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ON THE BEHAVIOUR OF DISTRIBUTIONS AT INFINITY, WIENER-TAUBERIAN TYPE RESULTS

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Abstract. The notions of quasiasymptotic and S-asymptotic behaviour of distributions are naturally connected with regularly varying functions. In this note we study the generalized S-asymptotic which enables us to enlarge the class of distributions which have generalized asymptotic behaviour at ∞. We obtain Wiener-Tauberian type assertions for appropriate distributions and functions with known S-asymptotic behaviour on a test function which Fourier transform is different from zero.

1. Introduction. There are several definitions of the asymptotic behaviour of distributions at infinity. They are related to the integral transformations of distributions [1], [2], [3], [7], [12]. The most applicable is the so-called quasiasymptotic of tempered distributions which has been much investigated and connected with the quantum field theory by Vladimirov, Drožžinov and Zavialov (see references in [11] and [10]. We defined and studied in [7] the so-called S-asymptotic of Schwartz distributions. We say that $f \in \mathcal{D}'(\mathbf{R})$ has the S-asymptotic at infinity related to some real-valued and positive function $c(h)$, $h \in (0,\infty)$, if there exists the limit in the sense of convergence in $\mathcal{D}'(\mathbf{R})$

$$\lim_{h\to\infty} f(x+h)/c(h) = g(x) \neq 0. \tag{1}$$

We write for short $f(x+h) \overset{s}{\sim} c(h)g(x)$, $h \to \infty$. It was proved in [7] that $g(x)$ must be of the form: $g(x) = C\exp(\alpha x)$ for some $C \in \mathbf{R}$ and $\alpha \in \mathbf{R}$.

Moreover, if $c(h)$, $h \in (0,\infty)$, is measurable then c must be of the form $c(h) = \exp(\alpha h)L(\exp h)$, $h > 0$, where L is Karamata's regularly varying function. Let us note that Karamata's function appear naturally in the notion of quasi-asymptotic behaviour of tempered distributions as well.

Many advantages of the S-asymptotic can be found in [5], [6] and [7]. But some of ordinary functions, for example $\exp(x^2)$, have no S-asymptotic at infinity.

The purpose of this note is to extend the notion of the S-asymptotic to the notion of the generalized S-asymptotic of distributions. This enables us to

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obtain Wiener-Tauberian type results for non-negative distributions and functions for which we know their S-asymptotic behaviour on a test function with the Fourier transform different from zero on $\mathbf{R}$.

2. Generalized S-asymptotic. Let $f \in \mathcal{D}'$ and $c \in C^\infty$ be such that $c(x) \neq 0$, $x > x_0$. If in the sense of convergence in $\mathcal{D}'$

$$\lim_{h\to\infty} f(x+h)/c(x+h) = 1, \tag{2}$$

then we say that f has the generalized S-asymptotic related to $c(x)$ as $x \to \infty$ and we write $f(x) \overset{g.s}{\sim} c(x)$, $x \to \infty$.

The generalized S-asymptotic is a local property of a distribution. Namely, if $f, g \in \mathcal{D}'$ in some neighbourhood of ∞ and $f(x) \overset{g.s}{\sim} c(x)$, $x \to \infty$, then $g(x) \overset{g.s}{\sim} c(x)$, $x \to \infty$.

LEMMA 1. *Let $c(h)$, $h \in (0,\infty)$, be a real-valued positive locally integrable function such that for some $f \in \mathcal{D}'$ the limit in* (1) *exists. There exist $\tilde{c}(x) \in C^\infty$ different from zero on $\mathbf{R}$, $\alpha \in \mathbf{R}$ and $A \in \mathbf{R}$, $A \neq 0$, such that $c(h)/\tilde{c}(x+h) \to A^{-1}\exp(-\alpha x)$, $h \to \infty$, in the sense of convergence in $\mathcal{E}$.*

Proof. We give a sketch of the proof. Let $c_0(x) = c(x)$, $x > 1$, $c_0(x) = 1$, $x \leq 1$, and let $\omega \in C_0^\infty$, $\operatorname{supp}\omega \subset [-1,1]$, $\omega(t) > 0$, $t \in (-1,1)$, $\int_{-1}^{1}\omega(t)\,dt = 1$. Put $\tilde{c}(x) = (c_0 * \omega)(x)$, $x \in \mathbf{R}$.

We have $\tilde{c} \in C^\infty$, $\tilde{c}(x) > 0$, $x \in \mathbf{R}$. Let K be a compact set in $\mathbf{R}$. By [**7**, Theorem 3.a] we have that, for some $\alpha \in \mathbf{R}$ and any $x \in K$ and $t \in [-1,1]$ $\tilde{c}(x+h-t)/c(h) \to \exp(\alpha(x-t))$, $h \to \infty$. This convergence is uniform on $K_0 = \{x - t;\ x \in K,\ t \in [-1,1]\}$. This implies that for any $\beta \in \mathbf{N}_0$

$$\begin{aligned}\tilde{c}^{(\beta)}(x+h)/c(h) &\to \int_{-1}^{1}\exp(\alpha(x-t))\omega^{(\beta)}(t)\,dt\\ &= A\alpha^\beta\exp(\alpha x), \quad h \to 0, \qquad \text{uniformly on } K,\end{aligned}$$

where $A = \int_{-1}^{1}\exp(-\alpha t)\omega(t)\,dt$. This implies that

$$\big(c(h)/\tilde{c}(x+h)\big)^{(\beta)} \to \big(A^{-1}\exp(-\alpha x)\big)^{(\beta)}, \quad h \to \infty, \quad \text{uniformly on } K$$

This proves the assertion.

From Lemma 1 and [**7**, Theorem 5] we get

PROPOSITION 2. *Let $f \in \mathcal{D}'$ and let $c(h)$, $h \in (0,\infty)$, be as in Lemma 1. Suppose that* (1) *holds. Then $f(x) \overset{g.s}{\sim} C\tilde{c}(x)$, $x \to \infty$, where $\tilde{c}(x)$ is determined in Lemma 1 and C is a suitable constant.*

3. Wiener-Tauberian type results. Let us denote by $\mathcal{W}'$ the subspace of $\mathcal{D}'$ consisting of those f for which $\{f(x+h);\ h \in \mathbf{R}\}$ is a bounded subset in $\mathcal{D}'$.

This space is the union of all the spaces of the type $\mathcal{W}'$ introduced in [**4**]. Let θ be a smooth function on $\mathbf{R}$ such that $\operatorname{supp}\theta \subset [a,\infty)$ for some $a \in \mathbf{R}$ and $\theta(x) = 1$ for $x > b$ for some $b > a$. Let $f(x) \overset{g.s}{\sim} c(x)$, $x \to \infty$. Then by [**9**, Theoreme VI] we have, $\theta f/c \in \mathcal{S}'$; $\theta(x+h)f(x+h)/c(x+h) \to 1$ in the sense of convergence in $\mathcal{S}'$, $h \to \infty$.

By [**4**, Theorem 1] we directly obtain

Proposition 3. *Let $f \in \mathcal{D}'$, $c \in C^\infty$, $c(x) \neq 0$, $x \in \mathbf{R}$, and $\varphi \in \mathcal{S}$ be such that the Fourier transform of φ, $\hat{\varphi}$, is different from 0 on $\mathbf{R}$. If $\theta f/c \in \mathcal{W}'$ and $\langle f(x+h)/c(x+h), \varphi(x)\rangle \to A \cdot B$, $h \to \infty$, where $A \neq 0$, $B = \int_{-\infty}^{+\infty} \varphi(x)\,dx$ $(= \hat{\varphi}(0) \neq 0)$, then $f(x) \overset{g.s}{\sim} A \cdot c(x)$, $x \to \infty$.*

We assume in the next proposition that $c(h)$, $h > 0$, and $\tilde{c}(x)$, $x > 0$, are as in Lemma 1.

Proposition 4. *Let f be a non-negative distribution such that $f/\tilde{c} \in \mathcal{W}'$. Let $\varphi \in \mathcal{D}$ and $\hat{\varphi}$ be different from 0 on $\mathbf{R}$ and such that*

$$\lim_{h\to\infty} \langle f(x+h)/c(h), \varphi(x)\rangle = C\langle 1, \varphi(x)\rangle, \qquad C \neq 0. \tag{3}$$

Then $f(x+h) \overset{s}{\sim} c(h) \cdot C$, $h \to \infty$.

Proof. By Lemma 1 we have $c(h)/\tilde{c}(x+h) - 1 \to 0$, $h \to \infty$, uniformly on $\operatorname{supp}\varphi$. Because of that, for a sequence of positive numbers ε_n, $n \in \mathbf{N}$, which tends monotonically to 0 there is an increasing sequence h_n, $n \in \mathbf{N}$, such that

$$-\varepsilon_n \varphi(x) \leq \big(c(h)/\tilde{c}(x+h) - 1\big)\varphi(x) \leq \varepsilon_n \varphi(x), \qquad x \in K, \quad h > h_n.$$

This implies (for $h > h_n$)

$$-\varepsilon_n \left\langle \frac{f(x+h)}{c(h)}, \varphi(x)\right\rangle \leq \left\langle \frac{f(x+h)}{c(h)}, \left(\frac{c(h)}{\tilde{c}(x+h)} - 1\right)\varphi(x)\right\rangle \leq \varepsilon_n \left\langle \frac{f(x+h)}{c(h)}, \varphi(x)\right\rangle,$$

and because of (3) we get

$$\left\langle \frac{f(x+h)}{c(h)}, \left(\frac{c(h)}{\tilde{c}(x+h)} - 1\right)\varphi(x)\right\rangle \to 0 \quad \text{as } h \to \infty.$$

This implies

$$\lim_{h\to\infty} \left\langle \frac{f(x+h)}{\tilde{c}(x+h)}, \varphi(x)\right\rangle = \lim_{h\to\infty} \left(\left\langle \frac{f(x+h)}{c(h)}, \varphi(x)\right\rangle \right. \tag{4}$$
$$\left. + \left\langle \frac{f(x+h)}{c(h)}, \left(\frac{c(h)}{\tilde{c}(x+h)} - 1\right)\varphi(x)\right\rangle \right) = C\langle 1, \varphi(x)\rangle.$$

Now Proposition 3 implies that for suitable $A \in \mathbf{R}$, $f(x) \overset{g.s}{\sim} A\tilde{c}(x)$, $x \to \infty$. This implies that (4) holds for any test function from S and the assertion is proved.

PROPOSITION 5. *Let f be a non-negative distribution, $f/\tilde{c} \in \mathcal{W}'$, and $\varphi \in \mathcal{D}$ such that $\hat{\varphi}$ is different from 0 on* $\mathbf{R}$. *Let*

$$\lim_{h\to\infty} \langle f(x+h)/c(h), e^{-\alpha x}\varphi(x)\rangle \to C\langle 1, \varphi(x)\rangle, \qquad \alpha \neq 0.$$

Then $f(x+h) \overset{s}{\sim} c(h)Ce^{\alpha x}$, $h \to \infty$.

Proof. We apply the preceding proof on $f(x)e^{-\alpha x}$, $x \in \mathbf{R}$.

Let $c(h) = h^{\beta}L(h)$, $h > 0$, where $\beta \in \mathbf{R}$ and $L(h)$, $h > 0$, is Karamata's slowly varying function for which we assume that it is monotonous and C^{∞} on $(0, \infty)$. By using Theorem 2 in [**2**] we have proved in [**6**, Part IV], the following result: "If $f \in L^1_{\text{loc}}$ and for some $m_0 \in \mathbf{N}_0$ and $x_0 \in \mathbf{R}$, $f(x)x^{m_0}$ is non-decreasing for $x > x_0$, then the assumption $f(x) \overset{s}{\sim} c(h) \cdot 1$, $h \to \infty$, implies $f(x) \sim c(x)$, $x \to \infty$ (in the ordinary sense). This assertion and Proposition 4 imply the following result.

PROPOSITION 6. *Let $f \in L^1_{\text{loc}}$ be such that for some $m_0 \in \mathbf{N}$ and $x_0 \in \mathbf{R}$, $f(x)x^{m_0}$ is non-negative and non-decreasing for $x > x_0$. If for some $\varphi \in \mathcal{S}$ Fourier transform is different from 0 on* $\mathbf{R}$,

$$\frac{1}{h^{\beta}L(h)} \int_{x_0}^{\infty} f(x+h)\varphi(x)\,dx \to C \int_{-\infty}^{\infty} \varphi(x)\,dx, \quad h \to \infty,$$

then, $f(x) \sim Cx^{\beta}L(x)$, $x \to \infty$.

Proof. We consider the function $x \to \theta(x)f(x)$ where $\theta \in C^{\infty}$, θ is non-decreasing, $\theta(h) \equiv 0$ for $x \leq x_0$, $\theta(x) \equiv 1$ for $x \geq x_0 + 1$, and apply the previous assertions.

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