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CONCERNING SPLITTABILITY AND PERFECT MAPPINGS

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Abstract. We consider the following question: let a space X admits a perfect mapping onto a space Y from some class $\mathcal{P}$ of topological spaces and let X be splittable over $\mathcal{P}$. Does X belong to $\mathcal{P}$?

0. Introduction

The notions of splittability, $\mathcal{P}$-splittability and $(\mathcal{M}, \mathcal{P})$-splittability introduced recently by A. V. Arhangel'skiĭ have been the subject of several papers: [**4**], [**6**], [**11**], [**12**]. The definitions are as follows:

Let $\mathcal{M}$ be a class of continuous mappings and $\mathcal{P}$ a class of topological spaces. A space X is called $(\mathcal{M}, \mathcal{P})$-*splittable* or $\mathcal{M}$-*splittable over* $\mathcal{P}$ if for every $A \subset X$ there exist some $Y \in \mathcal{P}$ and a mapping $f \in \mathcal{M}$ from X onto Y such that $f^{-1}f(A) = A$. When $\mathcal{M}$ is the class of all continuous (perfect) mappings we use the term *splittable over* $\mathcal{P}$ (*perfectly splittable over* $\mathcal{P}$) instead of $(\mathcal{M}, \mathcal{P})$-splittable (see [**4**]).

Clearly, if there is a continuous bijection from a space X onto a space $Y \in \mathcal{P}$, then X is splittable over $\mathcal{P}$ and we can say that X is absolutely splittable over $\mathcal{P}$ in this case. So splittability is a generalization of continuous bijections.

A number of theorems in general topology can be formulated in the following form: Let $\mathcal{P}$ be a topological property. If a space X admits a perfect mapping onto a space Y satisfying $\mathcal{P}$ and a one-to-one mapping onto a space Z satisfying $\mathcal{P}$, then X satisfies $\mathcal{P}$ (see, for example, [**5**], [**9**]). This suggests the following natural question which is the subject of this article: when splittability over $\mathcal{P}$ replaces one-to-one mappings in such theorems; more precisely: let a space X admit a perfect mapping onto a space from a class $\mathcal{P}$ and let X be splittable over a class Q. Does X belong to $\mathcal{P}$ or $\mathcal{Q}$?

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All spaces in this article are T_2 (unless stated otherwise) and all mappings are continuous and onto. Recall that a mapping $f : X \to Y$ is perfect if f is closed with $f^{-1}(y)$ compact for each $y \in Y$. We use the usual notation and terminology [**5**], [**9**] and give references (although not necessarily the original source) where the definitions of undefined concepts can be found.

Let us begin with the following simple but useful result [**11**]:

Lemma 0.1. *Let $\mathcal{M}$ be the class of all closed mappings. If a space X is $\mathcal{M}$-splittable over the class of Hausdorff (regular, Tychonoff, normal) spaces then X in Hausdorff (regular, Tychonoff, normal).*

In the sequel we shall use the following well known result.

Lemma 0.2. *If $f : X \to Y$ is a perfect mapping then for any subset $B \subset Y$ the restriction $f_B : f^{-1}(B) \to B$ is perfect.*

We shall often use

Lemma 0.3. *Let $\mathcal{P}$ be a class of topological spaces which is hereditary and finitely multiplicative. Suppose that a space X is splittable over $\mathcal{P}$ and admits a perfect mapping onto a space from $\mathcal{P}$. Then*

(i) *X is perfectly splittable over $\mathcal{P}$*;

(ii) *every $A \subset X$ admits a perfect mapping onto a space from $\mathcal{P}$.*

Proof. (i) Let $f : X \to Y \in \mathcal{P}$ be perfect. For every $A \subset X$ there are a space $Z \in \mathcal{P}$ and a mapping $g : X \to Z$ such that $g^{-1}g(A) = A$. Since f is a perfect mapping the diagonal product $\varphi = f\Delta g : X \to Y \times Z$ is also perfect — that is well known. Moreover, $\varphi^{-1}\varphi(A) = A$. This means that X is perfectly splittable over the class $\mathcal{P}$ as $Y \times Z \in \mathcal{P}$.

(ii) This follows from the fact that $\varphi(A) \in \mathcal{P}$ (because $\mathcal{P}$ is hereditary) together with Lemma 0.2.

1. Moore spaces and σ-spaces

Let us recall some definitions. A network for a space X is a collection $\mathcal{N}$ of subsets of X such that for every $x \in X$ and every open set U with $x \in U$ there is an $A \in \mathcal{N}$ such that $x \in A \subset U$. The net weight $\mathrm{nw}(X)$ of a space X is the least cardinality of a network for X. A cosmic space is a regular space with a countable network. A space X is a σ-space if it has a σ-discrete network. The definition of Moore spaces can be found in [**10**], for example.

Theorem 1.1. *If a space X is splittable over the class $\mathcal{P}$ of spaces of weight (net weight) $\leq \tau$ and admits a perfect mapping onto a space of weight (net weight) $\leq \tau$, then X has weight (net weight) $\leq \tau$.*

Proof. Let us note that the class $\mathcal{P}$ is hereditary and finitely multiplicative so that we can apply Lemma 0.3. Hence every subspace $A \subset X$ can be mapped by a perfect mapping onto a space of weight (net weight) $\leq \tau$. Then the theorem follows from the following result of Arhangel'skiĭ-Pytkeev (see [**3**] and [**14**]): if X is a Hausdorff space and every subspace of X admits a perfect mapping onto a space of weight (net weight) $\leq \tau$, then X itself has weight (net weight) $\leq \tau$.

Corollary 1.2. *If a Lindelöf p-space X [**5**] is splittable over the class of spaces of countable weight, then X has a countable base.*

A similar result can be formulated for perfectly Lindelöf spaces (= spaces which admit a perfect mapping onto a space with a countable network).

From Lemma 0.1 and Theorem 1.1 we get the following

Corollary 1.3. *If a space X is splittable over the class of cosmic spaces and admits a perfect mapping onto a cosmic space, then X is cosmic.*

Remark 1.4. Following [**4**] (see also [**11**]) denote by $\mathrm{w_{ps}}(X)$, X is a space, $\min\{\tau : X$ is perfectly splittable over the class of all spaces Y with $\mathrm{w}(Y) \leq \tau\}$. Then Theorem 1.1 gives us a method to prove: $\mathrm{w}(X) = \mathrm{w_{ps}}(X)$; this was proved in [**11**] by a different manner. Similarly we have $\mathrm{nw}(X) = \mathrm{nw_{ps}}(X)$ for every space X (see also [**12**]).

Theorem 1.5. *If a space X is splittable over the class $\mathcal{P}$ of σ-spaces and admits a perfect mapping onto a σ-space, then X is a σ-space.*

Proof. Let $f : X \to Y \in \mathcal{P}$ be a perfect mapping and let Y be an arbitrary subset of X. Then there exist a space $Z \in \mathcal{P}$ and a mapping $g : X \to Z$ such that $g^{-1}g(A) = A$. Put $\varphi = f\Delta g$. As in Lemma 0.3 φ is perfect and $\varphi^{-1}\varphi(A) = A$. The set $\varphi(A)$ is a σ-space because it is a subspace of $Y \times Z$ which is a σ-space. Thus $\varphi(A)$ is a strong Σ-space (see [**10**])) and consequently A is also a strong Σ-space as this property is an inverse invariant under perfect mappings. Therefore X is a hereditarily strong Σ-space. On the other hand, X is a perfect space. Indeed, if A is closed in X then $\varphi(A)$ is closed in (a perfect) space $Y \times Z$ so that $\varphi(A)$ is a G_δ-set. But then $A = \varphi^{-1}\varphi(A)$ is a G_δ-set in X. Now we have to apply the following result of Z. Balogh [**8**]: a perfect space X is a σ-space if and only if it is a hereditarily strong Σ-space. The theorem is proved.

Now we are going to prove that a similar result is true for the class of Moore spaces (this class is a subclass of the class of σ-spaces).

Theorem 1.6. *If a Tychonoff space X admits a perfect mapping f onto a Moore space Y and is splittable over the class of Moore spaces, then X is also a Moore space.*

Proof. Using notation from Lemma 0.3 and Theorem 1.5 we have that $\varphi(A)$ is a Moore space. It is known that the perfect inverse image (with completely regular domain) of a Moore space is a subparacompact [**5**], [**10**] p-space [**5**], [**10**]. So, every $A \subset X$ is a p-space, i.e. X is hereditarily p-space. Also, every $A \subset X$

is θ-refinable [**10**], because every subparacompact space is θ-refinable. The space X is perfect — that can be shown as in Theorem 1.5. In [**14**] Pytkeev has proved that hereditarily p-spaces are developable if and only if they are perfect. Hence X is a developable space and as X is (completely) regular, X is a Moore space. The theorem is proved.

It should be noted that in [**6**] it was proved (using a result of Balogh-Pytkeev [**8**], [**14**]) that every paracompact p-space splittable over the class of metrizable spaces is also metrizable.

2. Convergence properties

All undefined concepts can be found in [**1**], [[**2**], [**13**].

THEOREM 2.1. *If a space X admits a perfect mapping f onto a $\langle 3\text{-}FU\rangle$-space Y and is splittable over the class of countably compact FU-spaces, then X is a Fréchet-Urysohn space.*

Proof. Note that X is a k-space and prove first that $t(X) \leq \aleph_0$. According to a result of Rančin [**15**] for this it is enough to prove that for every compact $B \subset X$ one has $t(B) \leq \aleph_0$. We have that B is splittable over the class of spaces of countable tightness. As B is compact the tightness of B is also countable as was proved by Arhangel'skiĭ in [**4**].

Now, let T be any countable subset of X. Fix a continuous mapping $g : X \to Z$ onto some countably compact FU space Z such that $g^{-1}g(T) = T$. Consider $\varphi = f\Delta g$. The space $Y \times Z$ is an FU-space (see [**2**]) so that $\varphi(T)$ is also an FU-space and thus it is a k-space. As the property being a k-space is an inverse invariant under perfect mappings we have that T is a k-space (according to Lemma 0.2). Since every subspace of T is countable we have: every subspace of T is a k-space, so that T is an FU-space by the well known result of Arhangel'skiĭ. Hence we get: $t(X) \leq \aleph_0$ and every countable subset of X is an FU-space. From this it follows that X is a Fréchet-Urysohn space (see [**2**], [**13**]). The theorem is proved.

THEOREM 2.2. *If a space X admits a perfect mapping f onto a bisequential space Y and is splittable over the class of all bisequential spaces, then X is an $\aleph_0$-bisequential space.*

Proof. The proof is similar to the proof of Theorem 2.1. We use the same notation as in that theorem. The space $Y \times Z$ is bisequential, so that $\varphi(A)$ is bisequential. Thus $\varphi(A)$ is a bi-k-space. The property being a bi-k-space is an inverse invariant under perfect mappings so that A is bi-k in X. Hence every subspace of X is a bi-k-space. From a result of Arhangel'skiĭ [**1**] it follows that X is an $\aleph_0$-bisequential space.

Remark 2.3. In a similar way it can be proved: if a space X admits a perfect mapping onto an $\aleph_0$-bisequential space and is splittable over the class of strongly FU-spaces, then X is FU.

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