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REMEMBERING JOVAN KARAMATA

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1. As a scholar Karamata was self-taught. He wasn't a follower of any school; moreover, after completing his studies in Belgrade he didn't go to any foreign university at all for advanced study. We learned from a conversation with him that he began to scan scholarly papers when he was still a university student. He studied one of them, H. Weyl's well known paper *Über die Gleichverteilung von Zahlen mod. Eins*[1] in the utmost detail. He also analyzed and made a study of all the theorems mentioned in it. On the other hand, he was one of those mathematicians who learned a great deal from the then-famous problem book by G. Pólya and G. Szegő *Aufgaben und Lehrsätze aus der Analysis*. From Weyl's work one variant of the classical Weierstrass theorem remained impressed in his memory:

If $f(x)$ is a bounded function on $[a,b]$, with $-\infty < a < b < \infty$, and if $\alpha(x)$ is non-decreasing and the Riemann-Stieltjes integral $\int_a^b f(x)\,d\alpha(x)$ exists, then for every $\varepsilon > 0$ there exist polynomials $P(x)$ and $Q(x)$ such that

$$\inf f(x) - \varepsilon \le P(x) \le f(x) \le Q(x) \le \sup f(x) + \varepsilon$$

and

$$\int_a^b (Q(x) - P(x))\,d\alpha(x) < \varepsilon.$$

When Landau's monograph *Darstellung und Begründung einiger neuer Ergebnisse der Funktionentheorie* appeared in 1929, Karamata immediately set about studying Hardy-Littlewood's Tauberian theorem. He remembered, he told us, that Poincaré almost always made his discoveries on a characteristic special case first. Karamata immediately noticed that the Hardy-Litlewood's theorem is almost obvious for the special case of polynomials. That brought to mind the approximation theorem referred to above. In fact it all seemed extremely simple to him. He appended a number of historical remarks and similar theorems and asked his professor

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[1] Math. Annalen **77** (1916), 313–352.

Mihailo Petrović, who knew Landau well, for Landau's opinion. Landau rejected everything but the Hardy-Littlewood theorem and immediately sent the result to Mathematische Zeitschrift[2]. In his reply Landau expressed regret that he hadn't put off writing his monograph until sometime later, because the chapter on inverse theorems would have looked different if he had.

In a letter to Landau at that time (1931), Karamata anticipated the approximation problem of Wiener's type as being the central problem of general Tauberian theorems, before Wiener had published his theorems of the Tauber type.

Karamata's method of proof made its way into, and to this day remains in, the best known textbooks and monographs dealing with Tauberian theorems. To cite just a few: K. Knopp, *Theorie und Anwendung der unendlichen Reihen*, 1931; G. Doetsch, *Theorie und Anwendung der Laplace-Transformation*, 1937; E. C. Titchmarsch, *The Theory of Functions*, 1939; D. V. Widder, *The Laplace Transform*, 1946; G. H. Hardy, *Divergent Series*, 1949. More recently, in Favard's *Cours d'Analyse*[3], it can be found already as an exercise:

$$\sum_{n=0}^{\infty} a_n x^n \simeq (1-x)^{-1} \ (x \to 1-0) \text{ and } a_n \geq 0 \implies \sum_{k=0}^{n} a_k \simeq n \quad (n \to \infty)$$

Hint: $L(g) := \lim_{x\to 1-0}(1-x)\sum_{n=0}^{\infty} a_n x^n g(x^n)$ is a bounded linear functional on $C[0,1]$. Consequently, the formula $(*)$ $L(g) = \int_0^1 g(t)\,dt$, valid for every $g(t) = t^k$ $(k = 0, 1, 2, \ldots)$, will be valid for every polynomial g and also for every continuous function g. This will be then valid even for all continuous functions g which have a discontinuity of the first kind. For such a function g, $g(x) = 0$ $(0 \leq x < 1/e)$ and $g(x) = 1/x$ $(1/e \leq x \leq 1)$, formula $(*)$ reduces to the statement of the problem.

2. Karamata very quickly anticipated the extension of the Hardy-Littlewood theorem by means of slowly varying functions. The concept itself and the source for the slowly varying functions he found in G. Pólya and G. Szegő's already cited collection[4], and partly in R. Schmidt's thesis[5] as well. It was not his intention to create a theory of slowly varying functions, but rather, as he himself said, he wanted to give a simple characterization of that class of functions. The form of representation he discerned immediately from the typical examples of the slowly varying functions: the logarithm and the iterated logarithms. We should bear in

[2] J. Karamata, *Über die Hardy-Littlewoodsche Umkehrung des Abelschen Stetigkeitsatzes*, Math. Zeitschrift **32** (1930), 319–320.

[3] J. Favard, *Cours d'Analyse. Compléments et Exercices d'Analyse*, (Troisième Cahier, XI, 10°), École Polytechnique, Paris, 1962–1963.

[4] G. Pólya and G. Szegő; *Problems and Theorems in Analysis* (Chapter IV, Problems 150–154); Springer-Verlag, Berlin, 1972–76.

[5] R. Schmidt, *Über divergente Folgen und lineare Mittelbildungen*, Math. Zeitschrift **22** (1925), 89–152.

mind that at that time he was familiar with Hardy's book *Order of Infinity*. The greatest stumbling block for him there was the proof that pointwise convergence implies the uniformity of convergence, a fact that is also often given as an exercise today[6]. At that time Karamata was building a house and as the night watchman of the building site he spent every night thinking about that and nothing else. He himself says he came across Cauchy's functional equation by chance, and that got him started. Although he worked on this piece of work for nearly a year, he didn't attach any particular importance to it. This is confirmed by the fact that the first version of the paper was published in a virtually unknown journal[7]. He himself said he didn't want to return to this problem any more. On the wole he didn't much enjoy rummaging through his old papers. He had no interest at all in applications of slowly varying functions, except for the basic one in theorems of Tauberian type.

He wasn't even very knowledgeable about probability theory. That makes it all the stranger that he discovered so many properties of slowly varying functions that are so useful in applications. And during the first phase of his work (1927–1938) he mentioned slow variation only one other time[8], but that was in regard to sequences. Other applications of slowly varying functions, after 1945, were mostly developed by his students and, above all, by W. Feller[9].

3. There is a marked contrast between Karamata the scholar and Karamata the man. He led the life of an ordinary man and citizen of the first half of this century with all the compromises that the society of that time expected. While having virtually no interest in public life, he appears to have had an innate appreciation of beauty. But perhaps because of his upbringing he wasn't knowledgeable about art and took no interest in it. As for scholarship, on the other hand, he was deeply committed to its most beautiful ideals. If a result didn't please him, he threw the work out. He could polish a work for days and still not be satisfied with it. Except, perhaps, for the Hardy-Littlewood theorem, he displayed no satisfaction with any of his work. He did not like to talk about his scholarly works. He was extremely conscientious about everything he did in mathematics. His numerous journal reviews and reports were so informative that they were often cited in papers and books. L. Bieberbach in his book *Vorlesungen über Algebra* cites an observation of Karamata's that came from a review Karamata wrote for Zentralblatt für Mathematik. In the first issue of Publications Mathématiques de l'Université de Belgrade he derived a generalization of a Hardy-Littlewood inequality using an application of Stieltjes integral. Then he began to think that Hardy-Littlewood's theorem already

[6] J. Dieudonné, *Foundations of Modern Analysis, Vol. I*, Academic Press, New York, 1960, (Ch. VIII, p. 170).

[7] J. Karamata, (a) *Sur un mode de croissance régulière des fonctions*, Mathematica (Cluj) **4** (1930), 38–53. (b) *Sur un mode de croissance régulière. Théorèmes fondamentaux*, Bull. Soc. Math. France **41** (1933), 55–62.

[8] J. Karamata, *Über einen Konvergenzsatz des Herrn Knopp*, Math. Zeitschrift **40** (1935), 421–425.

[9] W. Feller, *An Introduction to Probability Theory and its Applications, Vol, I, II*, Wiley, New York, 1968–71.

contained his result, so he inserted an addendum in the already printed copies of the journal. However, Hardy and Littlewood did not think this was the case, and in the second edition of their *Inequalities* (1952) they included this result of Karamata's as a supplement to their theorem. Much later, Beckenbach did the same thing in his monograph on inequalities.

Karamata was always exhilarated by nice, elegant proofs and profound results. He made a detailed study of many such results (for example, Wiener's Tauberain theorems, Selberg-Erdős' elementary proof of the prime number theorem). In this respect he took his cue from Abel, who said "I preferred reading the great masters rather than the lesser ones". But he was capable of being quite blunt if he felt that a result was trivial. I recall him telling one of his students: "Every mathematician who works on that will discover that result, and every one you tell it to will prove it, but not everyone will publish it".

After the Second World War and an eight-year absence from mathematics, when Karamata saw the new directions the field was taking, he wasn't particularly negative about them. In so many words he said: How can I criticize what I don't understand? He shared a view he attributed to Marcel Riesz: I do not wish to work in new fields where my students will correct me and be amazed at my trivialities. Close to exhaustion by now, he realized his legacy would be the Hardy-Littlewod Tauberian theorem and the slowly varying functions. W. Feller, whom he knew well, sent him several pages of the second volume of his book on probability that had to do with the application of the slowly varying functions. But neither these citations nor numerous other ones stirred much emotion in him. Apropos of this, a friend of his from Switzerland said: "Yes, you're a blasé mathematician now; citations leave you unimpressed".

4. Everyone who worked with Karamata observed that his computation technique was faultless. In no time flat he could do several pages of calculations without making a single mistake, and in very legible handwriting at that. He could just as easily derive an entire proof in his head, again without any errors. One time when he was ill he sketched out an entire proof for *Ein Konvergenzsatz für trigonometrische Integrale*[10] in his head. The long, complicated calculations in that work weren't so difficult for him as to keep him tying together all the sections of the paper from memory. Zygmund and Calderon cite this work in their initial study of singular integrals.

5. The most significant period of Karamata's work was from 1927 to 1937, or perhaps between 1929 and 1933. He was obsessed with mathematics then. In order to be free, in order to be able to think solely and uninterruptedly about mathematics, he scheduled all his classes (he was an assistant professor then) on one day, from eight o'clock in the morning until eight in the evening. And those courses were: Introduction to Analysis, Elementary Algebra, Algebra, Foundations of Geometry, Descriptive Geometry, Introduction to the Theory of Functions, and

[10] Journal für die Reine und angewandte Mathematik, **178** (1937), 29–33.

Seminar. Although in ill health, he spoke without stopping, in a classroom where students and subjects followed each other in constant succession. One can see from this itself that when he worked he didn't let up. He made many innovations in the teaching process, which hadn't changed in decades. A great many of the students, accustomed to formality in the teaching process, were perplexed and dissatisfied. In the seminars, instead of the literature, textbooks and monographs, he used research papers and recent ones at that.

Karamata was always inquisitive about new fields of mathematics. Sometime about 1937 or 1938 he was with Marcel Riesz for two weeks. And Riesz told him about a number of new problems in the theory of potentials about bilinear forms and functions of several variables, and possibly about classical functional analysis and linear operators as well. But about 1940 he was working less. Financial conditions in Yugoslavia were difficult at that time. With a wife and three children he had financial difficulties. And he decided to set about solving his financial problems first so that he could then devote himself entirely to mathematics. This attempt ended in total failure. The war which came soon after that, tore him away from mathematics completely. All the bonds that linked him to mathematics, all his contacts with scholars — and he had many of them at that time — were severed. Isolated, he didn't attempt to maintain even the slightest contact with mathematics. As he said in 1945: "I forgot not only what I had been doing, but also many commonplace mathematical facts. I no longer knew what was going on in the field I had once known so well".

6. And yet, contact with new young people who found a mentor in him after the war, contributed to his return to mathematics.

During the Second World War the University of Belgrade was closed, which meant that in the autumn of 1945 five classes of students were ready to begin their studies. Thus the number of those deciding to study mathematics was also large. Through his classes Karamata left a profound impression on them. Within several years he enabled the best of them to progress from classroom problems to more advanced topics. Karamata's commitment to young talent in Belgrade didn't slacken even after he moved to the University of Geneva in 1951. On the contrary, it only intensified. During his frequent and regular visits to Belgrade he magnetically attracted young mathematics students who had never even seen him before, let alone heard him in the classroom. He would meet with them individually, always in regard to some mathematical problem, usually at the Hotel Majestic, or else — frequently — in groups, at the Mathematical Institute of the Serbian Academy of Sciences. The conversation topics would be related to some specific problem, or to general developments in mathematics. On one such occasion Karamata voiced a hunch that some internal connection must exist between the existence of Frullani's integrals and the regular variation of functions. And before his next arrival in Belgrade such a connection was in fact found — in the form of a new characterization of the regularly varying functions. But considerable work still lay ahead, because Karamata strived for perfection when work needed to be put in its final form: he strived for maximum precision without a single superfluous word, so long as the readability of the text itself didn't suffer in the process. More than once it seemed this goal had been achieved, but there was yet one more change — which proved to be justified. That was when his class of functions got the name by which it is known today[11]: the class of regularly varying functions. In the paper in which this class was introduced[7] and in subsequent papers he called them "fonctions à croissance régulière" (functions of regular growth).

[11] J. Karamata, *Fonctions à comportement régulier et l'intégral de Frullani* (with S. Aljančić), Recueil de Travaux, Inst. Math. Acad. Serbe Sci. 50 (1956), 239–248 (in Serbian with a French abstract).