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ON COMMON FIXED POINTS IN LOCALLY CONVEX TOPOLOGICAL VECTOR SPACE

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Abstract. We state a new common fixed point theorem for a continuous mapping and a multivalued uper semicontinuous mapping in locally convex topological vector spaces, under some additional conditions.

1. Introduction. The aim of this short paper is to use results given by Fan [**3**] and Browder [**1**], to prove the existence of a common fixed point for a single-valued mapping and a multivalued mapping in locally convex topological vector spaces.

For this sort of result the reader may be referred to [**2**], [**4**] and [**5**], for example.

Throughout, by E we will denote a Hausdorff topological vector space. Furthermore, if K is a nonempty subset of E, then 2^K will be the family of all subsets of K and $C(K)$ the family of all nonempty compact convex subsets of K.

Before proceeding to the statement of our result, we recall the concepts of algebraic boundary and multivalued upper semicontinuous mapping.

Definition 1 [**1**]. Let C be a closed convex subset of E. Then a point x of C is said to lie in the algebraic boundary $\delta(C)$ of C if there exists a finite dimensional subspace F of E such that x lies in the boundary of $C \cap F$.

Definition 2 [**4**]. A multivalued maping $T : K \to 2^K$, $K \subset E$, is said to be *upper semicontinuous* if for any closed subset F of K, $T^{-1}(F) = \{x \in K : T(x) \cap F \neq \varnothing\}$ is a closed subset of K.

2. Main result. The main result is inspired by an interesting paper by Itoh and Takahashi [**4**].

Next we combine a theorem of Browder [**1**] with the classic Fan's fixed point theorem [**3**], to obtain a common fixed point theorem for a continuous mapping f

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of X into E and an upper semicontinuous mapping T of X into 2^X, where X is a nonempty compact convex subset of E.

THEOREM 1. *Let E be a locally convex space and X a nonempty compact convex subset of E. Suppose that $f : X \to E$ is a continuous mapping and $T : X \to 2^X$ is an upper semicontinuous mapping, with $T(x) \in C(X)$ for all $x \in X$. Suppose further that the following condition is satisfied*:

$(*)$ *For each $x \in X$ and for every $u \in \delta(T(x))$ there exists an element v of $T(x)$ and a real number $\lambda > 0$ (both depending on x), such that $f(u)-u = \lambda(v-u)$.*

If $A = \{x \in X : f(x) = x\}$ is convex, then there exists an element w of X such that $w = f(x) \in T(w)$.

Before we prove the main result (Theorem 1 above), we recall results given by Fan [**4**] and Browder [**1**].

THEOREM 2 [**3**]. *If $K \in C(E)$, $T : K \to 2^K$ is an upper semicontinuous mapping and for any $x \in K$, $T(x) \in C(K)$, then $\{x \in K : x \in T(x)\}$ is nonempty compact.*

Remark 1. Theorem 2 is the classic “Fan's fixed point theorem” [**3**].

THEOREM 3 [**1**]. *Let $K \in C(E)$ and f be a continuous mapping of K into E. Suppose that for each u in $\delta(K)$, there exists an element v of K and a real number λ (both depending on u), such that $f(u) - u = \lambda(v - u)$.*

Then f has a fixed point in K.

Proof of Theorem 1. By theorem 3 we observe that, for each $x \in X$, f has a fixed point in $T(x)$. This shows that $x \in X$ implies $T(x) \cap A$ is nonempty. Now, we define a multivalued mapping S of A into 2^A, by $S(x) = A \cap T(x)$, $x \in A$. Since X is compact and f is continuous in X, it follows that A is compact; moreover by hypothesis A is convex. Hence, $S(x)$ is nonempty compact convex, for each $x \in A$.

Furthermore, since $F \subset A$ implies that $S^{-1}(F) = A \cap T^{-1}(F)$, and T is upper semicontinuous, we conclude that S is upper semicontinuous. Thus, by Theorem 2 there exists a fixed point w of S in A. For this w we have $f(w) = w \in T(w)$. The proof is now complete.

Remark 2. Theorem 1 remains true when the condition $(*)$ is replaced by the following one:

$(**)$ *If $x \in X$ and $u \in \delta(T(x))$, then $f(u) \in T(x)$.*

Indeed, if $x \in X$ and $u \in \delta(T(x))$, then $v = f(u) \in T(x)$ by the condition $(**)$. For these v and $\lambda = 1$ we have $f(u) - u = \lambda(v - u)$.

Remark 3. Note that this reduces to Fan's fixed point theorem if f is the identity on X. In this case if $x \in X$ and $u \in \delta(T(x))$, then the condition $(*)$ of the hypothesis of Theorem 1 is satisfied for $v = u$ and $\lambda > 0$, λ arbitrary.

Remark 4. Condition $(*)$ is not necessary for the conclusion of Theorem 1, since one adds the following hypothesis:

a) f is a single-valued mapping of X into X and

b) $f(T(x)) \subset T(f(x))$ for all $x \in X$ (see [**4**]).

Remark 5. In Theorem 1 if f is affine, then A is automatically convex.

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