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ON p-VALENT ANALYTIC FUNCTIONS WITH REFERENCE TO BERNARDI AND RUSCHEWEYH INTEGRAL OPERATORS

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Abstract. Let $T_n(h)$ be the class of analytic functions in the unit disk E of the form $f(z) = a_p z^p + \sum_{n=p+1}^{\infty} a_n z^n$, $p \geq 1$, which satisfy the condition, $\dfrac{(n+1)}{(n+p)} \dfrac{D^{n+1} f(z)}{D^n f(z)} \prec h(z)$, $z \in E$, where h is a convex univalent function in E with $h(0) = 1$. Then it is proved that f is preserved under the Bernardi integral operator under certain conditions. It is also shown that if $f \in T_0(h)$, it is preserved under the Ruscheweyh integral operator under certain conditions.

The Hadamard product $(f * g)(z)$ of two functions $f(z) = \sum_{m=0}^{\infty} a_m z^m$ and $g(z) = \sum_{m=0}^{\infty} b_m z^m$ is given by $(f * g)(z) = \sum_{m=0}^{\infty} a_m b_m z^m$. Let

$$D^{\alpha} f(z) = \frac{z}{(1-z)^{\alpha+1}} * f(z), \ (\alpha \geq 1).$$

Ruscheweyh [**7**] observed that $D^n f(z) = z\big(z^{n-1} f(z)\big)^{(n)}/n!$ when $n \in N \cup \{0\}$, where $N = \{1, 2, 3, \dots\}$. This symbol $D^n f(z)$, $n \in N \cup \{0\}$ was called the n-th Ruscheweyh derivative of $f(z)$ by Al-Amiri [**1**].

Let $S_p(A, B)$, denote the class of all functions of the form $f(z) = a_p z^p + \sum_{n=p+1}^{\infty} a_n z^n$, $p \geq 1$ which are regular in $E = \{z : |z| < 1\}$ and satisfy the condition,

$$z \frac{f'(z)}{f(z)} = p \frac{1 + Aw(z)}{1 + Bw(z)}, \quad z \in E,$$

where A, $B \in \mathbf{C}$, $|A| \leq 1$, $|B| \leq 1$, $A \neq B$ and $w(z)$ is regular in E and satisfies the Schwarz lemma conditions, that is $w(0) = 0$, $|w(z)| < 1$ in E.

The class $S_p(1, -1)$ is the class of p-valent starlike functions and $S_p(A, B)$ for A, $B \in \mathbf{C}$ will yield subclasses of spirallike p-valent functions. For A, B real and $0 \leq A < B \leq 1$, $S_p(A, B)$ will give a subclass of p-valent starlike functions.

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For $p=1$ such classes were investigated by many authors and were introduced by Janowski [**4**], when A, B are real and by Stankiewicz, Waniurski [**9**] when A, B are complex. Bernardi [**2**] showed that the function g defined by,

$$g(z)=(e+1)z^{-c}\int_0^z t^{c-1}f(t)\,dt,$$

where c is a positive integer, belongs to the class $S_1(1,-1)$ if f belongs to $S_1(1,-1)$.

Lakshma Reddy and Padmanabhan [**6**] showed that the function g defined by the Bernardi operator,

$$g(z)=(c+p)z^{-c}\int_0^z t^{c-1}f(t)\,dt,\quad c=1,2,\dots,$$

belongs to the class $S_p(A,B)$, A, B-real, $-1\le A<B\le -1$, if $f\in S_p(A,B)$.

Kumar and Shukla [**5**] obtained a generalization of the above result by considering the Ruscheweyh integral operator which is given by

$$g(z)=\left\{(c+p\alpha)z^{-c}\int_0^z t^{c-1}f^{\alpha}(t)\,dt\right\}^{1/\alpha};$$
$$c \text{ and } \alpha \text{ real},\ \alpha>0,\quad c\ge -p\alpha\frac{1+A}{1+B}.$$

Ryszard Kowal and Jan Stankiewicz [**8**] obtained the solution of this problem in the case when A, B are complex numbers. We propose to study the more general problem when f satisfies the condition that $\dfrac{n+1}{n+p}\,\dfrac{D^{n+1}f}{D^n f}$ is subordinate to a convex univalent function h. We allow c to be complex and make use of the method of differential subordination introduced in [**3**].

Let $T_n(h)$ be the class of analytic functions on E of the form $f(z)=a_pz^p+\sum_{n=p+1}^{\infty}a_nz^n$, $p\ge 1$ which satisfy the condition,

$$\left(\frac{n+1}{n+p}\right)\frac{D^{n+1}f(z)}{D^nf(z)}\prec h(z),\quad z\in E,$$

where h is a convex univalent function in E with $h(0)=1$ and the symbol $\prec$ denotes subordination.

Let $f\in T_n(h)$ and let g be given by,

$$g(z)=(c+p)z^{-c}\int_0^z t^{c-1}f(t)\,dt,\quad c\in\mathbf{C}. \tag{1}$$

Then we show that $g\in T_n(h)$ under certain conditions to be satisfied by c and h. In particular for $n=0$, the class $T_0(h)$ consists of functions f of the form $f(z)=a_pz^p+\sum_{n=p+1}^{\infty}a_nz^n$, satisfying $(1/p)zf'(z)/f(z)\prec h(z)$.

Let $f \in T_0(h)$ and define F by

$$F(z) = \left[(c+p\alpha)z^{-c}\int_0^z t^{c-1} f^{\alpha}(t)\,dt\right]^{1/\alpha}, \quad c \in \mathbf{C},\ \alpha > 0. \tag{2}$$

In this paper it is shown that F also belongs to $T_0(h)$ under certain conditions to be satisfied by c and h. We need the following theorem due to Eenigenburg, Miller, Mocanu and Reade.

Theorem A [**3**]. *Let $\beta,\ \gamma \in \mathbf{C}$, $h \in H(E)$ be convex univalent in E with $h(0) = 1$ and $\operatorname{Re}\bigl(\beta h(z) + \gamma\bigr) > 0$, $z \in E$ and let $p \in H(E)$, $p(z) = 1 + p_1 z + \cdots$. Then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z),$$

implies that $p(z) \prec h(z)$.

Theorem 1. *Suppose $f \in T_n(h)$ and g is given by* (1). *Then $g \in T_n(h)$ provided* $\operatorname{Re}\{(n+p)h(z) + (c-n)\} > 0$.

Proof: Differentiating (1) we get

$$g'(z) = -\frac{c(c+p)}{z^{c+1}}\int_0^z t^{c-1} f(t)\,dt + \frac{c+p}{z^c} z^{c-1} f(z),$$

which gives

$$zg'(z) + cg(z) = (c+p)f(z).$$

Therefore we have

$$D^n\bigl(zg'(z)\bigr) + D^n\bigl(cg(z)\bigr) = D^n\bigl((c+p)f(z)\bigr). \tag{3}$$

Using the fact that $D^n\bigl(zg'(z)\bigr) = z\bigl(D^n g(z)\bigr)'$, (3) reduces to

$$z\bigl(D^n g(z)\bigr)' + c\bigl(D^n g(z)\bigr) = (c+p)D^n f(z).$$

Again using the result

$$z\bigl(D^n g(z)\bigr)' = (n+1)\bigl(D^{n+1} g(z)\bigr) - n\bigl(D^n g(z)\bigr) \tag{4}$$

we get

$$(n+1)\frac{D^{n+1} g(z)}{D^n g(z)} + (c-n) = (c+p)\frac{D^n f(z)}{D^n g(z)}. \tag{5}$$

Set

$$P(z) = \frac{n+1}{n+p}\frac{D^{n+1} g(z)}{D^n g(z)}. \tag{6}$$

Then (4) takes the form

$$P(z)+\frac{c-n}{n+p}=\frac{c+p}{n+p}\frac{D^n f(z)}{D^n g(z)}. \tag{7}$$

Taking logarithmic derivatives and multiplying by z, we get,

$$\frac{zP'(z)}{P(z)+(c-n)/(n+p)}=z\frac{\left(D^n f(z)\right)'}{D^n f(z)}-z\frac{\left(D^n g(z)\right)'}{D^n g(z)}.$$

Using (4) and (6) this takes the form,

$$P(z)+\frac{zP'(z)}{(n+p)P(z)+(c-n)}=\frac{n+1}{n+p}\frac{D^{n+1} f(z)}{D^n f(z)}\prec h(z),$$

Since $f\in T_n(h)$.

Now, if $\operatorname{Re}\{(n+p)h(z)+(c-n)\}>0$, we can conclude by Theorem A that $P(z)\prec h(z)$, that is $P\in T_n(h)$.

Remark 1. For $n=0$ the above theorem reduces to the following result, namely, $g\in T_0(h)$, that is, $(1/p)zg'(z)/g(z)\prec h(z)$ whenever $f\in T_0(h)$ if $\operatorname{Re}\{ph(z)+c\}>0$. For the choice of $h(z)=(1+Az)/(1+Bz)$, this clearly includes the result in [**6**].

THEOREM 2. *Let $f\in T_0(h)$ and let F be given by* (2). *Then $F\in T_0(h)$ provided* $\operatorname{Re}\bigl(\alpha ph(z)+c\bigr)>0$.

Proof:

$$F^\alpha(z)=\frac{c+p\alpha}{z^c}\int_0^z t^{c-1}f^\alpha(t)\,dt.$$

Differentiation gives

$$\frac{z}{p}\frac{F'(z)}{F(z)}+\frac{c}{\alpha p}=\frac{f^\alpha(z)}{F^\alpha(z)}\frac{(c+p\alpha)}{p\alpha}.$$

Setting, $P(z)=zF'(z)/(pF(z))$, this reduces to

$$P(z)+\frac{c}{\alpha p}=\frac{z^c f^\alpha(z)}{p^\alpha\int_0^z t^{c-1}f^\alpha(t)\,dt}.$$

Taking logarithmic derivatives and multiplying by z, we obtain, after some simplification,

$$\frac{zP'(z)}{P(z)+c/(\alpha p)}=\frac{\alpha zf'(z)}{f(z)}-\alpha pP(z),$$

$$P(z)+\frac{zP'(z)}{\alpha pP(z)+c}=\frac{1}{p}z\frac{f'(z)}{f(z)}\prec h(z).$$

Provided $\operatorname{Re}\big(\alpha p h(z)+c\big)>0$, this implies that $P(z)\prec h(z)$ by Theorem A.

Remark: If we choose $h(z)=(1+Az)/(1+Bz)$, A, B real with $-1\le A<B\le 1$ and condition on c and h reduces to

$$\operatorname{Re} c > -\alpha p \operatorname{Re} h(z) > -\alpha p\frac{1-A}{1-B}.$$

This condition is clearly an improvement on the condition

$$c\ge -p\alpha\frac{1+A}{1+B}$$

in [**5**].

REFERENCES

[**1**] H. S. Al-Amiri, *On Ruscheweyh derivatives*, Ann. Polon. Math. **38** (1980), 87–94.

[**2**] S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc., **135** (1969), p. 429–446.

[**3**] P. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, *On a Briot-Bouquet differential subordination*, General Inequalities **3** (1983), (Birkhäuser Verlag, Basel), 339–348.

[**4**] W. Janowski, *Some extremal problems for certain families of analytic functions, I*, Ann. Polon. Math. **28** (1973), p. 297–326.

[**5**] V. Kumar, S. L. Shukla, *On p-valent starlike functions with reference to the Bernardi integral operator*, Bull. Austral. Math. Soc. **30** (1984), p. 37–43.

[**6**] G. L. Reddy, K. S. Padmanabhan, *On analytic functions with reference to Bernardi integral operator*, Bull. Austral. Math. Soc. **25** (1982), p. 387–396.

[**7**] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), 109–115.

[**8**] Ryszard Kowal, Jan Stankiewicz, *On p-valent functions with reference to the Bernardi Integral operator*, Zeszyty Naukowe Polytechniki Rzeszowskiej – Folia Scientiarum Universitatis Technicae Resovienis NR **33** Matematyka 1 Fizyka Z4 (1986) 17–24.

[**9**] J. Stankiewicz, J. Waniurski, *Some classes of functions subordinate to linear transformation and their applications*, Ann. Univ. Mariae Curie Sklodowska, Sec. A, **28** (1974), p. 85–94.

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