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THE FOURIER CHARACTER OF SERIES WITH SLOWLY VARYING CONVERGENCE MODULI

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Abstract. Let $\{c(n)\}$ be an asymptotically even complex null sequence such that for some $p \in (1,2]$

$$\overline{\lim_n} \sum_{|k|=n+1}^{[\lambda n]} |k|^{p-1} |\Delta c(k)|^p < \infty, \qquad \lambda > 1.$$

Then (i) the series $\sum_{|n|<\infty} c(n) e^{int}$ convergese a.e. and (ii) it is the Fourier series of its sum function if for every $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$, independent of n, such that

$$\int_{|t|\le\delta} \left| \sum_{|k|\ge n+1} \Delta c(k) E_k(t) \right| dt < \varepsilon,$$

for all n, where $E_k(t) = \sum_{j=0}^{k} e^{ijt}$.

Riemann's **[1]** representation theory influenced early attempts to identify the Fourier series among trigonometric series, and the ideas of Cantor **[2]** and Lebesque **[3]** dominated further efforts. The framework of subsequent studies in the Fourier character of the series (1) has been set by the de la Vallee Poussin **[4]** theorem: If the pointwise sum function f of the series

$$\sum_{|n|<\infty} c(n) e^{int} \tag{1}$$

exists everywhere in $T = \mathbf{R}/2\pi\mathbf{Z}$ save perhaps on a denumerable subset of T, and if f is integrable then the series (1) is the Fourier series of its sum function.

Nearly all results (see for instance **[5–10]**) based on de la Vallee Poussin's theorem display the following pattern: (i) the existence of f is guaranteed by certain regularity and/or speed conditions on $\{c(n)\}$; (ii) the integrability of f is implied by the same conditions applied to manipulated forms of (1). Consequently results based on de la Vallee Poussin's theorem are of quite restrictive nature, in particular regarding to the pointwise convergence of the series (1) to its sum function.

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A new approach to the Fourier character problem is motivated by recent studies in L^1-convergence of Fourier series **[11, 12]**, where a wide class of coefficients is defined through the convergence modulo $K_n^p(c) = \sum_{|k|\le n} |k|^{p-1}|\Delta c(k)|^p$, $p > 1$, of the series (1).

A sequence $\{R(n)\}$ of positive numbers is O-regularly varying if $\overline{\lim_n} \dfrac{R([\lambda n])}{R(n)}$ is finite for $\lambda > 1$. In particular $\{R(n)\}$ is $*$-regularly varying if $\lim_{\lambda\to 1+0} \overline{\lim_n} \dfrac{R([\lambda n])}{R(n)} = 1$, and if $\lim_n \dfrac{R([\lambda n])}{R(n)} = 1$, $\{R(n)\}$ is slowly varying.

Let $\{c(n)\}$ be a null sequence of complex numbers. If for some $p \in (1, 2]$ and some O-regularly varying sequence $\{R(n)\}$

$$K_n^p(c) = \lg R(n), \tag{2}$$

we say that the series (1) has slowly varying convergence modulo. (Notice that $\{R(n)\}$ is necessarily nondecreasing).

Here we shall show that the series (1) with slowly varying convergence modulo, is the Fourier series of its sum function f if and only if integrals of tails of the regularized sums are uniformly equicontinuous.

The control of the sine coefficients $\{c(n) - c(-n)\}$ of the series (1), can be done in two, essentially equivalent ways: (i) by splitting (1) into cosine and sine series, and assuming certain additional conditions for the sine coefficients; (ii) by working with the complex form of (1) and assuming some kind of mild evenness conditions for $\{c(n)\}$, i.e.:

$$\frac{1}{n}\sum_{k=1}^{n} |c(k) - c(-k)| \lg k = o(1), \qquad n \to \infty, \tag{3}$$

$$\lim_{\lambda\to 1+0} \overline{\lim_n} \sum_{k=n+1}^{[\lambda n]} |\Delta(c(k) - c(-k))| \lg k = 0. \tag{4}$$

We shall adopt the second way and call null sequences satisfying (3) and (4) asymptotically even.

In the next theorem we need the following denotations. The Fejer sums are denoted by $\sigma_n(c) = \sigma_n(c,t) = (n+1)^{-1}\sum_{k=0}^{n} S_k(c)$, where $\{S_k(c)\}$ are partial sums of (1). The regularized partial sum are denoted by

$$g_n(c) = g_n(c,t) = \sum_{|k|\le n-1} \Delta c(k) E_k(t) = S_n(c,t) - [c(n)E_n(t) + c(-n)E_{-n}(t)],$$

where $E_k = E_k(t) = \sum_{j=0}^{k} e^{ijt}$. For $p \in (1, 2]$ and $1/p + 1/q = 1$ we define the interval

$$T_n(\lambda) = \left(-\frac{\pi(\lambda-1)^{-q/(q+1)}}{n}, \frac{\pi(\lambda-1)^{-q/(q+1)}}{n}\right),$$

$\lambda > 1$. The $L^1(T)$-norm is denoted by $\|\cdot\|$.

THEOREM. *Let* (1) *be a series with slowly varying convergence modulo. Then*

(i) *the pointwise sum function f of the series* (1) *exists a.e. in T;*

(ii) *for asymptotically even coefficients the following statements are equivalent:*

(ii.1) *for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, independent of n, and such that*

$$C_n(c,\delta) = \int_{-\delta}^{\delta} \left| \sum_{|k|\geq n+1} \Delta c(k) E_k(t) \right| dt < \varepsilon$$

for all n.

(ii.2) *$c = \hat{f}$; and*

(iii) *for $c = \hat{f}$, $\|S_n(f) - f\| = o(1)$, $n \to \infty$, is equivalent to $\hat{f}(n) \lg |n| = o(1)$, $|n| \to \infty$.*

Proof. To prove (i) consider the nontrivial case $K_n^p(c) \to \infty$, as $n \to \infty$. Then (2) implies

$$1 \leq \overline{\lim_n} \frac{\lg R([\lambda n])}{\lg R(n)} \leq 1 + \overline{\lim_n} \frac{1}{\lg R(n)} \lg \overline{\lim_n} \frac{R([\lambda n])}{R(n)},$$

i.e. $\{K_n^p(c)\}$ is a slowly varying sequence. Hence the series $\sum_{|n|<\infty} |\Delta c(n)|^p$ converges. For $t \neq 0$, via Riesz' **[13]** and Carleson's **[14]** theorem, the series (1) converges a.e. in T to its sum function f.

The equivalence between (ii.1) and (ii.2) is a consequence of following estimations:

$$\left\| \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} S_k(c) - g_n(c) \right\| \leq \sum_{m=1}^{5} I_{mn}(\lambda), \qquad \lambda > 1,$$

where

$$\begin{aligned} I_{1n}(\lambda) &\leq \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \int_{T_n(\lambda)} \left| \sum_{|j|=n}^{k-1} \Delta c(j) E_j(t) \right| dt \\ &\leq C_1 (\lambda - 1)^{1/(q+1)} \left(\lg \frac{R([\lambda n])}{R(n)} \right)^{1/p}, \end{aligned}$$

by a uniform estimation of E_j, Jensen-Petrović inequality and the condition (3);

$$\begin{aligned} I_{2n} &\leq \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \int_{T - T_n(\lambda)} \left| \sum_{|j|=n}^{k-1} \Delta c(j) D_j(t) \right| dt \\ &\leq C_2 (\lambda - 1)^{1/(q+1)} \left(\lg \frac{R([\lambda n])}{R(n)} \right)^{1/p}, \end{aligned}$$

by Hölder's inequality followed by Hausdorff-Young inequality and the condition (3);

$$I_{3n}(\lambda) \leq C_3 \sum_{k=n}^{[\lambda n]} |\Delta(c(k) - c(-k))| \lg k,$$

straightforwardly;

$$I_{4n}(\lambda) \leq C_4 \left(\frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} |c(k)|^p \right)^{1/p}$$

by Hölder's and Hausdorff-Young's inequalities; and plainly

$$I_{5n}(\lambda) \leq \frac{C_5}{[\lambda n] - n} \sum_{k=n}^{[\lambda n]} |c(k) - c(-k)| \lg k.$$

(All five constants are absolute). From the above estimations and (2), (3) and (4) we have

$$\lim_{\lambda \to 1+0} \overline{\lim_{n}} \left\| \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} S_k(c) - g_n(c) \right\| = 0,$$

and by a standard argument if follows that $\lim_{\lambda \to 1+0} \lim_n \|\tau_n(c, \lambda) - f\| = 0$ is equivalent to $\|g_n(c) - f\| = o(1)$, $n \to \infty$. From

$$C_n(c, \delta) \leq \|f - g_n(c)\| \leq C_n(c, \delta) + C_6 \left(\sum_{|k| \geq n+1} |\Delta c(k)|^p \right)^{1/p},$$

where C_6 is an absolute constant and the series $\sum_{|n|<\infty} |\Delta c(k)|^p$ converges, it follows that (ii.1) is equivalent to $\|g_n(c) - f\| = o(1)$, $n \to \infty$. Hence (ii.1) is equivalent to $\lim_{\lambda \to 1+0} \overline{\lim}_n \|\tau_n(c) - f\| = 0$, and since the later is equivalent to $c = \hat{f}$, it follows that (ii.1) is equivalent to (ii.2).

The proof of (iii) goes as in [**12**].

When instead of L^p methods we use straightforward estimations we get the following result.

PROPOSITION. *Let $\{c(n)\}$ be a complex null sequence and for some $*$-regularly varying sequence $\{R(n)\}$ let the series*

$$\sum_{n=1}^{\infty} \frac{\lg^p [R(n)/R(n-1)]}{\lg^p (n+1)}$$

converge for some $p \in (1, \infty)$. If $\sum_{|k| \leq n} |\Delta c(k)| \lg |k| = \lg R(n)$ then the conclusions of Theorem hold.

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