau a_0^2)\|Nx_n\|^2 + \big(c(h) + 2a_+^{-1}\|f\| + 2\tau a_0^2\|f\| + \\ &\quad + a_0\|Av\|\big)\|Nx_n\| + c(h)\left(\delta\|x_{0n}\| + k(\delta)\right) + c_1(h),\end{aligned}$$

where $c_1(h) = (a_+^{-1} + \tau a_0^2)\|f\|^2 + a_0\|Av\|\,\|f\|$. Since $\tau a_0^2 < \gamma^{-1} - a_+^{-1}$, it follows that, for each $\delta > 0$ and some $c_2(h)$ large enough,

$$\begin{aligned}(h, x_n) &\le c(h)\left(\delta\|x_{0n}\| + k(\delta)\right) + c_1(h) \\ &\le c_2(h)\left(\delta\|x_n\| + k(\delta)\right).\end{aligned}$$

Suppose that $\|x_n\| \to \infty$ as $n \to \infty$, and for $r > 0$, set $w(r) = \inf_{\delta<0}\{\delta r + k(\delta)\}$ so that $w(r)/r \to 0$ as $r \to \infty$. Since $(h, x_n) \le w(\|x_n\|)c_2(h)$, the uniform boundedness principle implies that $\|x_n\|/w(\|x_n\|) \le C$, a contradiction. Hence, $\|x_n\| < r$ for some $r > 0$ and all n. Consequently, $H_n(t, x) = Ax + (1-t)P_nPx + tP_nNx \ne tP_nf$ on $[0,1] \times \partial B(0,r) \cap H_n$ for each n, and the Brouwer degree

$$\deg(A + P_nN, B(0,r) \cap H_n, P_nf) = \deg(A + P_nP, B(0,r) \cap H_n, 0) \ne 0.$$

Thus, $Ax_n + P_nNx_n = P_nf$ for some $x \in B(0,r) \cap H_n$ and each n, and consequently, $Ax + Nx = f$ for some $x \in D(A)$ by the pseudo A-properness of $A + N$. □

Now, condition (4.2) implies that $\limsup_{\|x\|\to\infty} \|Nx\|/\|x\| \le \gamma$, i.e. N is a quasibounded map with the quasinorm $|N| \le \gamma$. As pointed out in [**Br-Ni-1**], if N is potential and quasibounded, then a condition of type (4.2) holds. Moreover, if a is the smallest positive constant such that $(Ax,x) \ge -a^{-1}\|Ax\|^2$ on $D(A)$ and A is selfadjoint, then $Ax = \lambda x$ implies that $\lambda \ge -a$ and $-a$ is an eigenvalue of A. Hence, roughly speaking, the conditions $0 < \gamma < a$ and $\limsup_{\|x\|\to\infty} \|Nx\|/\|x\| \le \gamma$ mean that the nonlinearity N asymptotically stays away from the nonzero eigenvalues of A.

Note that $a_\pm$ in (4.1) exist since, by the boundedness of $A^{-1} : R(A) \subset H \to H$, $\|Ax_1\| \ge a\|x_1\|$ for some $a > 0$ and all $x_1 \in D(A) \cap R(A)$, and therefore $(Ax,x) = (Ax,x_1) \ge -\|Ax\|\,\|x_1\| \ge -a^{-1}\|Ax\|^2$ for all $x \in D(A)$. Here we used the fact that $R(A) = N(A)^\perp$. If A is sefadjoint and $a < \infty$ is the smallest positive constant such that $(Ax,x) \ge -a^{-1}\|Ax\|^2$ on $D(A)$, we have shown above that $-a$ is the largest eigenvlaue of A less than 0. More generally, suppose that $A : D(A) \subset H \to H$ is a normal linear map with closed range. If H_c is the complexification of the real Hilbert space H and $A_c : D(A_c) \subset H_c \to H_c$ is defined by $A_c(x+iy) = Ax + iAy$ on $D(A_e) = \{x + iy \mid x, y \in D(A)\}$, then Hetzer [**He**] has shown that

$$a = a(A) = \inf\{|\lambda|^2/(-\operatorname{Re}\lambda) \mid \lambda \in \sigma(A_c), \operatorname{Re}\lambda < 0\}.$$

In general a could be infinite, in which case A is a linear monotone map, and could belong to $\sigma(A_c)$ or be a regular value of A_c.

Theorem 4.1 has been proved by Brezis-Nirenberg [**Br-Ni-1**] when $N : H \to H$ is a monotone map and the partial inverse A^{-1} is compact using the Leray-Shauder and monotone operator theories. In view of the various examples of pseudo A-proper maps $A + N$ discussed in [**Mi-4-8**], this result holds also for many other classes of maps A and N, even when A^{-1} is not compact. For example, we have the following corollary.

Corollary 4.1 *Let A and N satisfy conditions* (4.1)–(4.2) *and $N = N_1 + N_2$ be such that N_1 is c-strongly monotone, k_1-ball contractive and N_2 is k_2 ball contractive, with k_1, k_2 sufficiently small, and continuous. Then $A+N$ is surjective.*

Proof. The map $A + N : D(A) \subset H \to H$ is A-proper by Proposition 2.6 in [**Mi-8**] (cf. also [**Mi-4**]). Moreover, since N is c-strongly monotone, it is well-known to be subjective and the conclusion follows from Theorem 4.1. □

For our second corollary, we need to recall the following result proven in [**Mi-7**].

Proposition 4.1 *Let $A : D(A) \subset H \to H$ be selfadjoint, $H^\pm$ be closed subspaces of H with $H = H^- \oplus H^+$ and $H^- \cap H^+ = \{0\}$ and $\Gamma = \{H_n, P_n\}$ be a scheme for H that satisfies* (2.21). *Suppose that $N : H \to H$ has a symmetric weak Gateaux derivative $N'(x)$ on H and there are symmetric maps $B^\pm \in L(H)$ such that $B^- \le N'(x) \le B^+$ for each $x \in H$ and*

(4.3) $$((A - B^-)x, x) \le 0 \qquad \text{for } x \in D(A) \cap H^-;$$

(4.4) $$((A - B^+)x, x) \ge 0 \qquad \text{for } x \in D(A) \cap H^+.$$

Then $\pm A + N : D(A) \subset H \to H$ *is pseudo A-proper w.r.t.* Γ.

In view of Proposition 4.1, we have the following special case of Theorem 4.1.

COROLLARY 4.2 *Let* $A : D(A) \subset H \to H$ *and* $N : H \to H$ *be as in Proposition* 4.1 *and satisfy conditions* (4.1)–(4.2). *Then* $\mathrm{Int}\,(R(A) + \mathrm{conv}\, R(A)) \subset R(\pm A + N)$ *and* $\pm A + N$ *is surjective if such is* N.

Next, we shall discuss some conditions that imply (4.3)–(4.4). Let $A : D(A) \subset H \to H$ be selfadjoint and suppose that

(4.5) $B^\pm = \sum_{i=1}^m \lambda_i^\pm P_i^\pm$ commute with A, where $P_i^\pm : H \to \ker(B^\pm - \lambda_i^\pm I)$ are orthogonal projections, $\lambda_1^\pm \le \cdots \le \lambda_m^\pm$ and are pairwise disjoint;

(4.6) $\bigcup_{i=1}^m [\lambda_i^-, \lambda_i^+] \subset \overline{\rho(A)}$, the closure of the resolvent set of A.

As we have seen in Section 2.2, $A - B^\pm$ have the spectral resolution

$$A - B^\pm = \sum_{i=1}^m \int_{-\infty}^{\infty} (\lambda - \lambda_i^\pm)\, dE_\lambda P_i^\pm.$$

Let δ be small enough such that $\mu_i^\pm = \lambda_1^\pm \mp \delta$ satisfy

(4.7) $$\bigcup_{i=1}^m [\mu_i^-, \mu_i^+] \subset \rho(A).$$

Then the operators $B_\delta^\pm = B^\pm \mp \delta I$ have $\mu_i^\pm$ as their eigenvalues and $\ker(B_\delta^\pm - \mu_i^\pm) = \ker(B^\pm - \lambda_i^\pm)$. Since $B_\delta^\pm$ commute with A, the spectral resolutions of $A - B_\delta^\pm$ are

(4.8) $$A - B^\pm = \sum_{i=1}^m \int_{-\infty}^{\infty} (\lambda - \mu_i^\pm) dE_\lambda P_i^\pm.$$

Define the orthogonal projections $P^\pm$ by

$$P^+ = \sum_{i=1}^m E(\mu_i^+, \infty) P_i^+ \quad \text{and} \quad P^- = \sum_{i=1}^m E(-\infty, \mu_i^-) P_i^-$$

and let $H^\pm = P^\pm(H)$. Note that by (4.7)

(4.9) $$P^+ = \sum_{i=1}^m E(\mu_i^-, \infty) P_i^+$$

and

$$\mathrm{dist}\left(\bigcup_{i=1}^m [\mu_i^-, \mu_i^+], \sigma(A) \right) \ge \delta.$$

Moreover, it follows from (4.8) that $((A - B_\delta^-)x, x) \leq -\delta\|x\|^2$ on $D(A) \cap H^-$ and $((A - B_\delta^-)x, x) \geq \delta\|x\|^2$ on $D(A) \cap H^+$. Hence, conditions (4.3)–(4.4) hold. Moreover, if we assume that $P_i^- = P_i^+$ for $1 \leq i \leq m$, then by (4.9), $P^+ = I - P^-$, i.e., $H^+ = (H^-)^\perp$. □

By the discussion above, we have

COROLLARY 4.3 *Let $A : D(A) \subset H \to H$ be selfadjoint, $H^\pm$ be closed subspaces of H with $H = H^- \oplus H^+$ and $H^- \cap H^+ = \{0\}$ and $\Gamma = \{H_n, P_n\}$ satisfy* (2.21). *Suppose that $N : H \to H$ has a symmetric weak Gateaux derivative $N'(x)$ on H and $B^- \leq N'(x) \leq B^+$ for $x \in H$ and some selfadjoint maps $B^\pm \in L(H)$. If conditions* (4.1)–(4.2) *and* (4.5)–(4.6) *hold, then* $\operatorname{Int}(R(A) + \operatorname{conv} R(A)) \subset R(\pm A + N)$ *and $\pm A + N$ is surjective if such is N.*

A particular case of Corollary 4.3 is when $B^- = \lambda_k I$ and $B^+ = \lambda_{k+1} I$ for two consective eigenvalues $\lambda_k < \lambda_{k+1}$ of A. If H^- (resp. H^+) is the subspace of H spanned by the eigenvectors of A corresponding to the eigenvalues $\lambda_i \leq \lambda_k$ (resp. $\lambda_i \geq \lambda_{k+1}$), then $H^+ = (H^-)^\perp$ and $H = H^- \oplus H^+$. Let $\Gamma = \{H_n, P_n\}$ be a scheme for H with $P_n A x = Ax$ and $P^\pm : H \to H^\pm$ be the orthogonal projections onto $H^\pm$. Then Γ satisfies (2.21) with $H_n = H_n^- \oplus H_n^-$ and $H_n^\pm = H_n \cap H^\pm$.

Now, since $B^- \leq N'(x) \leq B^+$ on H, the mean value theorem implies that $\|Nx - Ny\| \leq \max\{\|B^-\|, \|B^+\|\}\,\|x - y\|$ for $x, y \in H$. In particular, taking $B^- = \lambda_k I$ and $B^+ = \lambda_{k+1} I$, we get

$$\limsup_{\|x\|\to\infty} \|Nx\|/\|x\| \leq \max\{|\lambda_k|, |\lambda_{k+1}|\},$$

and, by (4.2) as noted above

$$\limsup_{\|x\|\to\infty} \|Nx\|/\|x\| \leq \gamma < a,$$

Hence, for N to interact with λ_k (or λ_{k+1}), it must be the zero eigenvalue.

4.2 A Perturbation Method. In this section we shall present another way of studying Eq. (1.1) by looking at the perturbed equations

$$Ax + Nx + \epsilon Gx = f, \quad \epsilon > 0 \tag{4.10}$$

where G is a bounded map. This approach consists of three basic steps. The first step is to establish the solvability of (4.10) for each $\epsilon \in (0, \epsilon_0)$; the second step is to obtain an a priori bound on the solutions, i.e. to show that the set $\{x_\epsilon \mid x_\epsilon$ is a solution of (4.10)$\}$ is bounded in a suitable space as $\epsilon \to 0$, and the final step is to show that a weak limit of $\{x_\epsilon\}$ is a solution of $Ax + Nx = f$.

The approach requires a closedness property of $A + N$, which is defined next.

Definition 4.1 Let X be a Banach space embedded in H. We say that $A + N : D(A) \cap X \to H$ satisfies condition $(*)$ if whenever $\{x_\epsilon \mid Ax_\epsilon + Nx_\epsilon + \epsilon Gx_\epsilon =$

$= f,\ 0 < \epsilon < \epsilon_0\}$ is bounded as $\epsilon \to 0$ (in X or H), then there is an $x \in D(A)$ such that $Ax + Nx = f$.

The following result gives various conditions on A and N which guarantee that condition $(*)$ holds.

PROPOSITION 4.2 *Let $A : D(A) \cap X \to H$ be a closed linear densely defined map and N be a nonlinear map. Suppose that either A^{-1} is compact and N monotone, or the conditions of Corollary 4.1 and Proposition 4.1 hold. Then $A + N$ satisfies condition $(*)$.*

Proof. (Sketch) Let $\{x_\epsilon \mid Ax_\epsilon + Nx_\epsilon + \epsilon Gx_\epsilon = f,\ 0 < \epsilon < \epsilon_0\}$ be bounded in a corresponding space. Then, using similar arguments as in the proofs of Propositions 5.1 in **[Mi-4]**, Corollary 4.1 and Proposition 4.1, one can show that the weak limit x of $\{x_\epsilon\}$ solves the equation $Ax + Nx = f$. □

The problem of getting a priori estimates for (4.10), i.e. of showing the boundedness of $\{x_\epsilon\}$, is harder to handle. Our basic result is **[Mi-6]**:

THEOREM 4.2 *Let $A : D(A) \subset H \to H$ be a linear densely defined selfadjoint map with $R(A) = N(A)^\perp$ and $a > 0$ be the largest number such that $(Ax, x) \geq -a^{-1}\|Ax\|^2$ on $D(A)$. Suppose that $N : H \to H$ is nonlinear, condition $(*)$ holds and*

(4.11) *There is a decomposition $f = f^* + f^{**}$ with $f^* \in R(A)$ such that for some $\gamma < a$ and a constant c*

$$(Nx - f^{**}, x) \geq \gamma^{-1}\|Nx\|^2 - c \text{ for } x \in H,$$

(4.12) *Eq. (4.10) with $G = I$ is solvable for each $0 < \epsilon < \epsilon_0$ and either*

(4.13) *the set $\{x_\epsilon \mid Ax_\epsilon + Nx_\epsilon + \epsilon x_\epsilon = f,\ \|Ax_\epsilon\| \leq C,\ \|Nx_\epsilon\| \leq C$ for all $0 < \epsilon < \epsilon_0$ and some $C\}$ is bounded in H, or*

(4.14) *$\|N(x_0 + x_1)\| \to \infty$ as $\|x_0\| \to \infty$, $x_0 \in N(A)$, uniformly for x_1 in bounded subsets of $R(A)$.*

Then Eq. (1.1) is solvable.

Proof. Let $\epsilon > 0$ be small and $Ax_\epsilon + Nx_\epsilon + \epsilon x_\epsilon = f$ for some $x_\epsilon \in D(A)$. We need to show that $\{x_\epsilon\}$ is bounded as $\epsilon \to 0$. Taking the inner product with x_ϵ, we find

$$\epsilon\|x_\epsilon\|^2 + (Nx_\epsilon - f, x_\epsilon) = -(Ax_\epsilon, x_\epsilon) \leq a^{-1}\|Ax_\epsilon\|^2,$$

and by (4.11),

$$\epsilon\|x_\epsilon\|^2 + \gamma^{-1}\|Nx_\epsilon\|^2 \leq -(Ax_\epsilon, x_\epsilon) + (f^{**}, x_\epsilon) + C\|Nx_\epsilon\| + C.$$

Since $A\nu = f^*$ for some $\nu \in D(A)$ and $(A(x_\epsilon - \nu), x_\epsilon - \nu) \geq -a^{-1}\|A(x_\epsilon - \nu)\|^2$, we get

$$\begin{aligned}\epsilon\|x_\epsilon\|^2 + \gamma^{-1}\|Nx_\epsilon\|^2 &\leq a^{-1}\|A(x_\epsilon - \nu)\|^2 - (A(x_\epsilon - \nu), \nu) + C\|Nx_\epsilon\| + C \\ &\leq a^{-1}\|Ax_\epsilon\|^2 + C\|Ax_\epsilon\| + C\|Nx_\epsilon\| + C.\end{aligned}$$

For each $\gamma < \beta < a$ there is a $C(\beta)$ such that

$$\gamma^{-1}\|Nx_\epsilon\|^2 - C\|Nx_\epsilon\| \geq \beta^{-1}\|Nx_\epsilon\|^2 - C(\beta).$$

Similarly, for each $\beta < \delta < a$ there is a $C(\delta)$ such that

$$\epsilon\|x_\epsilon\|^2 + \beta^{-1}\|Nx_\epsilon\| \leq a^{-1}\|Ax_\epsilon\|^2 + C\|Ax_\epsilon\| + C(\beta) \leq \delta^{-1}\|Ax_\epsilon\|^2 + C(\beta,\delta),$$

where $C(\beta,\delta) = C(\beta) + C(\delta)$. Since $Nx_\epsilon = f - Ax_\epsilon - \epsilon x_\epsilon$, the last inequality implies that

$$(4.15)\quad \epsilon\left[(1+\epsilon\beta^{-1})\|x_\epsilon\|^2 - 2\beta^{-1}\|f\|\,\|x_\epsilon\|\right] + \left(\beta^{-1} - \delta^{-1} - 2\epsilon(a\delta)^{-1}\right)\|Ax_\epsilon\|^2 - 2\delta^{-1}\|f\|\,\|Ax_\epsilon\| \leq C(\beta,\delta).$$

This implies that for some sufficiently small constants $C_i > 0$, $i = 1,2$, independent of ϵ,

$$\epsilon C_1\|x_\epsilon\|^2 + C_2\|Ax_\epsilon\|^2 \leq C_3,$$

and therefore $\|Ax_\epsilon\| \leq C$ for all small ϵ. Moreover, $\|Nx_\epsilon\| \leq C$ by (4.15) and consequently $\|x_\epsilon\| \leq C$ for all small ϵ if $A+N$ satisfies (4.13). Thus, $Ax_\epsilon + Nx_\epsilon = f - \epsilon x_\epsilon \to f$ as $\epsilon \to 0$ and $Ax + Nx = f$ for some $x \in D(A)$ by condition $(*)$.

Next, if instead of (4.13) we assume (4.14), then $x_\epsilon = x_{0\epsilon} + x_{1\epsilon}$ and $\|x_{1\epsilon}\| = \|A^{-1}Ax_\epsilon\| \leq C\|A^{-1}\|$ for all small ϵ. Moreover, $\|x_{0\epsilon}\| \leq C$ for such ϵ by (4.14), and consequently $\{x_\epsilon\}$ is bounded. Then, the conclusion follows as above. □

When N is a gradient map, we have the following sufficient and necessary conditions for the solvability of (1.1) (see [**Mi-6**]).

THEOREM 4.3 *Let $A : D(A) \subset H \to H$ be a linear selfadjoint map and $a > 0$ be the largest number such that $(Ax,x) \geq -a^{-1}\|Ax\|^2$ on $D(A)$. Assume that either (4.13) or (4.14) holds, $N = \partial F$ for some convex function $F : H \to R$ and*

$$(4.16)\qquad \limsup \|Nx\|/\|x\| < a/2, \quad \text{as } \|x\| \to \infty.$$

Suppose that either one of the following conditions holds

(4.17) *$A^{-1} : R(A) \subset H \to H$ is compact,*

(4.18) *$0 \in \sigma(A)$ and $\sigma(A) \cap (0,\infty) \neq \emptyset$ and consists of eigenvalues of finite multiplicities,*

(4.19) *There are closed subspaces $H^\pm$ of H with $H = H^- \oplus H^+$ and $H^- \cap H^+ = \{0\}$, a scheme $\Gamma = \{H_n, P_n\}$ satisfying (2.21) and N has a symmetric weak Gateaux derivative N' on H such that for some symmetric maps $B^\pm \in L(H)$ with $B^- \leq N'(x) \leq B^+$ on H we have*

$$\text{(i)}\quad ((A - B^-)x, x) \leq 0 \quad \textit{for } x \in D(A) \cap H^-;$$
$$\text{(ii)}\quad ((A - B^+)x, x) \geq 0 \quad \textit{for } x \in D(A) \cap H^+$$

Then, Eq. (1.1) is solvable if and only if $f = f^* + f^{**}$ with $f^* \in R(A)$ and $f^{**} \in R(N)$.

Proof. If x is a solution of $Ax + Nx = f$, then $f^* = Ax$ and $f^{**} = Nx$. Conversely, let $f = f^* + f^{**}$ with $f^* \in R(A)$ and $f^{**} \in R(N)$. Then (4.16) and Proposition A.4 in [**Br-Ni-1**] imply that N and $N_\epsilon = N + \epsilon I$ satisfy (4.2) and (4.11), respectively, for each $\epsilon > 0$ small. Moreover, N_ϵ is monotone and $\|N_\epsilon x\| \to \infty$ as $\|x\| \to \infty$ and therefore it is surjective.

Since $A + N_\epsilon$ is pseudo A-proper by Propositions 2.2 in [**Mi-4**] and 4.1, Eq. (1.1) is solvable for each small $\epsilon > 0$ by Theorem 4.1. Finally, since condition $(*)$ holds by Proposition 4.2, Eq. (1.1) is solvable by Theorem 4.2. □

When $H = L_2(Q, R^m)$ we can relax condition (4.14) in Theorem 4.2 (cf. [**Mi-6**]).

COROLLARY 4.4 *Let $Q \subset R^n$ be a bounded domain, $H = L_2(Q, R^m)$ and $Nu = D_u F(x, u)$ with $F : Q \times R^m \to R$ measurable in x and convex and C^1 in u. Then, condition* (4.14) *in Theorem* 4.2 *can be weakened to*

$$\int_Q F(x, u_0(x))dx \to \infty \quad \text{as } \|u_0\|_{L_2} \to \infty, \ u_0 \in N(A).$$

Remark 4.2 When $n = 1$, $a/2$ in (4.16) can be replaced by a by Proposition A.6 in [**Br-Ni-1**].

V. Periodic Solutions of Semilinear Wave Equations at Resonance

In this section we shall apply some of the abstract results from the previous section to semilinear (systems) of wave equations (3.1) allowing some type of interaction of the nonlinearity F with the spectrum of the linear problem.

We begin with strongly monotone and contractive nonlinearities. Suppose $\Omega = (0, T) \times Q$ and

(5.1) Let $F_i : \Omega \times R^m \to R^m$ be Caratheory functions, $i = 1, 2$, such that for some positive constants k_i and c, with k_1, k_2 sufficiently small,

$$(F_1(t, x, y) - F_1(t, x, z)) \cdot (y - z) \geq c|y - z|^2$$

and

$$|F_i(t, x, y) - F_i(t, x, z)| \leq k_i |y - z|$$

for a.e. $(t, x) \in \Omega$ and all $y, z \in R^m$.

(5.2) There are $a > 0$ and $h \in L_2(\Omega)$ such that for $\gamma_1 = 1$, and $\gamma_2 \in (0, 1)$

$$|F_i(t, x, y)| \leq a|y|^{\gamma_i} + h(t, x) \quad \text{for a.e. } (t, x) \in \Omega, \ y \in R^m.$$

Let L and V be as introduced in Section III. Then, we have shown there that a linear map $A : D(A) \subset L_2 \to L_2$, induced by (3.1), is selfadjoint and $R(A) = N(A)^\perp$. Moreover, condition (4.1) holds for some $a_\pm$ and a_0.

We have

THEOREM 5.1 *Let* (5.1)–(5.2) *hold with* $a < a_+/2$ *and* $m > 1$ *and* $F = \partial\psi$ *for some function* $\psi(t,x,y) : \Omega \times R^m \to R$ *measurable in* (t,x) *and differentiable and convex in* y. *Then there is a* T*-periodic weak solution* $u \in L_2$ *for each* $f \in L_2$.

Proof. Define $N_i u = F_i(t,x,u)$ for $u \in L_2$ and let $N = N_1 + N_2$. By (5.1), N_1 is c-strongly monotone and k_1-ball contractive and N_2 is k_2-ball contractive. Hence, $A + N$ is A-proper w.r.t. $\Gamma = \{H_n, P_n\}$ for L_2 with $P_n A u = Au$ for $u \in H_n$ by Proposition 2.6 in **[Mi-8]** (cf. also **[Mi-4]**). Moreover, $\|N_2 u\|/\|u\| \to 0$ as $\|u\| \to \infty$ and

$$(Nu,u) = (N_1 u, u) - \|N_2 u\|\,\|u\| \geq c\|u\|^2 - (\|N_1 0\| + \|N_2 u\|)\,\|u\|.$$

Thus, N is coercive, i.e. $(Nu,u)/\|u\| \to \infty$ as $\|u\| \to \infty$, and monotone and therefore, it is surjective.

Next, by Proposition A.4 in **[Br-Ni-1]**, there is a $\gamma < a_+$ such that

$$(Nu - Nv, u) \geq \gamma^{-1}\|Nu\|^2 - c(v) \quad \text{for all } u, v \in L_2$$

and some constant $c(v)$. Hence, by Theorem 4.1, the equation $Au - Nu = f$ is solvable for each $f \in L_2$, i.e., there is a T-periodic weak solution $u \in L_2$ of (3.1) for each $f \in L_2$. □

As it will be seen below, conditions on F can be greatly relaxed when $n = 1$. Next, we shall study the solvability of (3.1) for a given f.

THEOREM 5.2 *Let* $G(t,x,u) : \Omega \times R^m \to R$ *be* T*-periodic in* t, *measurable in* (t,x) *and convex and* C^1 *in* u *and* $F(t,x,u) = D_u G(t,x,u)$ *satisfy* (5.2). *Suppose that*

$$(5.3) \qquad \int_\Omega G(t,x,u)\,dt\,dx \to \infty \quad \text{as } \|u_0\|_{L_2} \to \infty,\ u_0 \in N(A).$$

Moreover, if $n > 1$ *assume that* $a < a_+/2$, (4.6) *holds and*

(5.4) $F \in C^1(R^m, R^m)$ *and has a symmetric derivative* $D_y F(t,x,y)$ *for a.e.* $(t,x) \in \Omega$ *and all* $y \in R^m$,

(5.5) *there exist two commuting* $m \times m$ *symmetric matrices* $b^\pm$ *such that* $b^- \leq D_y F(t,x,y) \leq b^+$ *for a.e.* $(t,x) \in \Omega$, $y \in R^m$

and, if $n = 1$, *assume that* $a < a_+$.

Then (3.1) *has a* T*-periodic weak solution* $u \in L_2$ *if and only if* $f = f^* + f^{**}$ *with* $f^* \in R(A)$ *and* $f^{**} \in R(N)$, *where* $Nu = F(t,x,u)$.

Proof. Let $n = 1$ and $a < a_+$. Then $A^{-1} : R(A) \subset L_2 \to L_2$ is compact and N satisfies condition (4.2) by Proposition A.1 in **[Br-Ni-1]**. If $n > 1$ and $a < a_+/2$, condition (4.2) holds by Proposition A.4 in **[Br-Ni-1]**. Let $\lambda_1^\pm \leq \cdots \leq \lambda_m^\pm$ be the eigenvalues of $b^\pm$ and set $B^\pm u = b^\pm u$. Then $B^\pm = \sum_{i=1}^m \lambda_1^\pm P_i^\pm$, where $P_i^\pm$ is the orthogonal projection onto $\ker(B^\pm - \lambda_i^\pm I)$. It is easy to show that, after a possible renumeration of the eigenvalues of $b^\pm$, we may assume that $P_i^+ = P_i^-$ for each i. Then, by our discussion in Section 4.1 the conclusion of the theorem follows from Theorem 4.3 and Corollary 4.4. □

COROLLARY 5.1 *Let $n = 1$ and $F(t,x,u) = D_uG(t,x,u)$ satisfy* (5.2) *with $h \in L_\infty(\Omega)$, $a < a_+$, and*

(5.6) $$G(t,x,u) \to \infty \quad as \quad |u| \to \infty \quad uniformly\ a.e.\ in\ \Omega.$$

Then there is 2π periodic weak solution $u \in L_2$ of

(5.7) $$\begin{cases} u_{tt} + u_{xxxx} + F(t,x,u) = f(t,x), & t \in R,\ x \in (0,\pi), \\ u(t,0) = u(t,\pi) = 0, & t \in R \\ u_{xx}(t,0) = u_{xx}(t,\pi) = 0 & \\ u(0,x) - u(2\pi,x) = u_t(0,x) = u_t(2\pi,x) = 0, & t \in R,\ x \in (0,\pi) \end{cases}$$

if and only if $f = f^* + f^{**}$ with $f^* \in R(A)$ and $f^{**} \in R(N)$.

Proof. Since $h \in L_\infty$ and G is convex in u, as in [**Ma-Wi**] there is $\delta > 0$ such that

(5.8) $$G(t,x,u) \geq \delta|u| - h(t,x) \text{ for a.e. } (t,x) \in \Omega,\ u \in R^m.$$

Since the L_1 and L_2 norms are equivalent on $N(A)$ (cf. [**Ba-Sa**]), this implies (5.3) and the conclusion follows from Theorem 5.2. □

When $f = 0$, we refer to [**Ma-Wi**] for related results obtained by a dual variational method. When $n = 1$, our results imply the following ones of Bahri-Brezis [**Ba-Br**] and Bahri-Sanchez [**Ba-Sa**] for the wave equation and (5.7), respectively.

COROLLARY 5.2 *Let $g : R \to R$ be continuous nondecreasing and for some $a < 3$, b and $\delta > 0$*

(5.9) $|g(u)| \leq a|u| + b$ *for all* $u \in R$,

(5.10) $f = f^* + f^{**} \in L_\infty(\Omega)$ *with* $f^* \in R(A)$ *and* $g(-\infty) + \delta \leq f^{**}(t,x) \leq g(+\infty) - \delta$ *a.e. in* Ω.

Then there is a weak solution of (5.7) *with F replaced by g, and of*

(5.11) $$\begin{cases} u_{tt} - u_{xx} + g(u) = f(t,x), & t \in R,\ x \in (0,\pi), \\ u(t,0) = u(t,\pi) = 0, & t \in R \\ u(t+2\pi,x) = u(t,x), & t \in R,\ x \in (0,\pi). \end{cases}$$

Proof. Consider first (5.7). We note that it is equivalent to

(5.12) $$Av + g(u^* + v) - f^{**} = Av + D_vG(t,x,v) = 0$$

where $u^* \in L_\infty(\Omega)$ is the solution of $Au = f^*$, $u = u^* + v$, and

$$G(t,x,v) = \int_0^v [g(u^* + s) - f^{**}(t,x)]\, ds.$$

Since (5.9)–(5.10) imply

$$(g(u) - f^{**}(t,x))\, u \geq \delta|u|/2 - C \text{ for } (t,x) \in \Omega,\ u \in R,$$

we have that (cf. [**Mi-6**]) (5.8) and therefore, (5.3) holds. Hence, the weak solvability of (5.7) follows from Theorem 5.2 applied to (5.12).

Next, consider (5.11). Conditions (5.9)–(5.10) and the monotonicity of g imply (4.11), (4.12) and $(*)$ and by Theorem 4.1 and Proposition 4.2. As shown by the proof in [**Br**], condition (4.13) also holds, and consequently the weak solvability of (5.11) follows from Theorem 4.2. □

Another condition on f^{**} is given in the following result (cf. [**Mi-8**]).

THEOREM 5.3 *Let the conditions of Theorem* 5.2 *hold. Suppose that* $f = f^* + f^{**} \in L_2(\Omega)$ *with* $f^* \in R(A)$ *and*

$$(F(t,x,y) - f^{**}(t,x)) \cdot y \geq c|F(t,x,y) - f^{**}(t,x)|\,|y| \tag{5.13}$$

for all $|y| \geq R$, *a.e.* $(t,x) \in \Omega$, *some* R, *and a sufficiently small* $c > 0$. *Then there is a T-periodic weak solution of* (3.1).

Proof. We shall first show that (5.2) and (5.13) imply condition (4.11). We have

$$\begin{aligned}(Nu - f^{**}, u) &= \int_{|u(t,x)|<R} (F(t,x,u(t,x)) - f^{**}(t,x)) \cdot u(t,x)\,dtdx \\ &\quad + \int_{|u(t,x)|\geq R} (F(t,x,u(t,x)) - f^{**}(t,x)) \cdot u(t,x)\,dtdx \\ &= I_1 + I_2.\end{aligned}$$

By (5.2),

$$\begin{aligned}|I_1| &\leq \int_{|u(t,x)|<R} (a|u(t,x)| + h(t,x) + |f^{**}(t,x)|)\,|u(t,x)|\,dtdx \\ &\leq aR^2 + R(\|h\| + \|f^{**}\|) = c_1.\end{aligned}$$

By (5.2) and (5.13), for $|u(t,x)| \geq R$

$$\begin{aligned}&(F(t,x,u(t,x)) - f^{**}(t,x)) \cdot u(t,x) \\ &\geq \frac{c}{a}\,|F(t,x,u(t,x)) - f^{**}(t,x)| \cdot (|F(t,x,u(t,x))| - h(t,x)) \\ &= \frac{c}{a}|F(t,x,u(t,x))|^2 - \frac{c}{a}|F(t,x,u(t,x))| \cdot (|f^{**}(t,x)| - h(t,x)) \\ &\quad - \frac{c}{a}|f^{**}(t,x)|h(t,x).\end{aligned}$$

Hence,

$$I_2 \geq \frac{c}{a}\|Nu\|^2 - \left(\frac{c}{a}\|f^{**}\| + \|h\|\right)\|Nu\| - \frac{c}{a}\|f^{**}\|\,\|h\|.$$

Since $C \leq \epsilon C^2 + c(\epsilon)$ for each $\epsilon > 0$, condition (4.11) follows easily from the estimates on I_1 and I_2.

Next, in view of Propositions $A.1$ and $A.4$ in [**Br-Ni-1**], condition (4.14) holds and therefore the equation $Ax + Nx + \epsilon x = f$ is solvable for each $\epsilon > 0$ as shown in the proof of Theorem 4.3. Hence by our discussion in Section 4.2, the conclusion follows from Theorem 4.2. ☐

Remark 5.1 It is easy to see that condition (5.10) implies condition (5.13).

REFERENCES

[Am-1] H. Amann (1979), *Saddle points and multiple solutions of differential equations*, Math. Zeitschr. **169**, 127–166.

[Am-2] H. Amann (1982), *On the unique solvability of semilinear operator equations in Hilbert spaces*, J. Math. Pures et Appl. **61**, 149–175.

[A-Z] H. Amann and E. Zehnder (1980), *Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations*, Annali Scuola Norm. Sup. Pisa, Ser. IV, **7**, 539–603.

[Ba-Br] A. Bahri and H. Brezis (1980), *Periodic solutions of a nonlinear wave equation*, Proc. Royal Soc. Edinburg, **A85**, 313b–320.

[Ba-Sa] A. Bahri and S. Sanchez (1981), *Periodic solutions of a nonlinear telegraph equation in one dimension*, Bull. Un. Mat. Ital. **5**, 18-B, 709–720.

[Br] H. Brezis (1983), *Periodic solutions of nonlinear vibrating strings and duality principles*, Bull. Amer. Math. Soc. **8**, 409–426.

[Br-Ni1] H. Brezis and L. Nirenberg (1978), *Characterizations of the ranges of some nonlinear operators and applications to boundary value problems*, Ann. Scuola Norm. Sup. Pisa **5**, 225–326.

[Br-Ni2] H. Brezis and L. Nirenberg (1978), *Forced vibrations for a nonlinear wave equation*, Comm. Pure Appl. Math. **31**, 1–30.

[Ce-Ka] L. Cesari and R. Kannan (1982), *Solutions of nonlinear hyperbolic equations at resonance*, Nonlinear Analysis, TMA, **6**, 751–805.

[Ch] K. C. Chang (1981), *Solutions of asymptotically linear equations via Morse theory*, Comm. Pure Appl. Math. **34**, 693–712.

[Co] J. M. Coron (1983), *Periodic solutions of nonlinear wave equations without assumption of monotonicity*, Math. Ann. **262**, 273–285.

[He] G. Hetzer (1984), *A spectral characterization of monotonicity properties of normal linear operators with an application to nonlinear telegraph equations*, J. Operator Theory **12**, 333–341.

[Ho] H. Hofer (1982), *On the range of a wave operator with non-monotone nonlinearity*, Math. Nachr. **106**, 327–340.

[Kr-Za] M. A. Krasnoselskiĭ and P. O. Zabreiko (1984), *Geometrical Methods of Nonlinear Analysis*, Springer-Verlag.

[L-L] S. Li and J. Q. Liu (1989), *Morse theory and asymptotic linear Hamiltonian system*, J. Differential Equations **78**, 57–73.

[Ma-1] J. Mawhin (1977), *Periodic solutions of nonlinear telegraph equations*, in: Bednarek-Cesari Ed., *Dynamical Systems*, Academic Press, 193–210.

[Ma-2] J. Mawhin (1981), *Compacticité, monotonie et convexite dans l'étude de problèmes aux limites semi-linéaires*, Lecture Notes, Univ. de Sherbrooke.

[Ma-3] J. Mawhin (1981), *Conservative systems of semi-linear wave equations with periodic-Dirichlet boundary conditions*, J. Diff. Equations **42**, 116–128.

[Ma-4] J. Mawhin (1982), *Nonlinear functional analysis and periodic solutions of semilinear wave equations*, in: Arlington, Lakshmikantham ed., *Nonlinear Phenomena in Math. Sci.*, Academic Press, 671–681.

[Ma-Wa] J. Mawhin and J. Ward (1983), *Asymptotic nonuniform nonresonance conditions in the periodic - Dirichlet problem for semilinear wave equations*, Annali Mat. Pura Appl. **135**, 85–97.

[Ma-Wi] J. Mawhin and M. Willem (1986), *Critical points of convex perturbations of some indefinite quadratic forms and semilinear boundary value problems at resonance*, Ann. de L'institut Henri Poincare — Analyse Nonlinéaire, **3** (6) 431–453.

[Mi-1] P. S. Milojević (1980), *Continuation theorems and solvability of equations involving nonlinear noncompact perturbations of Fredholm mappings*, Proc. 12th Seminario Brasileiro de Analize, ITA, Sao Jose dos Campos, 163–189.

[Mi-2] P. S. Milojević (1983), *Approximation — solvability results for equations involving nonlinear perturbations of Fredholm mappings with applications to differential equations*, in: G. Zapata, ed., *Proc. Int. Sem. Funct. Anal. Holom. Approx. Theory* (Rio de Janeiro, August 1979), Lecture notes in Pure and Appl. Math., Vol. **83**, M. Dekker, 305–358.

[Mi-3] P. S. Milojević (1982), *Continuation theory for A-proper and strongly A-closed mappings and their uniform limits and nonlinear perturbations of Fredholm mappings*, in: E. Barroso ed., *Proc. Int. Sem. Funct. Anal. Holom. Approx. Theory.* (Rio De Janeiro, August 1980), Mathematics Studies, Vol. **71**, North-Holland, Amsterdam, 299–372.

[Mi-4] P. S. Milojević (1986), *On the index and the covering dimension of the solution set of semilinear equations*, Proc. Symp. Pure Math. **45** (2), Amer. Math. Soc., 183–205.

[Mi-5] P. S. Milojević (1988), *Solvability of nonlinear operator equations with applictions to hyperbolic equations*, in: B. Stankovic et. al., eds., *Proc. Generalized Funct. Convergence Structures and their Appl.*, Plenum Press, NY, 245–250.

[Mi-6] P. S. Milojević (1989), *Solvability of semilinear hyperbolic equations at resonance*, in: A. R. Aftabizadeh, ed., *Proc. Diff. Eq. and Appl.*, Ohio University Press, pp. 216–221/

[Mi-7] P. S. Milojević, (1990), *Periodic solutions of semilinear hyperbolic equations at resonance*, J. Math, Anal. Appl. **146**, 546–569.

[Mi-8] P. S. Milojević (1989), *Solvability of semilinear operator equations and applications to semilinear hyperbolic equations*, in: P. S. Milojević, ed., *Nonlinear Functional Analysis*, pp. 95–178, Marcel Dekker Lecture Notes in Pure and Applied Mathematics.

[Pet] W. V. Petryshyn (1975), *On the approximation - solvability of equations involving A-proper and pseudo A-proper mappings*, Bull. Amer. Math. Soc. **81**, 221–312.

[Per] A. I. Perov (1953), *On the principle of the fixed point with two-sided estimates*, Dokl. Akad. Nauk SSSR **124**, 756–759.

[Ra] P. Rabinowitz (1967), *Periodic solutions of nonlinear hyperbolic partial differential equations*, Comm. Pure Appl. Math. **20**, 145–205.

[Ta-1] K. Tanaka (1985), *On the range of wave operators*, Tokyo, J. Math. **8** (2), 377–387.

[V] O. Vejvoda et al. (1981), *Partial Differential Equations: Time-periodic Solutions*, Noordhoff, Groningen.

[W] M. Willem (1981), *Density of the range of potential operators*, Proc. Amer. Math. Soc. **83**, 341–344.

Department of Mathematics,
New Jersey Institute of Technology,
Newark, New Jersey 07102, U.S.A.

(Received 10 01 1990)