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ON SOME HYPERSURFACES OF A RECURRENT RIEMANNIAN SPACE

Mileva Prvanović

Abstract. We consider quasiumbilical hypersurface of a recurrent Riemannian space and find the conditions for such hypersurface to be conformally recurrent. We prove (assuming some conditions) that the curvatutre tensor of such hypersurface has the form (3.10) and discuss the case when the hypersurface is conformally flat.

1. Introduction. Let $(\overline{M}, \overline{g})$ be an $(n+1)$-dimensional $(n > 3)$ Riemannian space covered by a system of coordinate neighbourhoods (U, y^α). Let (M, g) be a hypersurface of (M, g) defined in a local coordinate system by means of the system of parametric equations $y^\alpha = y^\alpha(x^i)$, where g is the induced metric. Here and in the sequel, Greek indices take the values $1, 2, \dots, n+1$, and Latin indices — the values $1, 2, \dots, n$. Let N^α be a local unit normal to (M, g) and let $B_i^\alpha = \partial y^\alpha / \partial x^i$. Then

$$g_{ij} = \overline{g}_{\alpha\beta} B_i^\alpha B_j^\beta, \tag{1.1}$$

$$\overline{g}_{\alpha\beta} N^\alpha B_i^\beta = 0, \qquad \overline{g}_{\alpha\beta} N^\alpha N^\beta = \varepsilon, \quad \varepsilon = \pm 1, \tag{1.2}$$

$$B_i^\alpha B_j^\beta g^{ij} = \overline{g}^{\alpha\beta} - \varepsilon N^\alpha N^\beta. \tag{1.3}$$

We denote by $\overline{R}_{\alpha\beta\gamma\delta}$, $\overline{R}_{\alpha\beta}$ and $\overline{R}$ the curvature tensor, the Ricci tensor and the scalar curvature of $(\overline{M}, \overline{g})$ respectively and by R_{ijkl}, R_{ij} and R the corresponding objects of the hypersurface. Let h be the second fundamental form of the hypersurface and let ∇ be the operator of the van der Waerden-Bortolotti covariant derivative. Then the Gauss and Codazzi equations for (M, g) of $(\overline{M}, \overline{g})$ can be written in the form (c.f.e.g. [**1**], p. 149)

$$\overline{R}_{\alpha\beta\gamma\delta} B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta = R_{ijkl} - \varepsilon(h_{il} h_{jk} - h_{ik} h_{jl}),$$
$$\overline{R}_{\alpha\beta\gamma\delta} N^\alpha B_j^\beta B_k^\gamma B_l^\delta = \nabla_l h_{jk} - \nabla_k h_{jl}.$$

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Also (c.f. [**1**, pp. 147–148])

$$\nabla_r B^\beta_j = \varepsilon h_{rj} N^\beta, \qquad \nabla_r N^\alpha = -h_{ra} g^{at} B^\alpha_t. \tag{1.4}$$

If there exist on (M, g) two functions α, β and a 1-form v such that

$$h_{ij} = \alpha g_{ij} + \beta v_i v_j, \tag{1.5}$$

(M, g) is said to be quasiumbilical ([**2**, p. 147]). If $\beta = 0$, (M, g) is an umbilical hypersurface. Miyazawa and Chūman [**3**] investigated totally umbilical subspaces of recurrent Riemannian space. Among others, they proved that such subspece is conformally recurrent (c.f. also [**4**]).

In this paper we consider quasiumbilical hypersurface of a recurrent Riemannian space assuming

$$\nabla_k v_i = \sigma g_{ik} + t_k v_i, \tag{1.6}$$

where t_k and σ are a covariant vector field and a scalar function on M respectively. Also, we suppose $v_a v^a \neq 0$.

The aim of this paper is to find the necessary and sufficient conditions for such hypersurface to be conformally recurrent. After that, we prove that if $\alpha \neq 0$, $\alpha n + \beta v_a v^a \neq 0$, the curvature tensor of the hypersurface has the form (3.10) and discuss the case when the hypersurface is conformally flat.

Using (1.5) and (1.6), we can rewrite the Gauss and Codazzi equations as follows

$$\begin{aligned} \overline{R}_{\alpha\beta\gamma\delta} B^\alpha_i B^\beta_j B^\gamma_k B^\delta_l = {} & R_{ijkl} - \varepsilon\alpha^2 (g_{il} g_{jk} - g_{ik} g_{jl}) \\ & - \varepsilon\alpha\beta (g_{il} v_j v_k + g_{jk} v_i v_l - g_{ik} v_j v_l - g_{jl} v_i v_k), \end{aligned} \tag{1.7}$$

$$\overline{R}_{\alpha\beta\gamma\delta} N^\alpha B^\beta_j B^\gamma_k B^\delta_l = g_{jk} w_l - g_{jl} w_k + v_j (v_k T_l - v_l T_k), \tag{1.8}$$

where

$$w_l = \alpha_l - \beta\sigma v_l, \quad T_l = \beta_l + 2\beta t_l, \quad \alpha_l = \frac{\partial\alpha}{\partial x^l}, \quad \beta_l = \frac{\partial\beta}{\partial x^l}. \tag{1.9}$$

As for (1.4), they became

$$\nabla_r B^\beta_j = \varepsilon(\alpha g_{rj} + \beta v_r v_j) N^\beta, \tag{1.10}$$

$$\nabla_r N^\alpha = -(\alpha B^\alpha_r + \beta v_r v^t B^\alpha_t). \tag{1.11}$$

From (1.6), we obtain

$$\nabla_l \nabla_k v_j - \nabla_k \nabla_l v_j = g_{kj}(\sigma_l - \sigma t_l) - g_{lj}(\sigma_k - \sigma t_k) + v_j(\nabla_l t_k - \nabla_k t_l).$$

Using the Ricci identity, we get

$$-v_a R^a{}_{jkl} = g_{jk} S_l - g_{lj} S_k + v_j M_{lk}, \tag{1.12}$$

where

$$\sigma_l = \frac{\partial\sigma}{\partial x^l}, \qquad S_l = \sigma_l - \sigma t_l, \qquad M_{lk} = \nabla_l t_k - \nabla_k t_l. \tag{1.13}$$

2. Quasiumbilical hypersurface of a recurrent Riemannian space. Applying the operator ∇_r to (1.7) and using (1.10), we obtain

$$\begin{aligned}
&\nabla_\rho \overline{R}_{\alpha\beta\gamma\delta} B^\rho_r B^\alpha_i B^\beta_j B^\gamma_k B^\delta_l \\
&\quad + \varepsilon(\alpha g_{ri} + \beta v_r v_i)\overline{R}_{\alpha\beta\gamma\delta} N^\alpha B^\beta_j B^\gamma_k B^\delta_l - \varepsilon(\alpha g_{rj} + \beta v_r v_j)\overline{R}_{\beta\alpha\gamma\delta} N^\beta B^\alpha_i B^\gamma_k B^\delta_l \\
&\quad + \varepsilon(\alpha g_{rk} + \beta v_r v_k)\overline{R}_{\gamma\delta\alpha\beta} N^\gamma B^\delta_l B^\alpha_i B^\beta_j - \varepsilon(\alpha g_{rl} + \beta v_r v_l)\overline{R}_{\delta\gamma\alpha\beta} N^\delta B^\gamma_k B^\alpha_i B^\beta_j \\
&= \nabla_r \big[R_{ijkl} - \varepsilon\alpha^2(g_{il}g_{jk} - g_{ik}g_{jl}) \\
&\quad -\varepsilon\alpha\beta(g_{il}v_jv_k + g_{jk}v_iv_l - g_{ik}v_jv_l - g_{jl}v_iv_k)\big].
\end{aligned}$$

Substituting (1.8) into this equation, we find

$$\begin{aligned}
&\nabla_\rho \overline{R}_{\alpha\beta\gamma\delta} B^\rho_r B^\alpha_i B^\beta_j B^\gamma_k B^\delta_l \\
&\quad + \varepsilon\alpha\big[(g_{ri}g_{jk} - g_{rj}g_{ik})w_l + (g_{rj}g_{il} - g_{ri}g_{jl})w_k \\
&\qquad + (g_{rk}g_{li} - g_{rl}g_{ik})w_j + (g_{rl}g_{kj} - g_{rk}g_{lj})w_i\big] \\
&\quad + \varepsilon\beta v_r\big[g_{jk}(v_iw_l + v_lw_i) + g_{il}(v_kw_j + v_jw_k) \\
&\qquad - g_{jl}(v_iw_k + v_kw_i) - g_{ik}(v_jw_l + v_lw_j)\big] \\
&\quad + \varepsilon\alpha\big[g_{ri}v_j(v_kT_l - v_lT_k) + g_{rj}v_i(v_lT_k - v_kT_l) \\
&\qquad + g_{rk}v_l(v_iT_j - v_jT_i) + g_{rl}v_k(v_jT_i - v_iT_j)\big] \\
&= \nabla_r\big[R_{ijkl} - \varepsilon\alpha^2(g_{il}g_{jk} - g_{ik}g_{jl}) \\
&\qquad - \varepsilon\alpha\beta(g_{il}v_jv_k + g_{jk}v_iv_l - g_{ik}v_jv_l - g_{jl}v_iv_k)\big].
\end{aligned}$$

In view of (1.6), this can be rewritten in the form

$$\begin{aligned}
\nabla_r R_{ijkl} &= \nabla_\rho \overline{R}_{\alpha\beta\gamma\delta} B^\rho_r B^\alpha_i B^\beta_j B^\gamma_k B^\delta_l + 2\varepsilon\alpha\alpha_r(g_{il}g_{jk} - g_{ik}g_{jl}) \\
&\varepsilon\big[(\alpha\beta)_r + 2\alpha\beta t_r - 2\beta^2\sigma v_r\big](g_{il}v_jv_k + g_{jk}v_iv_l - g_{ik}v_jv_l - g_{jl}v_iv_k) \\
&\quad + \varepsilon\alpha\big[(g_{ri}g_{jk} - g_{rj}g_{ik})\alpha_l + (g_{rj}g_{il} - g_{ri}g_{jl})\alpha_k \\
&\qquad + (g_{rk}g_{li} - g_{rl}g_{ik})\alpha_j + (g_{rl}g_{kj} - g_{rk}g_{lj})\alpha_i\big] \\
&\quad + \varepsilon\beta v_r\big[g_{jk}(v_i\alpha_l + v_l\alpha_i) + g_{il}(v_k\alpha_j + v_j\alpha_k) \\
&\qquad - g_{jl}(v_i\alpha_k + v_k\alpha_i) - g_{ik}(v_j\alpha_l + v_l\alpha_j)\big] \\
&\quad + \varepsilon\alpha\big[g_{ri}v_j(v_kT_l - v_lT_k) + g_{rj}v_i(v_lT_k - v_kT_l) \\
&\qquad + g_{rk}v_l(v_iT_j - v_jT_i) + g_{rl}v_k(v_jT_i - v_iT_j)\big].
\end{aligned}$$

Now, let us suppose that the $(\overline{M}, \overline{g})$ is a recurrent Riemannian space, i.e.

$$\nabla_\rho \overline{R}_{\alpha\beta\gamma\delta} = a_\rho \overline{R}_{\alpha\beta\gamma\delta}. \tag{2.1}$$

Taking into account (1.7), the preceding relation becomes

$$\begin{aligned}
\nabla_r R_{ijkl} &= a_r R_{ijkl} + \varepsilon(2\alpha\alpha_r - \alpha^2 a_r)(g_{il}g_{jk} - g_{ik}g_{jl}) \\
&\quad + \varepsilon\big[(\alpha\beta)_r + 2\alpha\beta t_r - 2\beta^2\sigma v_r - \alpha\beta a_r\big](g_{il}v_jv_k + g_{jk}v_iv_l - g_{ik}v_jv_l - g_{jl}v_iv_k)
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\alpha\big[(g_{ri}g_{jk}-g_{rj}g_{ik})\alpha_l+(g_{rj}g_{il}-g_{ri}g_{jl})\alpha_k\\
&\quad+(g_{rk}g_{il}-g_{rl}g_{ik})\alpha_j+(g_{rl}g_{kj}-g_{rk}g_{lj}\alpha_i)\big]\\
&+\varepsilon\beta v_r\big[g_{jk}(v_i\alpha_l+v_l\alpha_i)+g_{il}(v_k\alpha_j+v_j\alpha_k)\\
&\quad-g_{jl}(v_i\alpha_k+v_k\alpha_i)-g_{ik}(v_j\alpha_l+v_l\alpha_j)\big]\\
&+\varepsilon\alpha\big[g_{ri}v_j(v_kT_l-v_lT_k)+g_{rj}v_i(v_lT_k-v_kT_l)\\
&\quad+g_{rk}v_l(v_iT_j-v_jT_i)+g_{rl}v_k(v_jT_i-v_iT_j)\big],
\end{aligned}
\tag{2.2}
$$

where

$$a_r=a_\rho B^\rho{}_r.$$

From (2.2), we have

$$
\begin{aligned}
\nabla_rR_{jk}&=a_rR_{jk}\\
&+\varepsilon g_{jk}\big\{(2n\alpha+\beta v_av^a)\alpha_r+\alpha v_av^a\beta_r-\alpha[(n-1)\alpha+\beta v_av^a]a_r\\
&\quad+2\alpha\beta v_av^at_r+2\beta[\alpha_av^a-\beta\sigma v_av^a]v_r\big\}\\
&+\varepsilon v_jv_k\big[(n-2)\beta a_r+n\alpha\beta_r+2n\alpha\beta t_r-2(n-2)\beta^2\sigma v_r\\
&\quad-(n-2)\alpha\beta a_r\big]\\
&+\varepsilon\alpha g_{rj}\big[(n-2)\alpha_k+v_av^aT_k-T_av^av_k\big]\\
&+\varepsilon\alpha g_{rk}\big[(n-2)\alpha_j+v_av^aT_j-T_av^av_j\big]\\
&+\varepsilon v_rv_k\big[(n-2)\beta\alpha_j-\alpha T_j\big]+\varepsilon v_rv_j\big[(n-2)\beta\alpha_k-\alpha T_l\big],
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
\nabla_rR&=a_rR+\varepsilon\big\{[2(n^2+n-2)\alpha+2(n-1)\beta v_av^a]\alpha_r+2(n+1)\alpha v_av^a\beta_r\\
&\quad-[n(n-1)\alpha^2+2\alpha\beta(n-1)v_av^a]a_r+4(n+1)\alpha\beta v_av^at_r\\
&\quad+4[(n-1)\beta\alpha_av^a-(n-1)\beta^2\sigma v_av^a-\alpha T_av^a]v_r\big\}.
\end{aligned}
\tag{2.4}
$$

Now, let us consider the covariant derivative of the conformal curvature tensor C_{ijkl} of the hypersurface (M,g):

$$
\begin{aligned}
\nabla_rC_{ijkl}&=\nabla_rR_{ijkl}-\frac{1}{n-2}(g_{jk}\nabla_rR_{il}-g_{jl}\nabla_rR_{ik}+g_{il}\nabla_rR_{jk}-g_{ik}\nabla_rR_{jl})\\
&\quad+\frac{1}{(n-1)(n-2)}\nabla_rR(g_{jk}g_{il}-g_{jl}g_{ik}).
\end{aligned}
$$

Substituting (2.2), (2.3) and (2.4) into this relation, we obtain, after some calculation

$$
\begin{aligned}
\nabla_rC_{ijkl}&=a_rC_{ijkl}+\frac{4\alpha}{(n-1)(n-2)}\varepsilon(v_av^aT_r-T_av^av_r)(g_{il}g_{jk}-g_{ik}g_{jl})\\
&\quad-\frac{2\alpha}{n-2}\varepsilon T_r(g_{il}v_jv_k+g_{jk}v_iv_l-g_{ik}v_jv_l-g_{jl}v_iv_k)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\alpha}{n-2}\varepsilon\big[(g_{ri}g_{jk}-g_{rj}g_{ik})(T_a v^a v_l - v_a v^a T_l)\\
&\quad+(g_{rj}g_{il}-g_{ri}g_{jl})(T_a v^a v_k - v_a v^a T_k)\\
&\quad+(g_{rk}g_{il}-g_{rl}g_{ik})(T_a v^a v_j - v_a v^a T_j)\\
&\quad+(g_{rl}g_{jk}-g_{rk}g_{lj})(T_a v^a v_i - v_a v^a T_i)\big]\\
&+\frac{\alpha}{n-2}\varepsilon v_r\big[g_{jk}(v_i T_l + v_l T_i) + g_{il}(v_j T_k + v_k T_j)\\
&\quad-g_{jl}(v_i T_k + v_k T_i) - g_{ik}(v_j T_l + v_l T_j)\big]\\
&+\varepsilon\alpha\big[g_{ri}v_j(v_k T_l - v_l T_k) + g_{rj}v_i(v_l T_k - v_k T_l)\\
&\quad+g_{rk}v_l(v_i T_j - v_j T_i) + g_{rl}v_k(v_j T_i - v_i T_j)\big].
\end{aligned}
$$

If $\alpha = 0$, (2.5) reduces to

$$
\nabla_r C_{ijkl} = a_r C_{ijkl}, \tag{2.6}
$$

i.e. the hypersurface is conformally recurrent or (in the case $a_r = 0$) conformally symmetric.

If $\alpha \neq 0$, the condition (2.6) is satisfied if and only if

$$
\begin{aligned}
&\frac{4}{n-1}(v_a v^a T_r - T_a v^a v_r)(g_{il}g_{jk} - g_{ik}g_{jl})\\
&\quad-2T_r(g_{il}v_j v_k + g_{jk}v_i v_l - g_{ik}v_j v_l - g_{jl}v_i v_k)\\
&\quad+(g_{ri}g_{jk}-g_{rj}g_{ik})(T_a v^a v_l - v_a v^a T_l) + (g_{rj}g_{il}-g_{ri}g_{jl})(T_a v^a v_k - v_a v^a T_k)\\
&\quad+(g_{rk}g_{il}-g_{rl}g_{ik})(T_a v^a v_j - v_a v^a T_j) + (g_{rl}g_{jk}-g_{rk}g_{lj})(T_a v^a v_i - v_a v^a T_i)\\
&\quad+v_r\big[g_{jk}(v_i T_l + T_i v_l) + g_{il}(v_j T_k + v_k T_j) - g_{jl}(v_i T_k + v_k T_i) - g_{ik}(v_j T_l + v_l T_j)\big]\\
&\quad(n-2)\big[g_{ri}v_j(v_k T_l - v_l T_k) + g_{rj}v_i(v_l T_k - v_k T_l)\\
&\quad\quad+g_{rk}v_l(v_i T_j - v_j T_i) + g_{rl}v_k(v_j T_i - v_i T_j)\big] = 0.
\end{aligned}
$$

Transvecting this relation with $v^i v^l$, we find

$$
\begin{aligned}
&-\frac{2}{n-1}(g_{jk}v_b v^b - v_j v_k)(v_a v^a T_r - T_a v^a v_r)\\
&+(g_{rj}v_a v^a - v_r v_j)(v_b v^b T_k - T_b v^b v_k) + (g_{rk}v_a v^a - v_r v_k)(v_b v^b T_j - T_b v^b v_j) = 0.
\end{aligned}
$$

Transvecting the last relation with g^{rj}, we get

$$
\frac{(n+1)(n-2)}{n-1}v_b v^b(v_a v^a T_k - T_a v^a v_k) = 0,
$$

or

$$
T_k = \frac{T_a v^a}{v_b v^b}v_k,
$$

because $n > 3$ and $v_a v^a \neq 0$.

Conversely, if $T_k = f v_k$, where f is a scalar function, (2.5) reduces to (2.6). Thus we have

Theorem 1. *Let* $(\overline{M},\overline{g})$ *be a recurrent Riemannian space with* a_ρ *as a recurrence vector field. Let* (M,g) *be its quasiumbilical hypersurface satisfying* (1.6). *Then*:

if $\alpha = 0$, (M,g) *is a conformally recurrent manifold with* $a_r = a_\rho B^\rho_r$ *as a recurrence vector field*;

if $\alpha \neq 0$, (M,g) *is a conformally recurrent space if and only if* $T_k = fv_k$, *where* f *is a scalar function.*

In the following section we shall show that in the case $\alpha \neq 0$, $T_k = fv_k$, the additional informations about the hypersurface (M,g) can be obtained.

3. The case $\alpha \neq 0$, $\alpha n + \beta v_a v^a \neq 0$**.** In the case $T_k = fv_k$ the Codazzi equation (1.8) reduces to

$$\overline{R}_{\alpha\beta\gamma\delta}N^\alpha B^\beta_j B^\gamma_k B^\delta_l = g_{jk}w_l - g_{jl}w_k. \tag{3.1}$$

Applying the operator ∇_r to (3.1) and using (1.10) and (1.11) we get

$$\begin{aligned} g_{jk}\nabla_r w_l - g_{jl}\nabla_r w_k &= \nabla_\rho \overline{R}_{\alpha\beta\gamma\delta}N^\alpha B^\beta_j B^\gamma_k B^\delta_l B^\rho_r \\ &- \alpha\overline{R}_{\alpha\beta\gamma\delta}B^\alpha_r B^\beta_j B^\gamma_k B^\delta_l - \beta v_r v^t \overline{R}_{\alpha\beta\gamma\delta}B^\alpha_t B^\beta_j B^\gamma_k B^\delta_l \\ &- \varepsilon(\alpha g_{rk} + \beta v_r v_k)\overline{R}_{\alpha\beta\gamma\delta}N^\alpha B^\beta_j B^\delta_l N^\gamma + \varepsilon(\alpha g_{rl} + \beta v_r v_l)\overline{R}_{\alpha\beta\gamma\delta}N^\alpha B^\beta_j B^\gamma_k N^\delta. \end{aligned}$$

Substituting (2.1) and (3.1) into this relation, putting

$$\overline{R}_{\alpha\beta\gamma\delta}N^\alpha B^\beta_j B^\delta_l N^\gamma = P_{jl}, \tag{3.2}$$

and taking into account (1.7), we find

$$\begin{aligned} \alpha R_{rjkl} &= -\beta v_r v^a R_{ajkl} + \varepsilon\alpha^3(g_{rl}g_{jk} - g_{rk}g_{jl}) \\ &+ \varepsilon\alpha^2\beta(g_{rl}v_j v_k + g_{jk}v_r v_l - g_{rk}v_j v_l - g_{jl}v_r v_k) \\ &+ g_{jk}(\varepsilon\alpha^2\beta v_l v_r + \varepsilon\alpha\beta^2 v_a v^a v_r v_l + a_r w_l - \nabla_r w_l) \\ &- g_{jl}(\varepsilon\alpha^2\beta v_k v_r + \varepsilon\alpha\beta^2 v_a v^a v_r v_k + a_r w_k - \nabla_r w_k) \\ &- \varepsilon(\alpha g_{rk} + \beta v_r v_k)P_{jl} + \varepsilon(\alpha g_{rl} + \beta v_r v_l)P_{jk}. \end{aligned} \tag{3.3}$$

Substituting (1.12), we have

$$\begin{aligned} \alpha R_{rjkl} &= \varepsilon\alpha^3(g_{rl}g_{jk} - g_{rk}g_{jl}) \\ &+ \varepsilon\alpha^2\beta(g_{rl}v_j v_k + g_{jk}v_r v_l - g_{rk}v_j v_l - g_{jl}v_r v_k) \\ &+ g_{jk}[\varepsilon(\alpha^2\beta + \alpha\beta^2 v_a v^a)v_r v_l + \beta v_r S_l + a_r w_l - \nabla_r w_l] \\ &- g_{jl}[\varepsilon(\alpha^2\beta + \alpha\beta^2 v_a v^a)v_r v_k + \beta v_r S_k + a_r w_k - \nabla_r w_k] \\ &+ \beta v_r v_j M_{lk} - \varepsilon(\alpha g_{rk} + \beta v_r v_k)P_{jl} + \varepsilon(\alpha g_{rl} + \beta v_r v_l)P_{jk}. \end{aligned}$$

Interchanging the indices r and j and also k and l and adding the obtained relation to the preceding one, we get

$$2\alpha R_{rjkl} = 2\varepsilon\alpha^3(g_{rl}g_{jk} - g_{rk}g_{jl})$$

$$
\begin{aligned}
&+\varepsilon(3\alpha^2\beta+\alpha\beta^2 v_a v^a)(g_{rl}v_jv_k+g_{jk}v_rv_l-g_{rk}v_jv_l-g_{jl}v_rv_k)\\
&+g_{jk}(\beta v_r S_l+a_r w_l-\nabla_r w_l)-g_{jl}(\beta v_r S_k+a_r w_k-\nabla_r w_k)\\
&+g_{rl}(\beta v_j S_k+a_j w_k-\nabla_j w_k)-g_{rk}(\beta v_j S_l+a_j w_l-\nabla_j w_l)\\
&-\varepsilon(\alpha g_{rk}+\beta v_r v_k)P_{jl}+\varepsilon(\alpha g_{rl}+\beta v_r v_l)P_{jk}\\
&-\varepsilon(\alpha g_{jl}+\beta v_j v_l)P_{rk}+\varepsilon(\alpha g_{jk}+\beta v_j v_k)P_{rl}.
\end{aligned}
$$

Now, we replace the indices r, j, k, l with l, k, j, r respectively. Adding the obtained relation to the preceding one and taking into account that P_{ij} is a symmetric tensor, we find

$$
\begin{aligned}
4\alpha R_{rjkl} &= 4\varepsilon\alpha^3(g_{rl}g_{jk}-g_{rk}g_{jl})\\
&+2\varepsilon(3\alpha^2\beta+\alpha\beta^2 v_a v^a)(g_{rl}v_jv_k+g_{jk}v_rv_l-g_{rk}v_jv_l-g_{jl}v_rv_k)\\
&+g_{jk}[\beta(v_rS_l+v_lS_r)+a_rw_l+a_lw_r-(\nabla_r w_l+\nabla_l w_r)]\\
&-g_{jl}[\beta(v_rS_k+v_kS_r)+a_rw_k+a_kw_r-(\nabla_r w_k+\nabla_k w_r)]\\
&+g_{rl}[\beta(v_jS_k+v_kS_j)+a_jw_k+a_kw_j-(\nabla_j w_k+\nabla_k w_j)]\\
&-g_{rk}[\beta(v_jS_l+v_lS_j)+a_jw_l+a_lw_j-(\nabla_j w_l+\nabla_l w_j)]\\
&-2\varepsilon(\alpha g_{rk}+\beta v_r v_k)P_{jl}+2\varepsilon(\alpha g_{rl}+\beta v_r v_l)P_{jk}\\
&-2\varepsilon(\alpha g_{jl}+\beta v_j v_l)P_{rk}+2\varepsilon(\alpha g_{jk}+\beta v_j v_k)P_{rl}.
\end{aligned}
\tag{3.4}
$$

Now we have to determine P_{ij}. To do this, we interchange the indices r and j in (3.3) and add the obtained relation to (3.3). We get

$$
\begin{aligned}
&-\beta v_r v^a R_{ajkl}-\beta v_j v^a R_{arkl}\\
&+g_{jk}[\varepsilon(\alpha^2\beta+\alpha\beta^2 v_a v^a)v_lv_r+a_rw_l-\nabla_r w_l]\\
&-g_{jl}[\varepsilon(\alpha^2\beta+\alpha\beta^2 v_a v^a)v_kv_r+a_rw_k-\nabla_r w_k]\\
&+g_{rk}[\varepsilon(\alpha^2\beta+\alpha\beta^2 v_a v^a)v_jv_l+a_jw_l-\nabla_j w_l]\\
&-g_{rl}[\varepsilon(\alpha^2\beta+\alpha\beta^2 v_a v^a)v_jv_k+a_jw_k-\nabla_j w_k]\\
&-\varepsilon(\alpha g_{rk}+\beta v_r v_k)P_{jl}+\varepsilon(\alpha g_{rl}+\beta v_r v_l)P_{jk}\\
&-\varepsilon(\alpha g_{jk}+\beta v_j v_k)P_{rl}+\varepsilon(\alpha g_{jl}+\beta v_j v_l)P_{rk}=0.
\end{aligned}
$$

This, in view of (1.12), can be written in the form

$$
\begin{aligned}
&2\beta v_r v_j M_{lk}\\
&\quad+g_{jk}[\varepsilon(\alpha^2\beta+\alpha\beta^2 v_a v^a)v_lv_r+a_rw_l+\beta v_rS_l-\nabla_r w_l]\\
&\quad-g_{jl}[\varepsilon(\alpha^2\beta+\alpha\beta^2 v_a v^a)v_kv_r+a_rw_k+\beta v_rS_k-\nabla_r w_k]\\
&\quad+g_{rk}[\varepsilon(\alpha^2\beta+\alpha\beta^2 v_a v^a)v_jv_l+a_jw_l+\beta v_jS_l-\nabla_j w_l]\\
&\quad-g_{rl}[\varepsilon(\alpha^2\beta+\alpha\beta^2 v_a v^a)v_jv_k+a_jw_k+\beta v_jS_k-\nabla_j w_k]\\
&\quad-\varepsilon(\alpha g_{rk}+\beta v_r v_k)P_{jl}+\varepsilon(\alpha g_{rl}+\beta v_r v_l)P_{jk}\\
&\quad-\varepsilon(\alpha g_{jk}+\beta v_j v_k)P_{rl}+\varepsilon(\alpha g_{jl}+\beta v_j v_l)P_{rk}=0.
\end{aligned}
$$

Transvecting this with g^{kr}, we have

$$\begin{aligned}&\varepsilon(n\alpha^2\beta + n\alpha\beta^2 v_a v^a + \beta P)v_j v_l + n a_j w_l + n\beta v_j S_l - n\nabla_j w_l\\ &\quad - g_{jl}[\varepsilon(\alpha^2\beta + \alpha\beta^2 v_a v^a)v_b v^b - \varepsilon\alpha P + w_a a^a + \beta S_a v^a - \nabla_a w^a] + 2\beta v_j v^a M_{la}\\ &= \varepsilon(\alpha n + \beta v_a v^a)P_{jl} + \varepsilon\beta(v_j v^a P_{al} - v_l v^a P_{ja}),\end{aligned} \tag{3.5}$$

where $P = P_{ab}g^{ab}$. This function can be determined as follows.

Taking into account (1.3), we get from (3.2)

$$P = \overline{R}_{\alpha\gamma}N^\alpha N^\gamma.$$

On the other hand, transvecting (1.7) with g^{il} and using (1.3) and (3.2), we find

$$\overline{R}_{\beta\gamma}B^\beta_j B^\gamma_k - \varepsilon P_{jk} = R_{jk} - \varepsilon[(n-1)\alpha^2 + \alpha\beta v_a v^a]g_{jk} - \varepsilon(n-2)\alpha\beta v_j v_k,$$

from which, transvecting with g^{jk}, we get

$$P = \frac{\varepsilon}{2}(\overline{R} - R) + \frac{n(n-1)}{2}\alpha^2 + (n-1)\alpha\beta v_a v^a. \tag{3.6}$$

Now, let us return to (3.5). Interchanging the indices j and l in (3.5) and subtracting the obtained relation from (3.5), we find

$$\begin{aligned}&\varepsilon\beta(v_j v^a P_{al} - v_l v^a P_{ja}) = \beta(v_j v^a M_{la} - v_l v^a M_{ja})\\ &+ \frac{1}{2}n(a_j w_l - a_l w_j) + \frac{1}{2}n\beta(v_j S_l - v_l S_j) - \frac{n}{2}(\nabla_j w_l - \nabla_l w_j).\end{aligned}$$

Substituting this into (3.5), we find

$$\begin{aligned}&\varepsilon(\alpha n + \beta v_a v^a)P_{jl} = \varepsilon(n\alpha^2\beta + \alpha\beta^2 v_a v^a + \beta P)v_j v_l\\ &\quad + \frac{n}{2}(a_j w_l + a_l w_j) + \frac{n}{2}\beta(v_j S_l + v_l S_j) - \frac{n}{2}(\nabla_j w_l + \nabla_l w_j)\\ &\quad - g_{jl}[\varepsilon(\alpha^2\beta + \alpha\beta^2 v_a v^a)v_b v^b - \varepsilon\alpha P + w_a a^a + \beta S_a v^a - \nabla_a w^a]\\ &\quad + \beta(v_j v^a M_{la} + v_l v^a M_{ja}).\end{aligned}$$

Thus, supposing

$$\alpha n + \beta v_a v^a \neq 0,$$

we have

$$\begin{aligned}P_{jl} = C\left[v_j\left(\frac{n}{2}S_l + v^a M_{la}\right) + v_l\left(\frac{n}{2}S_j + v^a M_{ja}\right)\right]&\\ + D[a_j w_l + a_l w_j - (\nabla_j w_l + \nabla_l w_j)] + E v_j v_l + F g_{jl},&\end{aligned} \tag{3.7}$$

where we have put

$$C = \frac{\beta}{\alpha n + \beta v_a v^a}, \qquad D = \frac{n}{2(\alpha n + \beta v_a v^a)}, \tag{3.8}$$

$$E = \frac{n\alpha\beta(\alpha + \beta v_a v^a) + \beta P}{\alpha n + \beta v_a v^a},$$
$$F = \frac{\alpha P - \alpha\beta(\alpha + \beta v_a v^a) v_b v^b - \varepsilon w_b a^b - \varepsilon\beta S_a v^a + \varepsilon \nabla_a w^a}{\alpha n + \beta v_a v^a} \tag{3.9}$$

Substituting (3.7) into (3.4), we get

$$\begin{aligned} R_{rjkl} &= A(g_{rl} g_{jk} - g_{rk} g_{jl}) \\ &+ g_{rl}(B v_j v_k + v_j p_k + v_k p_j + L Q_{jk}) + g_{jk}(B v_r v_l + v_r p_l + v_l p_r + L Q_{rl}) \\ &- g_{rk}(B v_j v_l + v_j p_l + v_l p_j + L Q_{jl}) - g_{jl}(B v_r v_k + v_r p_k + v_k p_r + L Q_{rk}) \\ &+ N(v_j v_k Q_{rl} + v_r v_l Q_{jk} - v_r v_k Q_{jl} - v_j v_l Q_{rk}), \end{aligned} \tag{3.10}$$

where we have put

$$A = \varepsilon(\alpha^2 + F), \qquad B = \frac{\varepsilon}{2}\left(3\alpha\beta + \beta^2 v_a v^a + E + \frac{\beta}{\alpha} F\right) \tag{3.11}$$

$$p_l = S_l\left(\frac{\beta}{\alpha} + \varepsilon n C\right) + 2\varepsilon C v^a M_{la} \tag{3.12}$$

$$L = \frac{1}{\alpha} + 2\varepsilon D, \quad N = 2\varepsilon\frac{\beta}{\alpha} D, \quad Q_{rl} = a_r w_l + a_l w_r - \nabla_r w_l - \nabla_l w_r. \tag{3.13}$$

Thus, we have

Theorem 2. *Let* $(\overline{M}, \overline{g})$ *be a recurrent Riemannian space. Let* (M, g) *be its quasiumbilical hypersurface satisfying* (1.6). *If, in addition,*

$$\alpha \neq 0, \qquad n\alpha + \beta v_a v^a \neq 0, \qquad T_k = f v_k,$$

then the curvature tensor of the hypersurface can be expressed in the form (3.10), *where the functions* A *and* B *are given by* (3.11) *and* (3.9), *the functions* L *and* N *— by* (3.13) *and* (3.8), *while* p_i *is an* 1*-form determined by* (3.12), (3.8) *and* (1.13). *As for the function* P, *it is given by* (3.6).

If $p_l = 0$ and $Q_{ij} = 0$, (3.10) reduces to

$$R_{rjkl} = A(g_{rl} g_{jk} - g_{rk} g_{jl}) + B(g_{rl} v_j v_k + g_{rk} v_r v_l - g_{rk} v_j v_l - g_{jl} v_r v_k),$$

i.e. the hypersurface is a space of quasi-constant curvature. For example, if $\alpha = \text{const.} \neq 0$, and $\beta = 0$, the conditions $p_l = w_l = 0$, according to (1.9), (3.12) and (3.8), are satisfied. The condition $w_l = 0$ implies $Q_{ij} = 0$. On the other hand, if $\beta = 0$, then $B = 0$ and the hypersurface is a space of constant curvature. Thus

Theorem 3. *Umbilical hypersurface of the recurrent Riemannian space satisfying* $\alpha = \textit{const.} \neq 0$ *is a space of constant curvature.*

As a consequence of Theorem 3, we have (c.f. [**5**, Theorem 1])

Corollary. *Umbilical hypersurface of a Cartan-symmetric Riemannian space satisfying* $\alpha = \textit{const.} \neq 0$, *is a space of constant curvature.*

We get from (3.10)

$$\begin{aligned}R_{jk} &= g_{jk}[(n-1)A + 2p_a v^a + LQ_{ab}g^{ab} + Bv_a v^a] \\ &\quad + v_j v_k[(n-2)B + NQ_{ab}g^{ab}] + Q_{jk}[(n-2)L + v_a v^a N] \\ &\quad + (n-2)(p_j v_k + p_k v_j) - N(v_k v^a Q_{ja} + v_j v^a Q_{ak}),\end{aligned} \tag{3.14}$$

$$\begin{aligned}R &= n(n-1)A + 4(n-1)p_a v^a + 2(n-1)PQ_{ab}g^{ab} + 2(n-1)Bv_a v^a \\ &\quad + 2N(Q_{ab}g^{ab}v_p v^p - Q_{ab}v^a v^b).\end{aligned} \tag{3.15}$$

Substituting (3.10), (3.14) and (3.15) into the expression

$$\begin{aligned}C_{rjkl} &= R_{rjkl} - \frac{1}{n-2}(g_{jk}R_{rl} - g_{jl}R_{rk} + g_{rl}R_{jk} - g_{rk}R_{jl}) \\ &\quad + \frac{R}{(n-1)(n-2)}(g_{jk}g_{rl} - g_{jl}g_{rk})\end{aligned}$$

of the conformal curvature tensor, we find

$$\begin{aligned}C_{rjkl} = N\Big\{ &\frac{2}{(n-1)(n-2)}(g_{jk}g_{rl} - g_{jl}g_{rk})(Q_{ab}g^{ab}v_p v^p - Q_{ab}v^a v^b) \\ &+ Q_{rl}v_j v_k + Q_{jk}v_r v_l - Q_{jl}v_r v_k - Q_{rk}v_j v_l \\ &+ \frac{g_{jk}}{n-2}(v_r v^a Q_{al} + v_l v^a Q_{ar} - v_a v^a Q_{rl} - Q_{ab}g^{ab}v_r v_l) \\ &- \frac{g_{jl}}{n-2}(v_r v^a Q_{ak} + v_k v^a Q_{ar} - v_a v^a Q_{rk} - Q_{ab}g^{ab}v_r v_k) \\ &+ \frac{g_{rl}}{n-2}(v_k v^a Q_{ja} + v_j v^a Q_{ak} - v_a v^a Q_{jk} - Q_{ab}g^{ab}v_j v_k) \\ &- \frac{g_{rk}}{n-2}(v_j v^a Q_{la} + v_l v^a Q_{ja} - v_a v^a Q_{jl} - Q_{ab}g^{ab}v_j v_l)\Big\}.\end{aligned}$$

Thus, we have

Corrollary. *Let* $(\overline{M}, \overline{g})$ *be a recurrent Riemannian space and* (M, g) *its quasiumbilical hypersurface satisfying* (1.6)*. If, in addition,*

$$\alpha \neq 0, \qquad \alpha n + \beta v_a v^a \neq 0, \qquad T_k = f v_k, \qquad Q_{ij} = 0,$$

(M, g) *is conformally flat.*

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Institut za matematiku
Prirodno-matematički fakultet
21000 Novi Sad, Jugoslavija

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