PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 46 (60), 1989, pp. 145–162

THE GEOMETRY OF THE DUAL OF A VECTOR BUNDLE

Radu Miron, Stere Ianuş and **Mihai Anastasiei**

The differential geometry of the total space of a vector bundle has benefited by many interesting papers since the paper [**9**] by R. Miron has appeared. That paper has led to a deep study of some remarkable geometric structures. The main results from the geometry of the total space of vector bundle as well as some applications of it to General Relativity were published in a recent monograph (R. Miron, M. Anastasiei [**14**]).

Related to this geometry the geometry of the Lagrange spaces $L^n = (M, L)$ as well as the geometry of the generalized Lagrange spaces $M^n = (M, g_{ij}(x, y))$, (see [**11**], [**6**], [**1**], [**16**]) has been extensively developed.

Important applications of the theory of the spaces M^n in studying the effects of the gravitational field were pointed out by A. K. Aringazin and G. S. Asanov [**4**].

Let $\xi = (E, \pi, M)$ be a vector bundle and $\xi^* = (E^*, \pi^*, M)$ its dual. In this paper we study the differential geometry of the manifold E^* generalizing the results from the geometry of the total space T^*M of the cotangent bundle (T^*M, τ^*, M) of a manifold, [**15**], [**2**], or of a Hamilton space, [**12**], [**13**].

Our theory is of interest for the Hamiltonian theory of physical fields.

It is known that the main properties of T^*M are analogous to those of the total space TM of the tangent bundle (TM, τ, M). But there exist properties which are specific for T^*M. For instance, E. Calabi has remarked that on the total space of cotangent bundle of a complex projective space there exists a Kähler metric whose Ricci tensor identically vanishes.

The paper is organized as follows. In §1 the basic notations as well as the concept of nonlinear connection on E^* are introduced. In §§ 2–4 d-tensor fields and d-connections on E^* are considered. The main properties of the torsion and curvature of a d-connection are described, too. The equations of structure of a d-connections are derived in §5. In §6 h-metrics, v-metrics and (h, v)-metrics on

AMS Subject Classification (1980): Primary 53 C 05, 53 C 99

E^* are introduced and the d-connections compatible with them are studied. Also, Hamilton function is introduced and it is shown that it determines a v-metric on E^*. The Legendre transformation as a map $E \to E^*$ is studied in §7.

The terminology and notation are those from the monograph [**14**].

1. The dual vector bundle

Let $\xi = (E, \pi, M)$ be a real vector bundle, whose base M is an n-dimensional manifold, the type fiber F is an m-dimensional real linear space and the projection π is a differentiable map. We shall denote the dual of ξ by $\xi^* = (E^*, \pi^*, M)$. Its type fiber is F^*, the dual of F.

A trivialization of ξ induces a trivialization of ξ^*. Let $U \subset M$ be the domain of a chart of M and $e \in \pi^{-1}(U) \subset E$. Let us denote by (x^i, y^a) the coordinates of e such that (x^i), $1 \leq i \leq n$, are the coordinates of $\pi(e) = x$ and (y^a), $1 \leq a \leq m$, are the coordinates of the e in the fiber $E_x = \pi^{-1}(x)$. If a change of the bundle chart is performed one obtains (see [**14**])

$$\begin{aligned} \overline{x}^i &= \overline{x}^i(x^1, \cdots, x^n), \qquad \operatorname{rank}\left\| \frac{\partial \overline{x}^i}{\partial x^j} \right\| = n, \\ \overline{y}^a &= M^a_b(x) y^b, \qquad \operatorname{rank}\| M^b_a(x) \| = m. \end{aligned} \tag{1.1}$$

Here the Einstein summation convention is used and will always be used in this paper.

Let us consider $u \in \pi^{*-1}(U) \subset E^*$ such that $\pi^*(u) = x$ and let (x^i, p_a) be the canonical coordinates of u. If the local chart is changed these coordinates are transformed as follows:

$$\overline{x}^i = \overline{x}^i(x^1, \ldots, x^n), \qquad \operatorname{rank}\left\| \frac{\partial \overline{x}^i}{\partial x^j} \right\| = n, \qquad \overline{p}_a = \widetilde{M}^b_a(\overline{x}) p_b, \tag{1.2}$$

where the matrix $\big(\widetilde{M}^b_a(\overline{x})\big)$ is the inverse of the matrix $\big(M^b_a(x)\big)$. It follows immediately that locally we have $y^a p_a = \overline{y}^a \overline{p}_a$ because ξ and ξ^* are dual.

Let us denote

$$\partial_i = \frac{\partial}{\partial x^i}, \quad \dot{\partial}^a = \frac{\partial}{\partial p_a}. \tag{1.3}$$

These vector fields are transformed as follows:

$$\begin{aligned} \partial_i &= \frac{\partial \overline{x}^k}{\partial x^i} \overline{\partial}_k + \frac{\partial \widetilde{M}^a_b(\overline{x})}{\partial \overline{x}^k} \frac{\partial \overline{x}^k}{\partial x^i}\, p_a \overline{\dot{\partial}}^b \\ \dot{\partial}^a &= \widetilde{M}^a_b(\overline{x})\, \overline{\dot{\partial}}^b. \end{aligned} \tag{1.4}$$

By (1.4) we can define a global vector field $\widetilde{p}$ on E^* such that in a system of local coordinates $\widetilde{p} = p_a \dot{\partial}^a$.

Definition 1.1. The vector field $\widetilde{p}$ on E^* is called the *Liouville vector field*.

Let $\pi^{*T}: TE^* \to TM$ be the tangent map to π^*. Its kernel, denoted by VE^*, will be thought of as a distribution $u \to V_u E^*$ on E^*, called the *vertical distribution* of ξ^*. It is easy to see that $\pi^{*T}(\dot{\partial}^a) = 0$ for $a = 1, \dots, m$, hence $(\dot{\partial}^a)$ is a local basis for the vertical distribution. By the Frobenius theorem this distribution is integrable and its maximal integral submanifolds are exactly the fibers E_x^*, $x \in M$.

Definition 1.2. A *nonlinear connection* on E^* is a differentiable distribution N^* on E^* which is supplementary to the vertical distribution VE^*, i.e. $T_u E^* = N_u^* \oplus V_u E^*$ holds for every $u \in E^*$.

Proposition 1.1. *If M is a paracompact manifold then there exist nonlinear connections on E^*.*

Proof. One proceeds as in the case of the bundle (see [**14**]). Since the submersion π^* is differentiable we can associate to any vector field $A \in \mathcal{X}(M)$ a unique vector field A^h on E^* such that for every $u \in E^*$, $A_u^h \in N_u^*$ and $\pi^{*T}(A_u^h) = A_x$, $\pi^*(u) = x$. The vector field A^h will be called the horizontal lift of A with respect to the nonlinear connection N. Setting $\delta_i = (\partial_i)^h$, $i = 1, \dots, n$, it is obvious that $(\delta_1, \dots, \delta_n)$ is a local basis for the distribution N^* and that there exists a unique system of functions $N_{ai}: {\pi^*}^{-1}(U) \to R$, $(1 \le i \le n,\ 1 \le a \le m)$ such that

$$\delta_i = \partial_i + N_{ai}(x,p)\dot{\partial}^a. \tag{1.5}$$

The functions (N_{ai}) are called the *coefficients* of the nonlinear connection N^*. Sometimes N^* will be called the horizontal distribution on E^*.

As in the case of the nonlinear connections on E (see [**14**]) on can prove:

Proposition 1.2. *If a change of bundle charts is performed the following formulae hold*:

$$\delta_i = \frac{\partial \overline{x}^k}{\partial x^i}\overline{\delta}_k, \tag{1.6}$$

$$\overline{N}_{ai}(\overline{x},\overline{p}) = \widetilde{M}_a^b(\overline{x})\frac{\partial x^k}{\partial \overline{x}^i}N_{bk}(x,p) + p_b\frac{\partial \widetilde{M}_a^b}{\partial \overline{x}^i}. \tag{1.7}$$

Proposition 1.3. *If for a trivialization of ξ^* on the domain of each local chart on E^* a system of functions (N_{ai}) which are transformed by* (1.7) *is given, then there exists an unique nonlinear connection N^* on E^* whose coefficients are the given functions.*

It is clear that $(\delta_i, \dot{\partial}^a)$ is a local basis for $\mathcal{X}(E^*)$, which is adapted to the distribution N^* and to the distribution VE^*. If we set

$$\delta p_a = dp_a - N_{ai}(x,p)\,dx^i, \tag{1.8}$$

then $(dx^i, \delta p_a)$ is the basis dual to $(\delta_i, \dot{\partial}^a)$.

It is easy to see that

$$\delta\overline{p}_a = \widetilde{M}_a^b(\overline{x})\delta p_b. \tag{1.9}$$

Now we shall associate to N^* a 2-form ρ on M which is VE^*-valued:

$$\rho(A,B) = [A^h, B^h] - [A,B]^h. \tag{1.10}$$

It is VE^*-valued because A^h, B^h and $[A,B]^h$ are π^*-related to A, B and $[A,B]$ respectively, so that $\rho(A,B)$ is just the vertical component of $[A^h, B^h]$. But we know that N^* is integrable iff the vertical component of $[A^h, B^h]$ vanishes.

So we have:

Theorem 1.1. *The horizontal distribution N^* is integrable if and only if the 2-form ρ identically vanishes.*

Locally, we have:

$$\rho(\partial_i, \partial_j) = [\delta_i, \delta_j] = R_{aij}\dot{\partial}^a, \tag{1.11}$$

where

$$R_{aij} = \delta_i N_{aj} - \delta_j N_{ai}. \tag{1.12}$$

We also notice:

$$[\delta_i, \dot{\partial}^a] = -(\dot{\partial}^a N_{bi})\dot{\partial}^b, \quad [\dot{\partial}^a, \dot{\partial}^b] = 0. \tag{1.13}$$

2. *d*-tensor fields on E^*

For every vector field X on E^* we shall denote by X^H and X^V its projections on horizontal and vertical distribution, respectively. So we have

$$X = X^H + X^V, \tag{2.1}$$

where $X_u^H \in N_u^*$ and $X_u^V \in V_uE^*$ for every $u \in E^*$.

We shall say that X^H is a *horizontal vector field* and X^V is a *vertical vector field*.

If we put

$$X^H = X^i(x,p)\delta_i, \quad X^V = X_a(x,p)\dot{\partial}^a, \tag{2.2}$$

the following rules of transformation hold:

$$\overline{X}^i(\overline{x},\overline{p}) = \frac{\partial\overline{x}^i}{\partial x^j}X^j, \quad \overline{X}_a = \widetilde{M}_a^b(\overline{x})X_b. \tag{2.2'}$$

If ω is an 1-form on E^*, we have the decomposition

$$\omega = \omega^H + \omega^V, \tag{2.3}$$

where ω^H and ω^V are 1-forms on E^* defined by

(2.3') $$\omega^H(X) = \omega(X^H), \quad \omega^H(X^V) = 0,$$

(2.3") $$\omega^V(X) = \omega(X^V), \quad \omega^V(X^H) = 0, \qquad \forall X \in \mathcal{X}(E^*).$$

Locally, we have

(2.4) $$\omega^H = \omega_i(x,p)\,dx^i, \quad \omega^V = \omega^a(x,p)\delta p_a$$

and following laws of transformation hold:

(2.4') $$\overline{\omega}_i(\overline{x},\overline{p}) = \frac{\partial x^j}{\partial \overline{x}^i}\omega_j(x,p); \quad \overline{\omega}^a(\overline{x},\overline{p}) = M^a_b(x)\omega^b(x,p).$$

Definition 2.1. A tensor field $t \in \tau^r_s(E^*)$ with the property

(2.5) $$t(\overset{1}{\omega},\dots,\overset{r}{\omega},\underset{1}{X},\dots,\underset{s}{X}) = t(\overset{1}{\omega}{}^H,\dots,\overset{r}{\omega}{}^V,\underset{1}{X}{}^H,\dots,\underset{s}{X}{}^V),$$

where $\underset{1}{X},\dots,\underset{s}{X} \in \mathcal{X}(E^*)$ and $\overset{1}{\omega},\dots,\overset{n}{\omega} \in \mathcal{X}^*(E^*)$, we shall call *distinguished tensor field* or *d-tensor field*, on E^*. If we put

$$t^{i_1\dots a_1\dots}_{j_1\dots b_1\dots} = t(dx^{i_1},\dots,\delta_{j_1},\dots,\dot{\partial}^{a_1},\dots,\delta p_{b_1},\dots)$$

by (1.6) and (1.9), it follows

(2.6) $$\overline{t}^{i_1\dots a_1\dots}_{j_1\dots b_1\dots} = \frac{\partial \overline{x}^{i_1}}{\partial x^{h_1}}\cdots\frac{\partial x^{k_1}}{\partial \overline{x}^{j_1}}\cdots M^{a_1}_{c_1}\cdots\widetilde{M}^{d_1}_{b_1}\cdots t^{h_1\cdots c_1\cdots}_{k_1\cdots d_1\cdots}.$$

As an example we mention that the functions R_{aij} are the components of a d-tensor field. By (1.11) it follows that this d-tensor field vanishes iff the horizontal distribution N^* is integrable.

3. *d*-connections on E^*

When a nonlinear connection N^* on E^* is given, special linear connections on E^* can be considered.

Definition 3.1. The *distinguished connection* or *d-connection* on E^* is a linear connection D on E^* which preserves the distributions N^* and VE^* by parallelism.

Setting

(3.1) $$D^h_X = D_{X^H}, \quad D^v_X = D_{X^V}, \qquad \forall X \in \mathcal{X}(E^*),$$

gives

(3.1') $$D_X = D^h_X + D^v_X, \qquad \forall X \in \mathcal{X}(E^*).$$

Furthermore, D^h determines an algorithm of an h-covariant derivation and D^v determines an algorithm of a v-covariant derivation (cf. [**14**]).

We note the following properties of D^h and D^v, respectively:

$$\begin{cases} (D^h_X Y^H)^V = 0, \quad (D^h_X Y^V)^H = 0, \\ D^h_X Y = (D^h_X Y^H)^H + (D^h_X Y^V)^V, \quad D^h_X f = X^H f, \end{cases} \tag{3.2}$$

$$\begin{cases} (D^v_X Y^H)^V = 0, \quad (D^v_X Y^V)^H = 0, \\ D^v_X Y = (D^v_X Y^H)^H + (D^v_X Y^V)^V, \quad D^v_X f = X^V f, \end{cases} \tag{3.3}$$

where f is an arbitrary function on E^*.

If $t \in \mathcal{T}^r_s(E^*)$ is a d-tensor field on E^* then its h- and v- covariant derivatives are given by

(3.4)

$$(D^h_X t)(\overset{1}{\omega}, \dots, \underset{s}{X}) = X^H t(\overset{1}{\omega}, \dots, \underset{s}{X}) - t(D^h_X \overset{1}{\omega}, \dots, \underset{s}{X}) - \dots - t(\overset{1}{\omega}, \dots, D^h_X \underset{s}{X}),$$

$$(D^v_X t)(\overset{1}{\omega}, \dots, \underset{s}{X}) = X^V t(\overset{1}{\omega}, \dots, \underset{s}{X}) - t(D^v_X \overset{1}{\omega}, \dots, \underset{s}{X}) - \dots - t(\overset{1}{\omega}, \dots, D^v_X \underset{s}{X}),$$

The torsion Π of a d-connection D is completely determined by the following five d-tensor fields of torsions.

(3.5)

$$T^H(x,y) = [\Pi(X^H, Y^H)]^H, \qquad T^V(X,Y) = [\Pi(X^V, Y^V)]^V$$

$$R^0(X,Y) = -[\Pi(X^H, Y^H)]^V, [\Pi(X^V, Y^H)]^H, \quad P^1(X,Y) = [\Pi(X^V, Y^H)]^V$$

The curvature tensor field $\mathbb{R}$ of a d-connection D satisfies:

$$[\mathbb{R}(X,Y)Z^H]^V = 0, \quad [\mathbb{R}(X,Y)Z^V]^H = 0. \tag{3.6}$$

Hence it is completely determined by the following six d-tensor fields of curvature:

$$\begin{gathered} R(X,Y)Z = \mathbb{R}(X^H, Y^H)Z^H, \quad P(X,Y)Z = \mathbb{R}(X^V, Y^H)Z^H, \\ S(X,Y)Z = \mathbb{R}(X^V, Y^V)Z^H, \\ \widetilde{R}(X,Y)Z = \mathbb{R}(X^H, Y^H)Z^V, \quad \widetilde{P}(X,Y)Z = \mathbb{R}(X^V, Y^H)Z^V, \\ \widetilde{S}(X,Y)Z = \mathbb{R}(X^V, Y^V)Z^V. \end{gathered} \tag{3.7}$$

Every d-connection has a remarkable form with respect to the adapted basis, its coefficients having simple laws of transformations and giving a new characterisation of it.

THEOREM 3.1. *A d-connection D has, with respect to the adapted basis $(\delta_i, \dot{\partial}^a)$, the following form:*

$$\begin{gathered} D^i_{\delta_k} \delta_j = H^i_{jk}(x,p)\delta_i, \quad D_{\delta_k} \dot{\partial}^a = -\widetilde{H}^a_{bk} \dot{\partial}^b \\ D_{\dot{\partial}^c} \delta_i = C^{jc}_i(x,p)\delta_j, \quad D_{\dot{\partial}^c} \dot{\partial}^a = -\widetilde{C}^{ac}_b(x,p)\dot{\partial}^b, \end{gathered} \tag{3.8}$$

where the coefficients H^i_{jk} and $\widetilde{H}^a_{bk}$ have the following laws of transformation

$$\overline{H}^i_{jk} = \frac{\partial \overline{x}^i}{\partial x^h} \frac{\partial x^r}{\partial \overline{x}^j} \frac{\partial x^s}{\partial \overline{x}^k} H^h_{rs} + \frac{\partial \overline{x}^i}{\partial x^r} \frac{\partial^2 x^r}{\partial \overline{x}^j \partial \overline{x}^k}, \tag{3.9}$$

(3.9') $$\overline{\widetilde{H}}{}^a_{bk} = M^a_c \widetilde{M}^d_b \frac{\partial x^j}{\partial \overline{x}^k} \widetilde{H}^c_{dj} + M^a_c \frac{\partial \widetilde{M}^c_b}{\partial \overline{x}^k}$$

and C^{jc}_i, $\widetilde{C}^{ac}_b$ *are d-tensor fields.*

Proof. Since the d-connection D preserves by parallelism the distributions N^* and VE^*, the formulae (3.8) follow directly from (3.2) and (3.3) by using the basis $(\delta_i, \dot{\partial}^a)$. From (3.8) and (1.6), (1.9) one obtains (3.9) and (3.9') as well as

(3.9") $$\overline{C}{}^{jc}_i = \frac{\partial \overline{x}^j}{\partial x^r} \frac{\partial x^s}{\partial \overline{x}^i} M^c_a C^{ra}_s, \qquad \overline{\widetilde{C}}{}^{ac}_b = \widetilde{M}^d_b M^a_e M^c_f \widetilde{C}^{ef}_d,$$

which shows that C^{jc}_i and $\widetilde{C}^{ac}_b$ are d-tensor fields. QED.

THEOREM 3.2. *If on the domain of each local chart on* E^* *are given the functions* $\left(H^i_{jk}(x,p),\, \widetilde{H}^a_{bk}(x,p),\, C^{ic}_j(x,p),\, \widetilde{C}^{ac}_b(x,p)\right)$ *which transform by (3.9), (3.9') and (3.9") when the local chart is changed, then there exists a unique d-connection* D *on* E^* *whose local coefficients are given functions and which has the properties*:

$$D^h_{\delta_i} f = \delta_i f, \quad D^v_{\dot{\partial}^a} f = \dot{\partial}^a f \qquad \forall f \in \mathcal{F}(E^*)$$

Proof. For each local chart we can write (3.8). Then define the covariant derivative with respect to $X = X^i \delta_i + X_a \dot{\partial}^a$ by

(3.10) $$D_X = X^i D_{\delta_i} + X_a D_{\dot{\partial}^a}.$$

By standard arguments it follows that D is a linear connection, globally defined on E^* having as local coefficients just the given functions. The uniqueness is immediate.

THEOREM 3.3. *If the base manifold of the bundle* ξ^* *is paracompact, then there exist d-connections on* E^*.

Proof. Let N^* be a nonlinear connection on E^* having as local coefficients $N_{ai}(x,p)$ and let Γ be a linear connection on M, having as local coefficients $\Gamma^i_{jk}(x)$. Then the set of functions $\left(\Gamma^i_{jk}(x),\, \dot{\partial}^a N_{bi},\, 0,\, 0\right)$ satisfies the hypothesis of the Theorem 3.2 QED.

Next we shall give local expressions for the h-and v covariant derivatives of a d-tensor field.

If a d-tensor field t is locally given by

(3.11) $$t = t^{i\cdots a\cdots}_{j\cdots b\cdots} \delta_i \otimes \cdots \otimes dx^j \otimes \delta p_a \otimes \cdots \otimes \dot{\partial}^b \otimes \cdots,$$

for $X = X^H = X^i \delta_i$ we have

(3.12) $$D^h_X t = X^k t^{i\cdots a\cdots}_{j\cdots b\cdots|k} \delta_i \otimes \cdots \otimes dx^j \otimes \delta p_a \otimes \cdots \otimes \dot{\partial}^b \otimes \cdots$$

and for $X = X^V = X_a \dot{\partial}^a$ we have

(3.13) $$D^V_X t = X_c t^{i\cdots a\cdots}_{j\cdots b\cdots}|^c \delta_i \otimes \cdots \otimes dx^j \otimes \delta p_a \otimes \cdots \dot{\partial}^b \otimes \cdots,$$

where we have set

$$t^{j\cdots b\cdots}_{i\cdots a\cdots|k} = \delta_k t^{j\cdots b\cdots}_{i\cdots a\cdots} + H^j_{hk} t^{h\cdots b\cdots}_{i\cdots a\cdots} + \cdots + \widetilde{H}^b_{ck} t^{j\cdots c\cdots}_{i\cdots a\cdots} - H^h_{ik} t^{j\cdots b\cdots}_{h\cdots a\cdots} - \cdots - \widetilde{H}^c_{ak} t^{i\cdots b\cdots}_{j\cdots c\cdots} \tag{3.14}$$

$$t^{i\cdots a\cdots}_{j\cdots b\cdots}|^c = \dot{\partial}^c t^{i\cdots a\cdots}_{j\cdots b\cdots} + C^{ic}_h t^{h\cdots a\cdots}_{j\cdots b\cdots} + \widetilde{C}^{ac}_d t^{i\cdots d\cdots}_{j\cdots b\cdots} - C^{hc}_j t^{i\cdots a\cdots}_{h\cdots b\cdots} - \cdots - \widetilde{C}^{dc}_b t^{i\cdots a\cdots}_{j\cdots d\cdots}. \tag{3.15}$$

For instance, the h- and v-covariant derivatives of a horizontal vector field $X = X^i\delta_i$ are given by

$$X^i_{|k} = \delta_k X^i + H^i_{jk} X^j, \quad X^i|^a = \dot{\partial}^a X^i + C^{ia}_j X^j \tag{3.16}$$

and for a vertical vector field $\widetilde{X} = X_a\dot{\partial}^a$ these derivatives are given by:

$$X_{a|k} = \delta_k X_a - \widetilde{H}^b_{ak} X_b, \quad X_a|^b = \dot{\partial}^b X_a - \widetilde{C}^{cb}_a X_c. \tag{3.17}$$

Also we have

Proposition 3.1. *h- and v-covariant derivatives of the Liouville vector field $\widetilde{p} = p_a\dot{\partial}^a$ are*

$$p_{a|k} = D_{ak}, \quad p_a|^b = \delta^b_a - \widetilde{C}^{ob}_a, \tag{3.18}$$

where

$$D_{ak} = N_{ak} - \widetilde{H}^b_{ak} p_b \tag{3.19}$$

and o means the contraction by p_a.

It is obvious that D_{ak} are the local components of a d-tensor field. This will be called the deflection tensor field of the d-conection D.

4. Curvatures and torsion of a d-connection

The d-tensor fields of torsion and curvature of a d-connection D on E^* given by (3.5) and (3.7), respectively, have interesting forms in the adapted basis $(\delta_i, \dot{\partial}^a)$. Putting:

$$T^H(\delta_k, \delta_j) = T^i{}_{jk}\delta_i, \quad T^v(\dot{\partial}^c, \dot{\partial}^b) = S_a{}^{bc}\dot{\partial}^a, \quad R^o(\delta_i, \delta_k) = \widetilde{R}_{ajk}\dot{\partial}^a$$
$$T^H(\dot{\partial}^b, \delta_j) = \widetilde{C}^{ib}_j\delta_i, \quad P^1(\dot{\partial}^b, \delta_j) = P_{aj}{}^b\dot{\partial}^a$$

and taking into account (3.5) and (3.8) one obtains:

Proposition 4.1. *In the adapted basis $(\delta_i, \dot{\partial}^a)$ the d-tensor fields of torsion* (3.5) *have the coefficients*:

$$\begin{aligned} T^i_{jk} = H^i_{jk} - H^i_{kj}, \quad S_a{}^{bc} = -(\widetilde{C}^{bc}_a - \widetilde{C}^{cb}_a), \\ P_{aj}{}^b = -(\dot{\partial}^c N_{aj} - \widetilde{H}^c_{aj}), \quad \widetilde{R}_{aij} = R_{aij}, \quad \widetilde{C}^{ib}_j = C^{ib}_j. \end{aligned} \tag{4.2}$$

Proposition 4.2. *The d-connection D is without torsion iff the d-tensor fields $T^i{}_{jk}$, $S_a{}^{bc}$, $P_{aj}{}^b$, R_{aij}, C_j^{ib} vanish.*

Now, putting

$$(4.3)\qquad \begin{aligned} R(\delta_h,\delta_k)\delta_j &= R_j{}^i{}_{kh}\delta_i, & \widetilde{R}(\delta_h,\delta_k)\dot\partial^b &= -\widetilde{R}_a{}^b{}_{kh}\dot\partial^a,\\ S(\dot\partial^c,\dot\partial^b)\delta_j &= S_j{}^{ibc}\delta_i, & \widetilde{S}(\dot\partial^c,\dot\partial^b)\dot\partial^a &= -\widetilde{S}_d{}^{abc}\dot\partial^d,\\ P(\dot\partial^c,\delta_k)\delta_j &= P_j{}^i{}_k{}^c\delta_i, & \widetilde{P}(\dot\partial^c,\delta_k)\dot\partial^b &= -\widetilde{P}_a{}^b{}_k{}^c\dot\partial^a, \end{aligned}$$

a straightforward calculation leads to:

Proposition 4.3. *The d-tensor fields of curvature* (3.7) *have in the adapted basis* $(\delta_i,\dot\partial^a)$ *the following coefficients*:

$$(4.4)_1\qquad \begin{cases} R_j{}^i{}_{kh} = \delta_h H^i_{jk} - \delta_k H^i_{jh} + H^r_{jk}H^i_{rh} - H^r_{jh}H^i_{rk} + C_j^{ib}R_{bkh},\\ \widetilde{R}_a{}^b{}_{kh} = \delta_h \widetilde{H}^b_{ak} - \delta_k\widetilde{H}^a_{bk} + \widetilde{H}^c_{ak}\widetilde{H}^b_{ch} - \widetilde{H}^c_{ah}\widetilde{H}^b_{ck} + \widetilde{C}_a{}^{bc}R_{ckh}. \end{cases}$$

$$(4.4)_2\qquad \begin{cases} P_j{}^i{}_k{}^c = \dot\partial^c H^i_{jk} - \delta_k C_j^{ic} + H^r_{jk}C_r^{ic} - C_j^{rc}H^i_{rk} + C_j^{ia}(\dot\partial^c N_{ak}),\\ \widetilde{P}_a{}^b{}_k{}^c = \dot\partial^c\widetilde{H}^b_{ak} - \delta_k\widetilde{C}_a^{bc} + \widetilde{H}^d_{ak}\widetilde{C}_d^{bc} - \widetilde{C}_a^{dc}\widetilde{H}^d_{bk} + \widetilde{C}_a^{bd}(\dot\partial^c N_{dk}). \end{cases}$$

$$(4.4)_3\qquad \begin{cases} S_j{}^{abc} = \dot\partial^c C_j^{ib} - \dot\partial^b C_j^{ic} + C_j^{rb}C_r^{ic} - C_j^{rc}C_r^{ib},\\ \widetilde{S}_d{}^{abc} = \dot\partial^c\widetilde{C}_d^{ab} - \dot\partial^b\widetilde{C}_d^{ac} + \widetilde{C}_d^{fb}\widetilde{C}_f^{ac} - \widetilde{C}_d^{fc}\widetilde{C}_f^{ab}. \end{cases}$$

We notice the following more interesting forms of P and $\widetilde{P}$:

$$(4.5)\qquad \begin{aligned} P_j{}^i{}_k{}^c &= \dot\partial^c H^i_{jk} - C_j{}^{ic}{}_{|k} + C_j{}^{ia}P_{ak}{}^c\\ \widetilde{P}_a{}^b{}_k{}^c &= \dot\partial^c\widetilde{H}^b_{ak} - \widetilde{C}_a{}^{bc}{}_{|k} + \widetilde{C}_a{}^{bd}P_{dk}{}^c. \end{aligned}$$

The Ricci identity

$$[D_X,D_Y]Z = \mathbb{R}(X,Y)Z + D_{[X,Y]}Z,\qquad \forall X,Y,Z\in\mathcal{X}(E^*),$$

written in the adapted basis, leads to:

Proposition 4.4. *If $X^H = X^i\delta_i$ is a horizontal vector field, then the following Ricci identities hold*:

$$(4.6)\qquad \begin{aligned} X^i_{|k|h} - X^i_{|h|k} &= X^jR_j{}^i{}_{kh} - T^j{}_{kh}X^i_{|j} - R_{akh}X^i|^a,\\ X^i_{|k}|^c - X^i|^c{}_{|k} &= X^jP_j{}^i{}_k{}^c - C_k^{jc}X^i_{|j} - P_{ak}{}^cX^i|^a,\\ X^i|^b|^c - X^i|^c|^b &= X^jS_j{}^{ibc} - S_a{}^{bc}X^i|^a. \end{aligned}$$

Proposition 4.5. *If $X^V = X_a\dot\partial^a$ is a vertical vector field, then the following Ricci identities hold good:*

$$(4.7)\qquad \begin{aligned} X_{a|k|h} - X_{a|h|k} &= -X_d\widetilde{R}_a{}^d{}_{kh} - T^r{}_{kh}X_{a|r} - R_{dkh}X_a|^d\\ X_{a|k}|^b - X_a|^b{}_{|k} &= -X_d\widetilde{P}_a{}^d{}_k{}^b - C_k{}^{rb}X_{a|r} - P_{dk}{}^bX_a|^d\\ X_a|^b|^c - X_a|^c|^b &= -X_d\widetilde{S}_a{}^{dbc} - S_d{}^{bc}X_a|^d. \end{aligned}$$

As an application of these propositions, using (3.18) one obtains:

THEOREM 4.1. *For any d-connection D the following identities hold good:*

$$
\begin{aligned}
&D_{ak|h} - D_{ah|k} = -\widetilde{R}_a{}^o{}_{kh} - T^r_{kh} D_{ar} - R_{dkh}(\delta^d_a - \widetilde{C}^{od}_a),\\
&D_{ak}|^b + \widetilde{C}_a{}^{ob}{}_{|k} = -\widetilde{P}_a{}^o{}_k{}^b - C^{rb}_k D_{ar} - P_{dk}{}^b(\delta^d_a - \widetilde{C}^{od}_a),\\
&-\widetilde{C}^{ob}_a|^c + \widetilde{C}^{oc}_a|^b = -\widetilde{S}_a{}^{obc} - S_d{}^{bc}(\delta^d_a - \widetilde{C}^{od}_a).
\end{aligned} \tag{4.8}
$$

A d-connection for which $C_a{}^{ob} = 0$, $D_{ak} = 0$ is said to be of *Cartan type*. Using Theorem 4.1 one obtains:

PROPOSITION 4.6. *A d-connection of Cartan type has the properties*:

$$
\widetilde{R}_a{}^o{}_{kh} + R_{akh} = 0, \quad \widetilde{P}_a{}^o{}_k{}^b + P_{ak}{}^b = 0, \quad \widetilde{S}_a{}^{obc} + S_a{}^{bc} = 0. \tag{4.9}
$$

5. The equations of structure of a *d*-connection

Let $c\colon (a,b) \to E^*$ be a curve of class C^∞ on E^*. If $X \in \mathcal{X}(E^*)$ then its covariant derivative along c, with respect to the d-connection D is $D_{\dot{c}}X$, which will be also denoted by DX/dt.

The curve c is given locally by

$$
x^i = x^i(t), \quad p_a = p_a(t), \qquad t \in (a,b) \subset \mathbb{R}, \tag{5.1}
$$

where rank $\|dx^i/dt\| = 1$ and rank $\|p_a(t)\| = 1$.

The tangent vector c is represented in the adapted basis as

$$
\dot{c} = \frac{dx^i}{dt}\delta_i + \frac{\delta p_a}{dt}\dot{\partial}^a, \tag{5.2}
$$

so that we have

$$
\frac{DX}{dt} = \frac{dx^i}{dt} D^h_{\delta_i} X + \frac{\delta p_a}{dt} D^v_{\dot{\partial}^a} X. \tag{5.3}
$$

The covariant differential of X, with respect to D, is $(DX/dt)dt$. Hence by (5.3) one obtains:

PROPOSITION 5.1. *The covariant differential DX of a vector field X on E is expressed locally in the adapted basis $(\delta_i, \dot{\partial}^a)$ as follows*:

$$
DX = (D^h_{\delta_i} X)dx^i + (D^v_{\dot{\partial}^a} X)\delta p_a. \tag{5.4}
$$

If $X = X^H = X^i \delta_i$ we have

$$
DX^H = (DX^i)\delta_i \tag{5.4'}
$$

where

(5.4") $$DX^i = X^i_{|k}dx^k + X^i|^a\delta p_a.$$

If we put

(5.5) $$\omega_j{}^i = H^i_{jk}dx^k + C_j{}^{ia}\delta p_a$$

we obtain

(5.5') $$DX^i = dX^i + \omega^i_j X^j.$$

The 1-forms $\omega_j{}^i$ will be called the *h-forms of the d-connection D*. In the same way, for $X = X^V = X_a\dot{\partial}^a$, putting

(5.6) $$DX^V = DX_a\dot{\partial}^a,$$

one obtains from (5.4)

(5.6') $$DX_a = X_{a|k}dx^k + X_a|^b\delta p_b$$

and putting

(5.7) $$\widetilde{\omega}_a{}^b = \widetilde{H}^b_{ak}dx^k + \widetilde{C}_a{}^{bc}\delta p_c$$

one obtains

(5.8) $$DX_a = dX_a - \widetilde{\omega}_a{}^b X_b.$$

The 1-forms $\widetilde{\omega}_a{}^b$ will be called the *v-forms* of the d-connection D.

The differential of a function $f \in \mathcal{F}(E^*)$ has the form

(5.9) $$df = \delta_k f dx^k + \dot{\partial}^a f\delta p_a.$$

The exterior differential of the 1-forms δp_a, according to (1.8) has the following form:

(5.10) $$d(\delta p_a) = -\frac{1}{2}R_{aij}dx^i \wedge dx^j - (\dot{\partial}^b N_{ai})\delta p_b \wedge dx^i.$$

Taking into account previous formulae one obtains:

THEOREM 5.1. *The equations of structure of a d-connection D on E are*

(5.11) $$Dx^h \wedge \omega_h{}^i = \Omega^i, \quad d(\delta p_a) + \delta p_b \wedge \widetilde{\omega}_a{}^b = -\widetilde{\Omega}_a,$$

(5.12) $$d\omega_j{}^i - \omega_j{}^h \wedge \omega_h{}^i = -\Omega^i_j, \quad d\widetilde{\omega}^a_b - \widetilde{\omega}^c_b \wedge \widetilde{\omega}^a_c = -\widetilde{\Omega}^a_b,$$

where the 2-forms of torsion Ω^i, $\widetilde{\Omega}_a$ are given by

(5.13) $$\begin{aligned}\Omega^i &= (1/2)T^i{}_{jk}dx^j \wedge dx^k + C_j{}^{ia}dx^j \wedge \delta p_a,\\ \widetilde{\Omega}_a &= (1/2)R_{aij}dx^i \wedge dx^j + P_{ai}{}^b dx^i \wedge \delta p_b + (1/2)S_a{}^{bc}\delta p_b \wedge \delta p_c,\end{aligned}$$

and the 2-forms of curvature $\Omega_j{}^i$, $\widetilde{\Omega}_b{}^a$ are given by

$$\begin{aligned}\Omega^i_j &= (1/2)R_j{}^i{}_{hk}dx^h \wedge dx^k + P_j{}^i{}_h{}^a dx^h \wedge \delta p_a + (1/2)S_j{}^{icd}\delta p_c \wedge \delta p_d,\\ \widetilde{\Omega}^a_b &= (1/2)\widetilde{R}_b{}^a{}_{hk}dx^h \wedge dx^k + \widetilde{P}_b{}^a{}_h{}^c dx^h \wedge \delta p_c + (1/2)\widetilde{S}_b{}^{acd}\delta p_c \wedge \delta p_d.\end{aligned} \tag{5.14}$$

The equations of structure (5.11) and (5.12) allow us to deduce the Bianchi identities (fifteen in number) which are satisfied by any d-connection D.

These equations also allow us to obtain geometrical meaning for d-tensor fields of torsion and curvature.

6. *v*-and *h*-metrical structures on E^*

Let us consider a Hamilton function H on the total space E^* of the vector bundle ξ^* i.e. a function

$$H: E^* \to R \tag{6.1}$$

which is of the class C^∞ on $E^* \setminus \{0\}$ and continuous on the null section. For the case when ξ^* is the cotangent bundle we refer to [**12**], [**13**].

The function H defines a d-tensor field of type (2,0), symmetric, whose local components are given by

$$g^{ab}(x,p) = (1/2)\dot{\partial}^a\dot{\partial}^b H. \tag{6.2}$$

It is said that a Hamilton function H is *regular* if

$$\operatorname{rank}\|g^{ab}(x,p)\| = m \tag{6.2'}$$

on every domain of a local chart on E^*.

We shall assume there is given in advance a nonlinear connection N^* on E^*.

Definition 6.1. The *v-metric* on E^* is a d-tensor field G^V of the type (2,0) with the properties:

1° G^V is vertical i.e. $G^V(X,Y) = G^V(X^V, Y^V), \quad \forall X, Y \in \mathcal{X}(E^*)$.

2° G^V is symmetric.

3° The rank of G^V is equal to $\dim E_x$.

If we set

$$g^{ab}(x,p) = G^V(\dot{\partial}^a, \dot{\partial}^b) \tag{6.3}$$

it gives the following local form for G^V:

$$G^V = g^{ab}(x,p)\,\delta p_a \otimes \delta p_b \tag{6.4}$$

and, furthermore

$$g^{ab}(x,p) = g^{ba}(x,p), \quad \operatorname{rank}\|g^{ab}(x,p)\| = m. \tag{6.5}$$

We shall set $\|g_{ab}(x,p)\| = \|g^{ab}(x,p)\|^{-1}$.

By (6,2) and (6.2') a regular Hamiltonian function H defines a v-metric on E^*. Conversely, we have:

PROPOSITION 6.1. *A v-metric G^V is provided by a regular Hamilton function iff the d-tensor field whose local components are $\dot{\partial}^a g^{bc}(x,p)$ is totally symmetric.*

Proof. A straightforward calculation using (6.2).

Definition 6.2. A d-connection D on E^* is *compatible* with the v-metric G if

$$D_X G^V = 0, \qquad \forall X \in \mathcal{X}(E^*). \tag{6.6}$$

We remark that (6.6) can be expressed locally as

$$g^{ab}{}_{|k} = 0, \quad g^{ab}|^c = 0. \tag{6.6'}$$

THEOREM 6.1. *If $\big(\overset{\circ}{H}{}^i_{jk}, \overset{\circ}{\widetilde{H}}{}^a_{bk}, 0, 0\big)$ are the local coefficients of a fixed d-connection on E^*, then the d-connection whose local coefficients are $\big(\overset{\circ}{H}{}^i_{jk}, \widetilde{H}^a_{bk}, 0, \widetilde{C}_a{}^{bc}\big)$, where*

$$\begin{aligned} \widetilde{H}^a_{bk} &= \overset{\circ}{\widetilde{H}}{}^a_{bk} - (1/2) g_{bc} g^{ac}_{\overset{\circ}{|k}}, \\ \widetilde{C}_a{}^{bc} &= -(1/2) g_{ad}(\dot{\partial}^b g^{dc} + \dot{\partial}^c g^{bd} - \dot{\partial}^d g^{bc}) \end{aligned} \tag{6.7}$$

is compatible with the v-metric G.

Proof. One verifies (6.6') for the described d-connection, taking into account

$$g^{ac}_{\overset{\circ}{|k}} = \delta_k g^{ak} + g^{dc}\widetilde{H}^a_{dk} + g^{ad}\overset{\circ}{\widetilde{H}}{}^c_{dk}. \tag{6.8}$$

Definition 6.3. The *h-metric* on E^* is a d-tensor field G^H of type (0,2) having the properties:

1° G^H is horizontal i.e. $G^H(X,Y) = G^H(X^H, Y^H), \quad \forall X, Y \in \mathcal{X}(E^*)$,

2° G^H is symmetric.

3° The rank of G^H is equal to n in every point of E^*.

Locally we have

$$G^H = g_{ij}(x,p)\, dx^i \otimes dx^j, \tag{6.9}$$

where we have set

$$g_{ij}(x,p) = G^H(\delta_i, \delta_j). \tag{6.10}$$

Definition 6.4. A d-connection D on E^* is *compatible* with G^H if it satisfies

$$D_X G^H = 0, \qquad \forall X \in \mathcal{X}(E^*). \tag{6.11}$$

Locally, (6.11) can be written as follows:

$$g_{ij|k} = 0, \quad g_{ij}|^a = 0. \tag{6.12}$$

THEOREM 6.2. *The d-connection whose local coefficients are $\big(H^i_{jk}, \dot\partial^b N_{ak}, C_j{}^{ia}, 0\big)$, where*

$$\begin{cases} H^i_{jk} = (1/2)g^{ih}(\delta_k g_{jh} + \delta_j g_{kh} - \delta_h g_{jk}) \\ C_j{}^{ic} = (1/2)g^{ih}\dot\partial^c g_{hj} \end{cases} \tag{6.13}$$

is compatible with the h-metric G^H.

Proof. One verifies (6.12) by a straightforward calculation.

PROPOSITION 6.2. *If G^H is an h-metric and G^V is a v-metric on E^* then the tensor field G of the type (0,2) defined by*

$$G = G^H + G^V. \tag{6.14}$$

is a pseudo-Riemannian metric on E^* with respect to which the distributions N^* and VE^* are orthogonal.

Proof. G is symmetric because G^H and G^V are symmetric. Locally G is given by a matrix

$$\begin{Vmatrix} g_{ij}(x,p) & 0 \\ 0 & g^{ab}(x,p) \end{Vmatrix} \tag{6.15}$$

which is nondegenerate because G^H and G^V are so. The signature of G is constant. So G is a pseudo-Riemannian metric on E^*. By (6.14) the distributions N^* and VE^* are orthogonal with respect to it. QED.

Definition 6.5. A pseudo-Riemannian metric G given by (6.14) will be called an (h,v)-*metric* on E^*.

Remark 6.1. If G is a positive definite metric on E^*, then the metric induced by it on VE^* is positive definite, too. Let N^* be the distribution which is orthogonal to VE^* with respect to G. Then G restricted to VE^* and N^* gives a v-metric G^V and h-metric G^H, respectively, such that (6.14) holds good.

If G is a pseudo-Riemannian metric and the induced metric G^V on VE^* is pseudo-Riemannian, then N^* can still be defined so that G^H is pseudo-Riemannian and satisfies (6.14). Using the adapted basis $(\delta_i, \dot\partial^a)$ an (h,v)-metric G can be written as follows:

$$G = g_{ij}\, dx^i \otimes dx^j + g^{ab}\, \delta p_a \otimes \delta p_b. \tag{6.16}$$

Definition 6.6. A d-connection D is said to be *compatible* with an (h,v)-metric G if we have

$$D_X G = 0, \qquad \forall X \in \mathcal{X}(E^*). \tag{6.17}$$

The condition (6.17), by virtue of (6.16), is equivalent to:

$$g_{ij|k} = 0, \quad g_{ij}|^c = 0, \quad g^{ab}{}_{|k} = 0, \quad g^{ab}|^c = 0. \tag{6.18}$$

THEOREM 6.3. *If $\overset{\circ}{D}$ given locally by $\big(\overset{\circ}{H}{}^i_{jk}, \overset{\circ}{\widetilde{H}}{}^a_{bk}, \overset{\circ}{C}{}_j{}^{ic}, \overset{\circ}{\widetilde{C}}{}_a{}^{bc}\big)$ is a fixed d-connection on E, then the d-connection D with the coefficients*

$$\begin{aligned} &H^i_{jk} = (1/2)g^{ih}(\delta_k g_{jh} + \delta_j g_{hk} - \delta_h g_{jk}), \quad \widetilde{H}^a_{bk} = \overset{\circ}{\widetilde{H}}{}^a_{bk} - (1/2)g_{bc}g^{ca}_{\overset{\circ}{|k}}, \\ &C_j{}^{ic} = (1/2)g^{ih}\dot{\partial}^c g_{hj}, \quad \widetilde{C}_a{}^{bc} = -(1/2)g_{ad}(\dot{\partial}^b g^{dc} + \dot{\partial}^c g^{bd} - \dot{\partial}^d g^{bc}) \end{aligned} \tag{6.19}$$

is compatible with the (h,v)-metric G.

7. Legendre morphisms

Let us consider again the vector bundle $\xi = (E, \pi, M)$. A *Lagrangian* on E is a map $L\colon E \to R$ which is differentiable on $E \setminus \{0\}$ and continuous on null section. L is called a *regular Lagrangian* if with respect to any system of local coordinates (x^i, y^a) on E, the d-tensor field h defined by

$$h_{ab}(x,y) = \frac{\partial^2 \mathcal{L}}{\partial y^a \partial y^b}, \qquad \text{where } \mathcal{L} = (1/2)L, \tag{7.1}$$

is nondegenerate on $E \setminus 0$.

The vertical derivative of L, denoted by $d_V L$, is

$$(d_v L)_e = d(L_{|E_{\pi(e)}})|_e, \qquad \forall e \in E. \tag{7.2}$$

Considering the dual vector bundle ξ^* let us remark that VE^* can be identified with the bundle $(E \times_M E^*, \pi_1, E)$, where

$$E \times_M E^* = \{(e,u) \in E \times E^*, \pi(e) = \pi^*(u)\} \tag{7.3}$$

and $\pi_1\colon E \times_M E^* \to E$ is a projection.

It is obvious that $(d_V L)_e$ belongs to $E \times_M E^*$.

Following Liberman and Marle, [**7**], we set:

Definition 7.1. Let $\xi = (E, \pi, M)$ be a vector bundle endowed with a Lagrangian L. The Legendre morphism associated to L is a morphism $\Phi\colon E \to E^*$ defined by

$$\Phi = \pi_2 \circ d_v \mathcal{L}, \tag{7.4}$$

where $\pi_2: E \times_M E^* \to E^*$ is a projection.

Locally, we obtain

$$d_v\mathcal{L} = \frac{\partial \mathcal{L}}{\partial y^a} dy^a, \tag{7.5}$$

$$\Phi(x,y) = \left(x^i, p_a = \frac{\partial \mathcal{L}}{\partial y^a}\right). \tag{7.6}$$

PROPOSITION 7.1. *If L is a regular Lagrangian, then the Legendre morphism associated to it is a local diffeomorphism $\Phi: E \setminus \{0\} \to E^* \setminus \{0\}$.*

Proof. The Jacobi matrix of Φ in every point of $E \setminus \{0\}$ is $\begin{Vmatrix} \delta^i_j & 0 \\ * & h_{ab}(x,y) \end{Vmatrix}$ which is nonsingular, because L is regular. QED.

When the Legendre morphism Φ is a global diffeomorphism it is called *Legendre transformation*. In such case L is called *hyperregular Lagrangian*.

PROPOSITION 7.2. *Let L be a hyperregular Lagrangian on E and Z the Liouville field on E. Then the map $H = 2\mathcal{H}$ where*

$$\mathcal{H} = \big(i(Z)d\mathcal{L} - \mathcal{L}\big) \circ \Phi^{-1} \tag{7.7}$$

is a Hamilton function on E^*.

Proof. See [**7**].

Locally, the map $\widetilde{\mathcal{L}} = i(Z)d\mathcal{L} - \mathcal{L}$ is written

$$\widetilde{\mathcal{L}} = y^a \frac{\partial \mathcal{L}}{\partial y^a} - \mathcal{L}(x,y). \tag{7.8}$$

Next we have

$$d_v\widetilde{\mathcal{L}} = y^a dv\left(\frac{\partial \mathcal{L}}{\partial ya}\right) \tag{7.9}$$

$$d_v\mathcal{H} = y^a dp_a, \tag{7.10}$$

from which one obtains

$$y^a = \frac{\partial \mathcal{H}}{\partial p_a}. \tag{7.11}$$

Therefore Φ^{-1} is locally as follows

$$\Phi^{-1}: (x^i, p_a) \to \left(x^i, y^a = \frac{\partial \mathcal{H}}{\partial p_a}\right). \tag{7.12}$$

If we assume that L is only regular, the Legendre morphism can be inversed only locally and by (7.7) and (7.8) we can write

$$\mathcal{H}(x,p) = p_a y^a - \mathcal{L}(x,y), \tag{7.13}$$

where $y^a = y^a(x,p)$, for $a = 1, \dots, m$.

From the above considerations we get

Proposition 7.3. *Let L be a regular Lagrangian on $E \setminus \{0\}$ and U an open subset of $E \setminus \{0\}$ on which the Legendre morphism is a diffeomorphism. Then on $V = \Phi(U) \subset E^* \setminus \{0\}$ a regular Hamilton function H is obtained and Φ carries the v-metric tensor defined by L on U to the v-metric tensor defined by H on V.*

Proposition 7.4. *The Legendre transformation associated to a hyperregular Lagrangian L applies the v-metric h defined by L on E to the v-metric g defined by the Hamilton function H induced on E^*.*

Proposition 7.5. *If L is a regular Lagrangian, then locally we have*

$$\frac{\partial \mathcal{H}}{\partial x^i} = -\frac{\partial \mathcal{L}}{\partial x^i}. \tag{7.14}$$

Now we are interested in the effects of Φ on a nonlinear connection.

Theorem 7.1. *If L is a hyperregular Lagrangian, then the Legendre transformation Φ associated to it carries a nonlinear connection N on E to a nonlinear connection N^* on E^*. If N_i^a are the local coefficients of N and N_{ai} are the local coefficients of N^* on E^*, then we have*

$$N_{ai}(x,p) = -(N_i^b + \dot\partial^b \partial_i \mathcal{H}) h_{ba}, \tag{7.15}$$

where $\mathcal{H}$ is the Hamilton function induced on E^* and h_{ab} are the coefficients of the v-metric induced by L on E.

Proof. Taking into account (7.6) one can see that the differential $d\Phi$ acts on the canonical basis as follows

$$d\Phi(\partial_i) = \partial_i + \frac{\partial^2 \mathcal{L}}{\partial y^a \partial x^i} \dot\partial^a = \partial_i - (\dot\partial^b \partial_i \mathcal{H}) h_{bc} \dot\partial^c \tag{7.16}$$

$$d\phi(\dot\partial_a) = h_{ab} \dot\partial^b, \tag{7.17}$$

so that on (δ_i) $i = 1, \dots, n$, $d\Phi$ acts as

$$d\Phi(\delta_i) = d\Phi(\partial_i - N_i^a \dot\partial_a) = \partial_i - (N_i^b + \dot\partial^b \partial_i \mathcal{H}) h_{ba} \dot\partial^a. \tag{7.18}$$

Therefore the distribution N is mapped by Φ to the distribution N^* and (7.15) holds good. QED.

Proposition 7.6. *Let L be a hyperregular Lagrangian and Φ the Legendre transformation associated to it. If ${R^a}_{ij}$ and R_{aij} are the integrability tensors of the nonlinear connection N and N^*, respectively, then*

$$R_{aij} \circ \Phi^{-1} = h_{ab} R^b_{ij}, \text{ holds good }. \tag{7.19}$$

Proof. We have $[\delta_i, \delta_j] = R^a{}_{ij}\dot{\partial}_a$, $(\dot{\partial}_a = \partial/\partial y^a)$. Since Φ is a diffeomorphism, $[d\Phi(\delta_i), d\Phi(\delta_j)] = R^a{}_{ij} d\Phi(\dot{\partial}_a) = R^a{}_{ij} h_{ab}\dot{\partial}^b$. On the other hand $[d\Phi(\delta_i), d\Phi(\delta_j)] = (R_{aij}\dot{\partial}^a) \circ \Phi^{-1}$. QED.

COROLLARY 7.6. *The distribution N is integrable if and only if the induced distribution N^* is integrable.*

REFERENCES

[**1**] M. Anastasiei, *Models of Finsler and Lagrange geometry*, The Proc. of IV$^{\text{th}}$ National Seminar on Finsler and Lagrange spaces. Univ. Braşov, (Romania) 1986, 43–56.

[**2**] M. Anastasiei, *Nonlinear connection in Hamilton spaces*, (to appear).

[**3**] M. Anastasiei, *Some models in geometry of Hamilton spaces*, (to appear).

[**4**] A. K. Aringazin and G. S. Asanov, *Problems of Finslerian theory of gauge fields and gravitation*, Rep. Math. Physics **25** (1988), 35–93.

[**5**] S. Ianuş, *Some almost product structures on manifold with linear connection*, Kodai Math. Sem. Rep. **23** (1971), 305–310.

[**6**] Y. Ichijo, R. Miron, *On some structures defined on vector bundles*, J. Math. Tokushima Univ. **20** (1986), 13–26.

[**7**] P. Libermann, Ch. M. Marle, *Géométrie symplectique. Bases theoriques de la mecanique. T. I–IV*, Publ. Math. de l'Univ. Paris VII, 1986.

[**8**] M. Matsumoto, *Foundations of Finsler Geometry and Special Finsler Spaces*, Kaiseisha Press, Shigaken, 1986.

[**9**] R. Miron, *Vector bundles. Finsler geometry*, Proc. of the National Seminar on Finsler spaces. II, Braşov, (Romania) 1983, 147–188.

[**10**] R. Miron, *Techniques of Finsler geometry in the theory of vector bundles*, Acta. Sci. Math. **49** (1985), 119–129.

[**11**] R. Miron, *On the Finslerian theory of relativity*, Tensor N. S. **44** 1987, 63–81.

[**12**] R. Miron, *Hamilton geometry*, Seminarul de Mecanică, Univ. Timişoara (Romania) Preprint Nr. 3, 1987, 54 pg.

[**13**] R. Miron, *Sur la géométrie des espaces d'Hamilton*, C. R. Acad. Sci. Paris, t. **306**, S. I, 1988, 195–198.

[**14**] R. Miron, M. Anastasiei, *Vector Bundles. Lagrange Spaces. Applications to Relativity*, (in Romanian), Edit. Acad. R. S. Romania, 1987.

[**15**] R. Miron, S. Watanabe, S. Ikeda, *On the geometry of total space of the cotangent bundle*, (to appear).

[**16**] T. Sakaguchi, *Subspaces in Lagrange spaces*, Ph. Thesis, Univ. Iaşi (Romania), 1988, 167 pg.

Universitatea "Al. I. Cusa" Iaşi
Facultatea de Matematică
6600 Iaşi, Romania

(Received 25 10 1988)

Universitatea din Bucureşti
Facultatea de Matematică
Str. Academiei Nr. 14
Bucureşti, Romania