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REGULARLY VARYING DISTRIBUTIONS

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Abstract. The notions of regularly varying function and distribution in a cone $\Gamma \subset \mathbf{R}^n$ are given. Also some properties of these two classes are proved, such as: Every regularly varying function defines a regular distribution which is a regularly varying distribution; a regularly varying distribution is a sum of derivatives of regularly varying functions. A regularly varying distribution has a S-asymptotic and the whole theory of the S-asymptotic of distributions can be applied to the class of regularly varying distributions including applications to partial differential equations and to integral transformations.

1. Introduction. In the classical mathematics the notion of Karamata's regularly varying function has been of great help in analysing the asymptotical behaviour of solutions of mathematical models of many real systems. Seneta's book [**9**] presents the first complete theory of the regularly varying functions in one variable. Monograph [**2**] enlarges and supplements [**9**]. In the last years many authors proposed definitions of regularly varying functions in multi dimensional case ([**4**], [**5**], [**7**], [**17**] and [**19**]). We shall mention three characteristic definitions given in [**4**], [**17**] and [**19**].

If solutions of our mathematical models are distributions (generalized functions), then we have the same problem, to define asymptotical behaviour, now for distributions. The actual development of the quantum physics has given a special instigation for such studies (see [**3**], [**18**]). In the last two decades many definitions of asymptotic behaviour of distributions have been presented, elaborated and applied especially to the integral transformations of distributions. We shall mention three of them: Equivalence at infinity [**6**], quasiasymptotic [**17**] and S-asymptotic (shift asymptotic) [**8**].

In this paper we shall, first, give a definition of a regularly varying function in a cone $\Gamma \subset \mathbf{R}^n$ and, second, a definition of a regularly varying distribution in the cone Γ. Then we shall continue with some properties of these two classes. Let us remark that in [**1**] authors have introduced "regularly varying tempered distributions", but this is only a generalization of the quasiasymptotic. The class of

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q-strictly admissible distributions generalizes directly Karamata's class of regularly varying functions to the space of tempered distributions. This class is a useful tool in the quantum field theory and suits well for the Abelian and Tauberian type theorems for integral transformations of distributions (see [**17**]).

The property that a function is regularly varying at infinity, in one-dimension, is a local property. We wanted to preserve this characteristic in the definition of a regularly varying distribution. Therefore we used the S-asymptotic and not the quasiasymptotic. Quasiasymptotic is not a local property.

2. Regularly varying functions of several variables. Γ will be a convex acute cone with the vertex at zero and with the nonempty interior; $B(0,r)$ the open ball in $\mathbf{R}^n$ with the center at zero and with the radius $r>0$. If $x=(x_1,\dots,x_n)\in\mathbf{R}^n$ and $a=(a_1,\dots,a_n)\in\mathbf{R}^n$, we denote by $\|x\|^2=\sum_{i=1}^n x_i^2$ and by $(a,x)=\sum_{i=1}^n a_ix_i$. A cone is acute if and only if $\mathrm{Cl}(\mathrm{ch}\,\Gamma)$ does not contain an integral straight line. For a function f, $f^\vee(x)=f(-x)$. For $i=(i_1,\dots,i_n)\in\mathbf{N}^n$, $|i|=\sum_{k=1}^n i_k$.

Definition 1. Let $h_1,h_2\in\Gamma$. We say that $h_1\geq h_2$ in Γ if $h_1\in h_2+\Gamma$.

Definition 2. For a complex valued function $G(h)$, $h\in\Gamma$, $\lim_{h\in\Gamma,\,h\to\infty}G(h)=A$ if for every $\varepsilon>0$ there exists $h(\varepsilon)\in\Gamma$ such that $|G(h)-A|<\varepsilon$, when $h\geq h(\varepsilon)$ in Γ.

From definition 2 it follows that if $\lim_{h\in\Gamma,\,h\to\infty}G(h)=A$, then $\lim_{h\in\Gamma,\,h\to\infty}G(h+h_0)=A$ for every $h_0\in\Gamma$, as well.

PROPOSITION 1. *Suppose that $B(a,r)\subset\mathrm{int}\,\Gamma$. We denote by $\Gamma_1=\bigcup_{\lambda>0}\lambda B(a,r)$. If $\lim_{h\in\Gamma,\,h\to\infty}G(h)=A$, then $\lim_{h\in\Gamma,\,h\to\infty}G(x+h)=A$, as well, for every $x\in\mathbf{R}^n$.*

Proof. By definition 2, for every $\varepsilon>0$ there exists $h(\varepsilon)\in\Gamma$, such that $|G(h)-A|<\varepsilon$, $h\in h(\varepsilon)+\Gamma$. Now, $x-h(\varepsilon)+\beta_0a\in B(\beta_0a,\beta_0r)$ if $\beta_0>\|x-h(\varepsilon)\|/r$, β_0 depends on ε. Hence, $x-h(\varepsilon)+\beta_0a\in\Gamma_1\subset\Gamma$. Since Γ is a convex cone, $x-h(\varepsilon)+\beta_0a+\Gamma_1\subset\Gamma+\Gamma_1\subset\Gamma$ and $x+\beta_0a+\Gamma_1\subset h(\varepsilon)+\Gamma$. It follows that $|G(x+h)-A|<\varepsilon$ for $h\in\beta_0a+\Gamma_1$.

Remark. 1. If $\Gamma=\mathrm{int}\,\Gamma$, then Γ_1 in Proposition 1 can be the whole Γ, because a can be any element from Γ.

2. If $B(a,r)$ is such that for any coordinate-axis X_i, $i=1,\dots,n$, the distance between $B(a,r)$ and X_i is positive, $d(B,X_i)\geq\alpha>0$, then $|h_i|\to\infty$, $i=1,\dots,n$, when $h\in\Gamma_1$, $h\to\infty$; Γ_1 is constructed as in Proposition 1.

Let us show that. If $u\in h_0+\Gamma_1$ for a $h_0\in\Gamma_1$, then $u=\lambda_0a+\lambda_0z+\lambda a+\lambda y$, where $z,y\in B(0,r)$; $\lambda_0,\lambda>0$. Now, it is easy to see that $|u_i|\geq(\lambda+\lambda_0)\cdot(|a_i|-r)\geq(\lambda+\lambda_0)\alpha$ and for any $M>0$ we can find h_0 with the property $|u_i|\geq M$. Hence, if $h\to\infty$, $h\in\Gamma_1$, then $|h_i|\to\infty$, $i=1,\dots,n$.

Definition 3. A function $P : \mathbf{R}^n \to \mathbf{R}$ is said to be regularly varying function at infinity in Γ (r.v. function in Γ) if:

a) for every compact set $K \subset \mathbf{R}^n$, P is measurable and locally integrable on $K + \Gamma$; $P(x+h) > 0$, $x \in K$, $h \geq h_0$ in Γ;

b) for every $x \in \mathbf{R}^n$

$$\lim_{h\in\Gamma,\, h\to\infty} P(x+h)/P(h) = \exp(a \cdot x) \tag{1}$$

for an $a \in \mathbf{R}^n$; a is called the index of variation.

If a) and b) are valid only for x belonging to a fixed compact set K, P is said to be the *locally r.v. function in* Γ.

Proposition 2. *P is a r.v. function in Γ with index a if and only if $P(x) = \exp(a \cdot x)R(x)$, where R is a r.v. function in Γ with index $a = 0$.*

Proof. The proof follows from the fact that $\exp(a \cdot x)$ is positive and continuous function for every $a \in \mathbf{R}^n$.

Remark. If we substitute $x = (x_1, \ldots, x_n)$ by $(\ln y_1, \ldots, \ln y_n)$, $h = (h_1, \ldots, h_n)$ by $(\ln t_1, \ldots, \ln t_n)$ and $P(\ln t_1 y_1, \ldots, \ln t_n y_n)$ by $Q(t_1 y_1, \ldots, t_n y_n)$, then for $\Gamma = \mathbf{R}^n_+$ Definition 3 gives the definition of Diamond [**4**]. He treated the case $n = 2$.

Proposition 3. *If x belongs to a compact set K, limit* (1) *is uniform on K.*

Proof. By Proposition 2 it is enough to prove Proposition 3 in the case $a = 0$. The idea of the proof is just the same as for one variable (see [**9**]). However, we shall give the proof completely to point out the modifications in the technique of proving.

We denote by $f(x) = \ln R(x)$, then we have to prove that

$$\lim_{h\in\Gamma,\, h\to\infty} [f(x+h) - f(h)] = 0 \tag{2}$$

is uniform for $x \in K$.

Suppose that limit (2) is not uniform for $x \in I_p^n$, $I_p = [-p, p]$; $K \subset I_p^n$. Then, there exist an $\varepsilon > 0$ and two sequences $\{x_m\} \subset I_p^n$, $\{h_m\} \subset \Gamma$, such that

$$|f(x_m + h_m) - f(h_m)| \geq \varepsilon, \qquad m \in \mathbf{N}. \tag{3}$$

Let us construct two sequences of sets:

$$\begin{aligned} U_i &= \{y \in I_p^n,\ |f(y + h_m) - f(h_m)| < \varepsilon/2,\ m \geq i\} \\ V_i &= \{z \in I_q^n,\ |f(z + x_m + h_m) - f(x_m + h_m)| < \varepsilon/2,\ m \geq i\}, \end{aligned} \tag{4}$$

where we take q such that $(1 + p/q)^n = 3/2$.

The function f is measurable on $I^n_{p+q}+\Gamma$. Hence, U_i and V_i are measurable sets. These two sequences of sets are, by construction, monotone nondecreasing. We know that (2) is true for every $x\in\mathbf{R}^n$, therefore it follows that

$$\lim_{i\to\infty}\mu(U_i)=\lim_{i\to\infty}\mu(V_i)=\mu(I^n_q)=(2q)^n,$$

where $\mu(A)$ is the measure of the set $A\subset\mathbf{R}^n$. Now, there exists n' such that $\mu(U_{n'})>3(2q)^n/4$ and $\mu(V_{n'})>3(2q)^n/4$. The set $W_{n'}=V_{n'}+x_{n'}$ has the same measure as $V_{n'}$. Both, $U_{n'}$ and $V_{n'}$ lie in I^n_{p+q}; $\mu(I^n_{p+q})=(2q)^n(1+p/q)^n=3(2q)^n/2$.

Suppose, now, that $U_{n'}$ and $W_{n'}$ are disjoint, then we have $3(2q)^n/2\geq\mu(U_{n'})+\mu(W_{n'})>3(2q)^n/2$, but this is impossible. Therefore there exists $x\in(U_{n'}\cap W_{n'})$, $x-x_{n'}\in V_{n'}$. Hence, by relation (4), for this x we have

$$|f(x+h_{n'})-f(h_{n'})|<\varepsilon/2 \qquad \text{and}$$
$$|f(x_{n'}+x-x_{n'}+h_{n'})-f(x_{n'}+h_{n'})|<\varepsilon/2.$$

Using these two relations, we have

$$|f(x_{n'}+h_{n'})-f(h_{n'})|<\varepsilon$$

contradicting relation (3).

3. Regularly varying distribution. *Definition* 4. A distribution $T\in(D')$ for which there exist $\varphi_0\in(D)$ and $h_0\in\Gamma$ such that $\langle T(x+h),\varphi_0(x)\rangle\neq0$, when $h\geq h_0$ in Γ, is said to be regularly varying distribution in Γ (r.v. distribution in Γ) if for every $y\in\mathbf{R}^n$ and every $\varphi\in(D)$

$$\lim_{h\in\Gamma,\,h\to\infty}\frac{\langle T(x+y+h),\varphi(x)\rangle}{\langle T(x+h),\varphi_0(x)\rangle}=C(\varphi_0)\langle\exp(a\cdot(x+y)),\varphi(x)\rangle, \tag{5}$$

where $C(\varphi_0)$ is a constant depending on φ_0; $C(\varphi_0)\exp(a\cdot x)$ is called the limit distribution.

PROPOSITION 4. *Suppose $T\in(D')$. If there exists a continuous function $c(h)>0$, $h\geq h_0$ in Γ such that for every $\varphi\in(D)$*

$$\lim_{h\in\Gamma,\,h\to\infty}\langle T(x+h)/c(h),\varphi(x)\rangle=\langle U,\varphi\rangle,\quad U\neq0. \tag{6}$$

then, T is a r.v. distribution in Γ_1, where Γ_1 is from Proposition 1.

In fact, we supposed that T has the *S-asymptotic in* Γ related to $c(h)$ with the limit U (see [**8**] and [**10**]).

Proof. Suppose that T has the property given in relation (6). We proved in [**8**] that U is of the form $U(x)=C\exp(a\cdot x)$ and for $\varphi_0\in(D)$ for which $\langle\exp(a\cdot x),\varphi_0(x)\rangle>0$, there exists $h_0\in\Gamma$ such that $\langle T(x+h)/c(h),\varphi_0(x)\rangle>0$ for $h\geq h_0$ in Γ. Hence, $\langle T(x+h),\varphi_0(x)\rangle>0$, $h\geq h_0$ in Γ.

We put $G(h,\varphi) = \langle T(x+h)/c(h), \varphi(x)\rangle = (T * \varphi^{\vee})(h)/c(h)$, then by Proposition 1, for every $\varphi \in (D)$, we have

$$\lim_{h\in\Gamma_1, h\to\infty} \frac{\langle T(x+y+h), \varphi(x)\rangle}{\langle T(x+h), \varphi_0(x)\rangle} = \lim_{h\in\Gamma_1, h\to\infty} \frac{c(y+h)}{c(h)} \frac{G(y+h,\varphi)}{G(h,\varphi_0)}$$
$$= \frac{\langle \exp(a\cdot x), \varphi(x)\rangle}{\langle \exp(a\cdot x), \varphi_0(x)\rangle} \exp(a\cdot y).$$

PROPOSITION 5. *Suppose $T \in (D')$. If T is a r.v. distribution in Γ with the limit $C(\varphi_0)\exp(a\cdot x)$, then T has S-asymptotic related to $c(h) = (T * \varphi_0^{\vee})(h)$ with the limit $C(\varphi_0)\exp(a\cdot x)$ in Γ, that is*

$$\lim_{h\in\Gamma, h\to\infty} \langle T(x+h)/c(h), \varphi(x)\rangle = \langle C(\varphi_0)\exp(a\cdot x), \varphi(x)\rangle.$$

The proof is obvious.

Remark. 1. If Γ is an open cone, $\Gamma = \operatorname{int}\Gamma$, then the distribution T is a r.v. distribution in Γ if and only if there exists a continuous function $c(h) > 0$, $h \in \Gamma$ such that T has S-asymptotic related to $c(h)$ with the limit $U \neq 0$.

2. Proposition 5 makes possible to use the whole theory of the S-asymptotic including applications to partial differential equations and integral transformations of distributions (see **[10]**–**[15]**). Using different functions $c(h) > 0$ and the notion of the S-asymptotic, we can precise the behaviour of a r.v. distributions at infinity on a cone Γ.

PROPOSITION 6. *If P is a r.v. function in Γ, it defines a regular distribution which is a r.v. distribution in Γ_1.*

Proof. By Proposition 3.

$$\lim_{h\in\Gamma, h\to\infty} \int_{\mathbf{R}^n} \frac{P(x+h)}{P(h)} \varphi(x)\,dx = \int_{\mathbf{R}^n} \exp(a\cdot x)\varphi(x)\,dx.$$

Therefore, there exist $\varphi_0 \in (D)$ and $h_0 \in \Gamma$ such that

$$\int_{\mathbf{R}^n} P(x+h)\varphi_0(x)\,dx > 0, \qquad h \geq h_0 \text{ in } \Gamma.$$

To bring the proof to an end, we shall use Proposition 1

$$\lim_{h\in\Gamma_1, h\to\infty} \int_{\mathbf{R}^n} P(x+y+h)\varphi(x)\,dx \Big/ \int_{\mathbf{R}^n} P(x+h)\varphi_0(x)\,dx$$
$$= \lim_{h\in\Gamma_1, h\to\infty} \frac{P(y+h)}{P(h)} \int_{\mathbf{R}^n} \frac{P(x+y+h)}{P(y+h)}\varphi(x)\,dx \Big/ \int_{\mathbf{R}^n} \frac{P(x+h)}{P(h)}\varphi_0(x)\,dx$$
$$= \langle \exp(a\cdot(y+x)), \varphi(x)\rangle / \langle \exp(a\cdot x), \varphi_0(x)\rangle.$$

A locally integrable function f can define a r.v. distribution without being a r.v. function. Such a function, in one-dimensional case, is the following: $f = 1 + \psi$, where ψ differs from zero only on intervals $I_n = (n - 2^{-n}, n + 2^{-n})$, $n = 2, 3, \ldots$, where $\psi(x) = n$, $x \in I_n$.

Proposition 6 shows that the notion of r.v. distribution generalizes the notion of r.v. function in a natural way.

For the next proposition we need the following structural theorem [**14**].

STRUCTURAL THEOREM. *If $T \in (D')$ has an S-asymptotic related to the continuous function $c(h) > 0$ with the limit $U \neq 0$, then for the ball $B(0,r)$ there exist numerical functions F_i, $|i| \leq m$, continuous on $B(0,r) + \Gamma$, such that for every $|i| \leq m$, $F_i(x+h)/c(h)$ converges uniformly for $x \in B(0,r)$ when $h \in \Gamma$, $h \to \infty$, and the restriction of the distribution T on $B(0,r) + \Gamma$ can be given in the form $T = \sum_{|i|\leq m} D^i F_i$, where D^i are partial derivatives in the sense of distributions.*

In the proof of that theorem we showed that all F_i are of he form $F_i(x) = (T * \varphi_i * \psi_i)(x)$, where $\varphi_i, \psi_i \in (D^m_\Omega)$, Ω is a relatively compact open neighborhood of zero in $\mathbf{R}^n$. $F_i(x+h)/c(h)$ converges to $(U * \varphi_i * \psi_i)$ when $h \in \Gamma$, $h \to \infty$, and x belongs to $B(0,r)$.

PROPOSITION 7. *If $T \in (D')$ is a r.v. distribution in Γ with the limit $C(\varphi_0)\exp(a \cdot x)$, $C(\varphi_0) \neq 0$, then*

a) *for the ball $B(0,r)$ there exist numerical functions F_i, $|i| \leq m$, continuous on $B(0,r) + \Gamma$, such that for every $|i| \leq m$ and $c(h) = (T * \varphi_0^\vee)(h)$, $F_i(x+h)/c(h)$ converges uniformly for $x \in B(0,r)$, when $h \in \Gamma$, $h \to \infty$, and the restriction of the distribution T on $B(0,r) + \Gamma$ can be given in the form $T = \sum_{|i|\leq m} D^i F_i$;*

b) *F_i, $|i| \leq m$, are locally r.v. functions in Γ.*

Proof. By Proposition 5, T has S-asymptotic related to $c(h) = (T * \varphi_0^\vee)(h)$ and with the limit $C(\varphi_0)\exp(a \cdot x)$ in Γ. The first part of the Proposition 7 follows from the cited Structural theorem. Now we shall prove the second part.

By the remarks after the cited Structural theorem it follows that properties a) in Definition 3 for the compact set $\operatorname{Cl} B(0,r)$ are valid. Only the property b) in Definition 3 remains.

$$\lim_{h\in\Gamma,\, h\to\infty} \frac{F_i(x+h)}{F_i(h)} = \lim_{h\in\Gamma,\, h\to\infty} \frac{F_i(x+h)}{c(h)} \Big/ \frac{F_i(h)}{c(h)} = \exp(a \cdot x).$$

Proposition 7 characterizes the class of r.v. distributions.

The next proposition shows that regular variation of a distribution is a local property.

PROPOSITION 8. *Suppose that distributions T_1 and T_2 are equal on $h_1 + \Gamma_1$, $\Gamma_1 = \bigcup_{\lambda>0} \lambda B(a,r)$, $h_1 \in \Gamma_1$. If T_1 is a r.v. distribution in Γ_1 with the limit $C(\varphi_0)\exp(a \cdot x)$, then T_2 has the same property.*

Proof. By our supposition, $\langle T_1(x+h), \varphi_0(x)\rangle > 0$, $h \geq h_0$, in Γ_1. Let K_1, K_2, K_3 and K be compact sets in $\mathbf{R}^n$, such that $\operatorname{supp}\varphi_0 \subset K_1$, $\operatorname{supp}\varphi \subset K_2$,

$K_1, K_2 + K_3 \subset K$. By the proof of Proposition 1, we can find β_0 such that $x + h \in h_1 + \Gamma_1$ when $h \in \beta_0 a + \Gamma_1$ and $x \in K$. Hence, $\langle T_1(x+h), \psi(x)\rangle = \langle T_2(x+h), \psi(x)\rangle$ for $\psi = \varphi$ and $\psi = \varphi_0$; $h \geq \beta_0 a + h_0$ in Γ_1 and $x \in K$. Now, the proof follows from this equality.

At the end of this section we shall illustrate how r.v. distributions can be used in Abelian type theorems for the Stieltjes transformation of distributions in the one-dimensional case. First, we have to define such a transformation (see [**15**]).

Let η_ω be the well known smooth function with the properties: $0 \leq \eta_\omega(x) \leq 1$, $x \in \mathbf{R}$; $\eta_\omega(x) = 1$, $x \in (-\omega, \omega)$; $\eta_\omega(x) = 0$, $|x| \geq 3\omega$; $|D^k \eta_\omega(x)| \leq C_k \omega^{-k}$, $x \in \mathbf{R}$, $k > 0$.

Definition 4. The Stieltjes transformation of a distribution T (S_ρ-transformation) is defined by the limit

$$\lim_{\omega \to \infty} \langle T(x), \eta_\omega(x)(s+x)^{-(\rho+1)}\rangle = S_\rho(T)(s), \qquad s \in (\mathbf{C} \setminus \mathbf{R}) \tag{7}$$

if it exists for some $\rho \in \mathbf{R}$.

PROPOSITION 9. *Suppose that T is a r.v. distribution, $\Gamma = (a, \infty)$ and for $r \geq 0$ and $s_0 \in (\mathbf{C} \setminus \mathbf{R})$*

(i) *The distribution $T(x)/(s_0 + x)^r$ belongs to (B');*

(ii) *$c(h)/(s_0 + x + h)^r$, $h \in \mathbf{R}_+$, $x \in K$ is a bounded set for any compact set K.*

Then T has S_ρ-transform for all $\rho > r$, and

$$\lim_{h \to \infty} S_\rho(T)(s-h)/\langle T(x+h), \varphi_0(x)\rangle = \langle C(\varphi_0), (s+x)^{-\rho-1}\rangle = 0$$

for $\rho > r$.

For the proof see [**15**].

4. Regularly varying tempered distribution. Since a tempered distribution $T \in (S')$ is also in (D'), Definition 4 can be applied to it as well. But for the application to partial differential equations and some other applications a stronger definition regarding tempered distributions is more useful.

Definition 5. A distribution $T \in (S')$ is said to be regularly varying tempered distribution in Γ (r.v. tempered distribution in Γ) if T satisfies all the suppositions of Definition 4, but for a $\varphi_0 \in (S)$ and every $\varphi \in (S)$.

The following proposition illustrates the advantage of the notion of a r.v. tempered distribution. But we have first to introduce a differential operator.

Let $P(i\,D_x)$ be a linear differential operator, with real coefficients, of degree m and in n dimensions

$$P(i\,D_x) = \sum_{|i| \leq m} a_i D_x^i, \qquad i = (i_1, \dots, i_n), \quad x = (x_1, \dots, x_n).$$

P_0 is the principal part of $P(i\,D_x)$. If $P_0(y) \neq 0$ for any real $y \neq 0$, then $P(i\,D_x)$ is called an elliptic operator. If the elliptic operator is also homogeneous, it coincides with its principal part.

PROPOSITION 10. *If $P(i\,D_x)$ is an elliptic and homogeneous linear differential operator, then the equation*

$$P(i\,D_x)U(x) = f(x), \qquad f \in (O'_c)$$

has always a solution which is a r.v. tempered distribution in $\Gamma = \{\rho w,\, \rho > 0\}$ for every $w \in \mathbf{R}^n$, $\|w\| = 1$.

For the proof see [**16**].

4. Some comments. We can not say that there is a satifactory definition of regularly varying functions in multidimensional case. The same situation is with Definition 3, as well. Enlarging the definition of the regularly varying function of one variable to the case of several variables, one have to choose between two extreme possibilities: to keep all the properties of r.v. functions of one variable, but to narrow the class of r.v. functions of several variables down to a very limited one; or to obtain wide class of r.v. functions of several variables, but lacking some of the properties. Of course, all depends on usefulness of such notion in solving mathematical models of real systems. Two mentioned definitions ([**4**] and [**19**] are characteristic in this sense.

Definition A of P. Diamont. A function $R(x, y)$ is said to be regularly varying at infinity if it is real valued, positive and measurable on $\{x \geq A,\ y \geq B\}$ for some $A, B > 0$, and for each positive α, β

$$\lim_{x,y\to\infty} R(\alpha x, \beta y)/R(x, y) = \alpha^a \beta^b \tag{8}$$

for some real numbers a and b.

This definition gives a very restrictive class of r.v. functions, but it keeps all the properties of r.v. functions of one variable. For example, the function $R(x, y) = x + y$ is not a r.v. function by this definition. The limit $\lim_{x,y\to\infty}(\alpha x + \beta y)/(x + y)$ does not exist because for every $\lambda > 0$ and $y = \lambda x$, we have $R(\alpha x, \beta y)/R(x, y) = (\alpha + \lambda\beta)/(1 + \lambda)$.

Definition B of A. L. Yakimiv. A function P is said to be regularly varying at infinity on the cone $\Gamma \subset \mathbf{R}^n$ if for x, $\|x\| \geq r > 0$, it is positive, measurable and for a fixed $e \in \Gamma \setminus \{0\}$ and every $x \in \Gamma \setminus \{0\}$

$$\lim_{t\to\infty} P(tx)/P(te) = w(x, e). \tag{9}$$

The class of r.v. functions defined by Yakimiv is, by all means, wider and contains the class defined by Diamond. This is not only because Diamond used just a special cone $\Gamma = \mathbf{R}^2_+$, but also because Yakimiv's limit is weaker and the limit function $w(x,e)$ is not specified. The limit function $w(x,e)$ from Yakimiv's definition is measurable and homogeneous on Γ. In order to have the continuity of $w(x,e)$ and some other properties of r.v. functions of one variable, he had to suppose, in addition to the properties from Definition B, the following condition: If $\{x_t\} \subset \Gamma \setminus \{0\}$ and $x_t \to x \in \Gamma \setminus \{0\}$, when $t \to \infty$, then

$$\lim_{t\to\infty} [P(tx_t) - P(tx)]/P(te) = 0. \tag{10}$$

It seems to me that the class of strictly admissible functions (see [**17**], p. 108) generalizes in the best way the Karamata's class of r.v. functions to multidimensional case. We gave our Definition 3 thinking on the possibility to enlarge it to distributions and to keep the local property of this notion. That is the reason why we chosed such a definition. Moreover, the S-asymptotic is also a notion which has its origin in the quantum field theory (see [**10**]). Hence, r.v. distributions have a real meaning.

Let us remark that in one-dimensional case, if a function f is a r.v. function by Karamata's definition, namely if

$$\lim_{t\to\infty} f(tx)/f(t) = x^{\alpha}, \qquad x > 0, \quad \alpha \in \mathbf{R}$$

then f is r.v. function by Definition 3, with $a = 0$, as well. The function f is of the form $f(x) = x^{\alpha} L(x)$, where L is a slowly varying function. Now, for $x \in \mathbf{R}$

$$\lim_{y\to\infty} \frac{f(x+y)}{f(y)} = \lim_{y\to\infty} \frac{(x+y)^{\alpha}}{y^{\alpha}} \cdot \frac{L(x+y)}{L(y)} = \lim_{v\to\infty} \frac{L(\ln uv)}{L(\ln v)} = 1$$

because $L(\ln x)$ is a slowly varying function at infinity together with L.

On the other hand, if P is a r.v. functon by Definition 3 and in one-dimension, it is of the form $P(x) = \exp(a \cdot x) L(\exp x)$, where L is a slowly varying function. This follows from Proposition 2; the function R has to satisfy

$$\lim_{y\to\infty} \frac{R(x+y)}{R(y)} = 1, \quad \text{or} \quad \lim_{v\to\infty} \frac{R(\ln uv)}{R(\ln v)} = 1.$$

The natural question arises: For what reason we chose just the function $\exp(a \cdot x)$ in Definition 3. The answer lies in

Proposition 11. *If the function P satisfies condition* a) *from Definition 3 for a cone Γ, $\operatorname{int}\Gamma \neq \varnothing$, and*

$$\lim_{h\in\Gamma,\, h\to\infty} P(x+h)/P(h) = Q(x) \neq 0,$$

then $Q(x) = \exp(a \cdot x)$, $a \in \mathbf{R}^n$.

Proof. Function P defines a regular distribution which has S-asymptotic with the limit $\neq 0$. We know that Q has the form $Q(x) = \exp(a \cdot x)$ (see Proposition 9 in [**10**]).

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