PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 46 (60), 1989, pp. 173–182

ON HYPERBOLIC HERMITE MANIFOLDS

H. R. Farran and **M. S. Zanoun**

Abstract. We introduce hyperbolic Hermite structures on manifolds. Each such structure induces a pair of null distributions on the manifold. The aim of the paper is to begin the study of the geometry of these structures, and to classify them according to the parallelism and integrability properties of the distributions. In fact, we show that there are fifteen different classes of hyperbolic Hermite manifolds.

1. Introduction and notation

The study of geometric structures on differentiable manifolds is the main concern of differential geometry. Almost Hermitian structures and almost product Riemannian structures are among the most important of these, and they have one feature in common, namely the existence of a (1,1) tensor field F and a (0,2) tensor field g on the manifold. In the almost Hrmitian case $F^2 = -I$ (I is the identity) and $g(FX, FY) = g(X, Y)$ for all vector fields X, Y while in the case of almost product Riemannian manifolds we have $F^2 = I$ and $g(FX, FY) = g(X, Y)$.

Hyperbolic Hermite manifolds, the subject of our study, present a third family of structures related to the above two families in the sense that, here again, there exist tensor fields F, g as before, but the conditions satisfied this time are $F^2 = I$ and $g(FX, FY) = -g(X, Y)$. Throughout the paper, all manifolds M, distributions P, Q and tensor fields F, g are smooth (i.e. C^∞). Our manifolds are always connected and paracompact.

2. Hyperbolic Hermite manifolds

By a *non-trivial almost product structure* on M, we mean a (1,1) tensor field F, $F \neq \pm I$ and $F^2 = I$, where I is the identity tensor field. It is well known that the existence of such an F is equvalent to the existence of a pair of complementary distribution P and Q on M such that $(\dim P)(\dim Q) \neq 0$. These distributions are respectively given by the projection tensor fields $p = [I + F]/2$ and $q = [I - F]/2$. The pair (M, F) is called an *almost product manifold.*

AMS Subject Classification (1980): Primary 53 C 15

Definition 2.1. A hyperbolic Hermite manifold is an almost product manifold (M, F) together with a pseudo-Riemannian metric g such that $g(FX, FY) = -g(X, Y)$ for all vector fields X, Y on M.

This manifold will be denoted by the triple (M, g, F), and the pair (g, F) is called a *hyperbolic Hermite structure.*

Proposition 2.2. *If (M, g, F) is a hyperbolic Hermite manifold, then the distributions P and Q given by F are both null and of the same dimension (where, by a null distribution P we mean that $\forall x \in M$ and all $X, Y \in P_x$ we have $g_x(X, Y) = 0$).*

Proof. The proof that P and Q are null is a direct application of the condition $g(FX, FY) = -g(X, Y)$ and the fact that $X \in P$ $(X \in Q)$ if and only if $FX = X$ $(FX = -X)$. Now let $\dim M = m$, $\dim P = p$, $\dim Q = q$. Since P is null, then $P \subset P^{\perp}$, where $P_x^{\perp} = \{X \in T_xM \mid g_x(X, Y) = 0, \forall Y \in P_x\}$, $x \in M$ any point. This gives $p \leq m - p$ and hence $p \leq m/2$. Similarly, Q is null and hence $q \leq m/2$. But P and Q are complementary, i.e. $p + q = m$ and we therefore conclude that $p = q = m/2$. As an immediate corollary we have:

Corollary 2.3. *A hyperbolic Hermite manifold is even-dimensional.*

Proposition 2.4. *Let (M, g) be a pseudo-Riemannian manifold with two complementary null distributions P and Q. If F is the almost product structure given by P and Q then (M, g, F) is hyperbolic Hermite.*

Proof: Show by direct calculation that $g(FX, FY) = -g(X, Y)$.

Suppose that M is a manifold with two complementary distributions$P$ and Q of the same dimension, and let F be the almost product structure given by P and Q. Now we may ask whether there exists a pseudo-Riemannian metric g on M such that P and Q are null. That is, using the above proposition, (M, g, F) will be hyperbolic Hermite. The answer to this proposition is, in general, negative. As an example take M to be the product of a non-orientable manifold by itself, (i.e.$M = N \times N$, N non-orientable). Clearly M admits a pair of complementary distributions of the same dimension. Since M is non-orientable, it will follow from corollary 2.8 that there exists no g such that (M, g, F) is hyperbolic Hermite.

Proposition 2.5. *Let (M, g, F) be a hyperbolic Hermite manifold of dimension $m = 2k$. Then the metric g is of signature (k, k).*

Proof. Let g be of signature $(r, m - r)$. Since P is null and of dimension k then $k \leq s$ where $s = \min\{r, m - r\}$ [**1**]. Now, without loss of generality we take $s = r$ so $k \leq r$, i.e. $m/2 \leq r$. But $r = \min\{r, m - r\}$, so $r \leq m - r$ and $r \leq m/2$. Hence $r = m/2 = k$.

Again here, starting with a pseudo-Riemannian manifold (M, g) of signature (k, k), although it is known that M admits an almost product structure F whose distributions are k-dimensional [**3**], yet, in general there is no guarantee that such an F exists with (M, g, F) is hyperbolic Hermite. For example, we take $M = N \times N$, N

is non-orientable. If α, β are respectively positive definite and negative definite metrics on N, then the product metric $g = \alpha \times \beta$ is a metric on M of signature (k, k). Using corollary 2.8 again we conclude that there is no F such that (M, g, F) is hyperbolic Hermite.

Let Ω be an almost symplectic structure and F an almost product structure on a manifold M. Then Ω anf F are said to be compatible if $\forall x \in M$, P_x and Q_x are isotropic (i.e. null) with respect to Ω.

Theorem 2.6. *Let (M, g, F) be a hyperbolic Hermite manifold. Then there exists an almost symplectic structure Ω on M, compatible with the tensor field F.*

Proof. Given g and F, we define Ω by $\Omega(X, Y) = g(FX, Y)$ for all vector fields X, Y on M. It is clear that Ω is bilinear, non-degenerate, and skew symetric. Also, if $X, Y \in P$ then $\Omega(X, Y) = g(FX, Y) = g(X, Y) = 0$, and similary if $X, Y \in Q$ then $\Omega(X, Y) = -g(X, Y) = 0$.

Corollary 2.7. *Every hyperbolic Hermite manifold admits an almost complex structure.*

Corollary 2.8. *Every hyperbolic Hermite manifold is orientable.*

Theorem 2.9. *Let (M, Ω) be an almost symplectic manifold. If F is an almost product structure compatible with Ω, then M admits a pseudo-Riemannian metric g such that (M, g, F) is hyperbolic Hermite.*

Proof. If we define a tensor field g by $g(X, Y) = \Omega(FX, Y)$, then it can be verified by direct calculations that g is a pseudo-Riemannian metric such that (M, g, F) is hyperbolic Hermite.

Now, theorems 2.6 and 2.9 together can be given by:

Corollary 2.10. *An almost product manifold (M, F) admits a hyperbolic Hermite metric if and only if it admits an almost symplectic structure compatible with F.*

Corollary 2.11. *The cotangent bundle of any manifold admits a hyperbolic Hermite structure.*

We end this section by the following theorem, which will be needed later in the paper.

Theorem 2.12. *Let (M, g, F) be a hyperbolic Hermite manifold, and let $\Omega(X, Y) = g(FX, Y)$ be the corresponding almost symplectic structure. Then*:

(i) $\Omega(X, Y) = -\Omega(FX, FY)$

(ii) $(\nabla_X \Omega)(Y, Z) = g\big((\nabla_X F)Y, Z\big) = -g\big(Y, (\nabla_X F)Z\big)$

(iii) $(\nabla_X \Omega)(Y, Z) = (\nabla_X \Omega)(FY, FZ)$

(iv) $(\nabla_X \Omega)(FY, Z) = (\nabla_X \Omega)(Y, FZ)$

(v) $(\nabla_X \Omega)(Y, Z) = -(\nabla_X \Omega)(Z, Y)$

(vi) $(d\Omega)(X, Y, Z) = (\nabla_X \Omega)(Y, Z) + (\nabla_Y \Omega)(Z, X) + (\nabla_Z \Omega)(X, Y)$, *where ∇ is the Levi-Civita connection given by g and X, Y, Z any vector fields on M.*

The proof involves long calculations and will be omitted.

3. Integrability

Recall that a distribution D on a manifold M is integrable if it is tangent to a foliation F. That is to say, for every point $x \in M$, the tangent subspace T_xF_x to the leaf F_x through x is equal to $D_x \subset T_xM$. In this section we study the integrability of the distributions P and Q of a hyperbolic Hermite manifold (M, g, F). The following theorem gives integrability in terms of conditions involving the covariant derivative of F with respect to the Levi-Civita connection given by g. To simplify the notation, from now on, FX will be denoted by $\overline{X}$.

THEOREM 3.1. *Let (M, g, F) be a hyperbolic Hermite manifold and ∇ the Levi-Civita connection given by g. Then*:

(a) *The distribution P is integrable if and only if*

$$\begin{aligned}&(\nabla_{\overline{X}}F)Y - (\nabla_{\overline{Y}}F)X - (\nabla_Y F)\overline{X} + (\nabla_X F)\overline{Y}\\ &+(\nabla_X F)Y - (\nabla_{\overline{Y}}F)\overline{X} - (\nabla_Y F)X + (\nabla_{\overline{X}}F)\overline{Y} = 0.\end{aligned} \tag{1}$$

(b) *The distribution Q is integrable if and only if*

$$\begin{aligned}&(\nabla_{\overline{X}}F)Y - (\nabla_{\overline{Y}}F)X - (\nabla_Y F)\overline{X} + (\nabla_X F)\overline{Y}\\ &-(\nabla_X F)Y + (\nabla_{\overline{Y}}F)\overline{X} + (\nabla_Y F)X - (\nabla_{\overline{X}}F)\overline{Y} = 0\end{aligned} \tag{-1}$$

for all vector fields X, Y on M.

Proof. First, assume that (1) holds. We want to show that P is integrable. Take any two vector fields $X, Y \in P$. We prove that $[X, Y] \in P$. Since $\overline{X} = X$ and $\overline{Y} = Y$ we have from (1)

$$4\big((\nabla_X F)Y - (\nabla_Y F)\big) = 0 \quad \text{or} \quad (\nabla_X F)Y - (\nabla_Y F)X = 0,$$

but $FX = X$ gives $(\nabla_Y F)X + F(\nabla_Y X) = \nabla_Y X$ and $FY = Y$ gives $(\nabla_X F)Y + F(\nabla_X Y) = \nabla_X Y$

Now by substraction we get $F(\nabla_X Y - \nabla_Y X) = \nabla_X Y - \nabla_Y X$ or $F[X, Y] = [X, Y]$ (since ∇ is torsion-free). Using Frobenius theorem we conclude that P is integrable.

Conversely, let P be integrable and X, Y any vector fields on M. Let $V = X + \overline{X}$ and $W = Y + \overline{Y}$. Then $V, W \in P$. Since P is integrable, an argument similar to the above gives $(\nabla_V F)W - (\nabla_W F)V = 0$ and (1) follows when we substitute for V, and W their values in therms of $X, \overline{X}, Y$ and $\overline{Y}$.

The proof for Q is similar and will be omitted. Now, clearly conditions (1) and (-1) give the following.

Corollary 3.2. *Let (M, g, F) be a hyperbolic Hermite manifold and ∇ the Levi-Civita connection given by g. Then both P and Q are integrable if and only if*:

$$(\nabla_{\overline{X}}F)Y - (\nabla_{\overline{Y}}F)X - (\nabla_Y F)\overline{X} + (\nabla_X F)\overline{Y} = 0$$

for all vector field X, Y on M.

4. Parallelism

In this section we discuss the parallelism, semi-parallelism and relative parallelism of the distributions P and Q of a hyperbolic Hermite manifold (M, g, F) with respect to the Levi-Civita connection ∇ given by g. For the definitions of these parallelism notions we refer the reader to [**9**]. In the following lemma, $H(M)$ denotes the Lie algebra of vector fields on M.

Lemma 4.1. *Let (M, F) be an almost product manifold with P, Q the two complementary distributions given by F. If ∇ is any affine connection on M then*:

(i) *P is parallel if and only if $(\nabla_X F)Y = 0$, $\forall X \in H(M)$, $Y \in P$*

(ii) *P is semi-parallel if and only if $(\nabla_X F)Y = 0$, $\forall X, Y \in P$*

(iii) *P is parallel relative to Q if and only if $(\nabla_X F)Y = 0$, $\forall X \in Q, Y \in P$.*

The proof depends on the fact that $FX = X$ if and only if $X \in P$, and the results of Walker [**5**] and Willmore [**8**] on these notion of parallelism. Of course, the dual statements for Q in this lemma are obvious.

Theorem 4.2. *Let (M, g, F) be a hyperbolic Hermite manifold with P and Q the corresponding distributions. If ∇ is the Levi-Civita connection given by g then*:

(i) *P is parallel (and hence integrable) if and only if*

$$(\nabla_Y F)\overline{X} + (\nabla_Y F)X = 0 \tag{2}$$

(ii) *Q is parallel (and hence integrable) if and only if*

$$(\nabla_Y F)X - (\nabla_Y F)\overline{X} = 0 \tag{-2}$$

(iii) *P is semi-parallel (and hence integrable) if and only if*

$$(\nabla_Y F)X + (\nabla_Y F)\overline{X} + (\nabla_{\overline{Y}}F)X + (\nabla_{\overline{Y}}F)\overline{X} = 0 \tag{3}$$

(iv) *Q is semi-parallel (and hence integrable) if and only if*

$$(\nabla_Y F)X - (\nabla_Y F)\overline{X} - (\nabla_{\overline{Y}}F)X + (\nabla_{\overline{Y}}F)\overline{X} = 0 \tag{-3}$$

(v) *P is parallel relative to Q if and only if*

$$(\nabla_Y F)X - (\nabla_{\overline{Y}}F)X + (\nabla_Y F)\overline{X} - (\nabla_{\overline{Y}}F)\overline{X} = 0 \tag{4}$$

(vi) *Q is parallel relative to P if and only if*

$$(\nabla_Y F)X + (\nabla_{\overline{Y}} F)X - (\nabla_Y F)\overline{X} - (\nabla_{\overline{Y}} F)\overline{X} = 0 \tag{-4}$$

where X, Y are arbitrary vector fields on M.

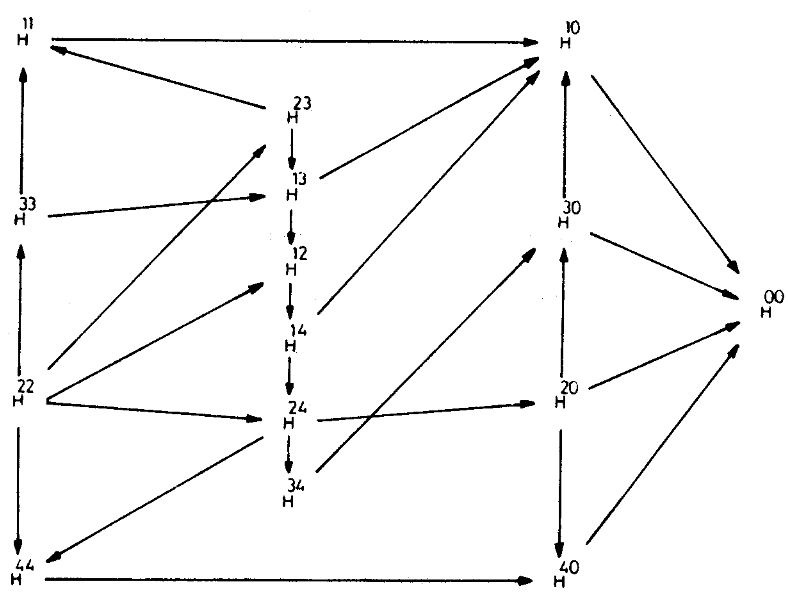
The proof for conditions ± 2, ± 3, ± 4 follows directly from lemma 4.1 above. As for integrability, it can be deduced from theorem 3.1.

Corollary 4.3. *Both distributions P and Q are parallel (and hence integrable) if and only if $(\nabla_X F)Y = 0$ for all vector fields X, Y on M.*

5. Hyperbolic Hermite structures

Let $I = \{0, 1, 2, 3, 4\}$. We say that a hyperbolic Hermite structure (g, F) on a manifold M (and hence the manifold (M, g, F)) is of type (r, s), $r, s \in I$, if (M, g, F) satisfies the conditions (r) and $(-s)$ of theorems 3.1 and 4.2. The case $r = 0$ $(s = 0)$ means that $P(Q)$ is arbitrary. Thus altogether, we have twenty five hyperbolic Hermite structures. However, the conditions (i) and (-i) give the same information. The only distinction between them concerns the distribution on which the information is given. Thus, topologically and geometrically, a manifold M admits a hyperbolic Hermite structure (g, F) of type (r, s) if and only if it admits a structure (g, F') of type (s, r) and F' is simply—F. This means that we must subtract the ten dual cases of the twenty five possibilities found above to obtain fifteen different classes of hyperbolic Hermite structures.

Theorem 5.1. *If H^{rs} denotes the class of hyperbolic Hermite manifolds of type (r, s) then inclusions among these classes are given by the diagram in which arrows indicate inclusions.*



The proof follows from the fact that parallelism implies semi-parallelism and relative parallelism. Also semi-parallelism implies integrability.

We have seen in corollary 2.11 that the cotangent bundle of any manifold admits a hyperbolic Hermite structure. Now since the distibution tangent to the fibres is oviously integrable then this structure is of the type $(1,0)$. The proposition below gives a stronger result.

PROPOSITION 5.2. *The cotangent bundle T^*M of any smooth manifold M admits a hyperbolic Hermite structure of type* $(2,0)$.

Proof. Let M be any smooth manifold and A any atlas on it. If A^c denotes the natural atlas on T^*M induced by A, then the change of coordinates in A^c must be given by $x^* = X(x)$, $y^* = (dX(x)^{-1})^t y$, where $x^* = X(x)$ is the change of coordinates in A. But these changes of coordinates are of the Walker type [**1**], [**4**], and hence T^*M admits a metric g such that the foliation given locally by $x = \text{constant}$ (i.e. foliation by fibres) is parallel and null. If g is of signature (r,s), then, using the fact that the dimension m of each null fibre is less than or aqual to the minimum of $\{r,s\}$ we conclude that $m = r = s$ and g is of signature (m,m). Now if P is the distribution tangent to the fibres, then a connection in the fibre bundle $\pi\colon T^*M \to M$ can always be chosen such that the horizontal subspace Q_x for every $x \in T^*M$ lies in the null cone of g_x. This means that T^*M has a pair of complementary null distibutions P and Q. If F is the almost product structure given by P and Q, then using Proposition 2.4 we conclude that (T^*M, g, F) is hyperbolic Hermite and from the very construction of g we obtain that (g,F) is of type $(2,0)$.

Definition 5.3. A hyperbolic Hermite manifold of type $(2,2)$ is called *hyperbolic Kähler.*

The class H^{22} of hyperbolic Kähler manifolds is contained in every other class H^{rs}. Thus a hyperbolic Kähler manifold is a pseudo Riemannian $2k$-manifold (M,g) with a pair of null k-distributions P and Q that are parallel and hence integrable. Moreover, if F is any one of the two almost product structures given by P and Q, then the almost symplectic structure $\Omega(X,Y) = g(FX,Y)$ is symplectic. This can be seen from (ii) and (vi) of theorem 2.12 where we get

$$d\Omega(X,Y,Z) = g((\nabla_X F)Y,Z) + g((\nabla_Y F)Z,X) + g((\nabla_Z F)X,Y),$$

where X,Y,Z are arbitrary vector fields on M. But it follows from corollary 4.3 that $(\nabla_X F)Y = 0$ for all $X,Y \in H(M)$ and hence $d\Omega = 0$.

PROPOSITION 5.4. *If (M,g) is a locally Euclidean m-manifold, then the cotangent bundle T^*M admits a hyperbolic Kähler structure.*

Proof. Since (M,g) is locally Euclidean, there exists an atlas A on M such that the coordinate transformations have locally constant Jacobian (in fact, this belongs to the Euclidean group $E(r,m-r) = O(r,m-r)$. R^m, where $(r,m-r)$ is the signature of g.) Also, the components of g are constant in every chart in A.

Now if A^C is the canonical atlas on T^*M induced by A then the transformations of coordinates in A^C are given by

$$\begin{aligned} x^* &= Ax + B \qquad A \in O(r, m-r) \\ y^* &= (A^{-1})^t y \end{aligned}$$

This means that the foliation $\mathcal{F}$ of T^*M tangent to the fibres and given by x =constant admits a complementary foliation $\mathcal{F}'$ given by y = constant. Now we define a metric h on T^*M such that for every chart $\eta\colon U \to U'$ of A^C and every point $p(x, y) \in U$, h_p is given by the matrix

$$M(p) = \begin{bmatrix} K & I \\ I & 0 \end{bmatrix}$$

where I is the identity $m \times m$ matrix and K is an $m \times m$ matrix whose entries K_{ij} are given by $K_{ij} = -2\Gamma^s_{ij} y_s$ and Γ^s_{ij} are the coefficients of the Levi-Civita connection given by g. The fact that h, defined this way, gives a metric on T^*M follows from [**6**], and the fact that $\mathcal{F}$ is parallel and null with respect to h follows from [**4**]. Now recall that the Christoffel coefficients Γ^s_{ij} are given by

$$\Gamma^s_{ij} = \frac{1}{2} \sum g^{sk} \left(\frac{\partial g_{ik}}{\partial x_i} + \frac{\partial g_{ik}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k} \right).$$

Since g_{ij} = constant, we obtain $\Gamma^s_{ij} = 0$ for all i, j, s, and hence the metric h is given at every point by the matrix

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

which means that both foliations $\mathcal{F}$ and $\mathcal{F}'$ are parallel and null with respect to h. Thus if P and Q are the distributions tangent to $\mathcal{F}$ and $\mathcal{F}'$ respectively and F is the almost product structure given by them, then (T^*M, h, F) is a hyperbolic Kähler manifold.

Notice that (T^*M, h) itself is also locally Euclidean. This raises the question of whether one can find examples of a hyperbolic Kähler manifold (M, g, F) where (M, g) is not locally Euclidean. Before we give such an example we need the following proposition.

PROPOSITION 5.5. *Every 2-dimensional hyperbolic Hermite manifold (M, g, F) is hyperbolic Kähler.*

Note that in this case, the integral submanifolds of P and Q are null geodesics.

Proof. We only need to show that the 1-distributions P and Q given by F are parallel with respect to the Levi-Civita connection ∇ given by g. First notice that if X is any null vector field and Y any other vector field then $g(X, \nabla_Y X) = 0$. Now suppose that $X \in P$ and $\nabla_Y X \notin P$ for some vector field Y. Then there is a point $x \in M$ such that $(\nabla_Y X)(x) \notin P(x)$. We thus obtain $(\nabla_Y X)(x) =$

$A+B,\ A \in P(x),\ B \in Q(x),\ B \neq 0$. Now since $(\nabla_Y X)(x)$ is orthogonal to $X(x)$ it is orthogonal to every vector in $P(x)$ (since P is one-dimensional). That is

$$g(A+B, L) = 0 \qquad \forall L \in P(x)$$
$$g(A, L) + g(B, L) = 0$$

and hence $g(B, L) = 0$ since $g(A, L) = 0$.

Now if $L + N \in T_X M$ is arbitrary, $L \in P(x)$, $N \in Q(x)$ then $g(B, L+N) = g(B, L) + g(B, N) = 0$ (since $B, N \in Q(x)$ which is null). This proves that the non-zero vector B is orthogonal to all vectors $T_X M$, which contradicts the non-degeneracy of g. We thus obtain that $\nabla_Y X \in P$ for all vector fields Y. P is, therefore, parallel and Q is, similarly parallel.

Example 5.6. To give an example of a hyperbolic Kähler manifold (M, g, F) such that (M, g) is not locally Euclidean, we use the above proposition and take any 2-dimensional hyperbolic Hermite manifold that is not locally Euclidean. For example, take $M = R^2$ with $e_1 = (1,0)$, $e_2 = (0,1)$ the usual basis. Define a metric g on M such that for every point $(x, y) \in M$ we define $g_{(x,y)}$ by the matrix

$$G(x,y) = \begin{bmatrix} e^{x+y}/2 & 0 \\ 0 & -e^{x+y}/2 \end{bmatrix}$$

Clearly (M, g) is not locally Euclidean. Now if we take P, Q to be the 1-distributions induced by $e_1 + e_2$, $e_1 - e_2$ then P and Q will be null. If F is the almost product structure given by P and Q then propositions 2.4 and 5.5 give that (M, g, F) is hyperbolic Kähler.

We end the paper by the following example of a hyperbolic Hermite manifold of type $(0,0)$.

Example 5.7. Let N_1 and N_2 be any two smooth manifolds and g_1, g_2 two metrics defined on T^*N_1, T^*N_2 respectively to make each of them a hyperbolic Hermite manifold of type $(2,0)$ as in proposition 5.2 . Now $T(T^*N_1) = P_1 \oplus Q_1$ and $T(T^*N_2) = P_2 \oplus Q_2$. Take as (M, g) the pseudo-Riemannian product of (T^*N_1, g_1) and (T^*N_2, g_2). Then $TM = P_1 \oplus Q_1 \oplus P_2 \oplus Q_2$, and the two distributions $P = P_1 \oplus Q_2$ and $Q = Q_1 \oplus P_2$ on M define an almost product structure F on M. Now since P_1 and Q_2 are both null with respect to g and $g(X, Y) = 0$ for all $X \in P_1$, $Y \in Q_2$ we conclude that P is null with respect to g. Similary Q is null and hence (M, g, F) is hyperbolic Hermite. By the very constructions of P and Q we conclude that it is of type $(0,0)$.

REFERENCES

[**1**] S. A. Robertson and H. R. Farran, *Parallel foliations*, (to appear).

[**2**] S. A. Robertson and P. M. D. Furness, *Parallel framings and foliations on pseudo-Riemannian manifolds*, J. Differential Geom. **9** (1974), 409–422.

[**3**] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, 1965.

[**4**] A. G. Walker, *Canonical form for a Riemannian space with a parallel field of null planes*, Quart. J. Math. Oxford Ser. (2) **1** (1950), 69–79.

[**5**] A. G. Walker, *Connections for parallel distributions in the large I and II*, Quart. J. Math. Oxford Ser. (2) **6** (1955), 303–308, and (2) **6** (1958), 221–231.

[**6**] A. G. Walker and E. M. Patterson, *Riemann extensions*, Quart. J. Math. Oxford Ser. (2) **3** (1952), 19–28.

[**7**] A. Weinstein, *Sympletic manifolds and their Lagrangian submanifolds*, Advances in Math. **6** (1971), 329–346.

[**8**] T. J. Willmore, *Parallel distributions on manifolds*, Proc. London Math. Soc. (3) **6** (1956), 191–204.

[**9**] K. Yano, *Differential Geometry on Complex and Almost Complex Spaces*, Pergamon Press, 1965.

Mathematics Department
Kuwait University
P. O. Box 5969
13060-Safat-Kuwait, Kuwait

(Received 07 10 1988)