

## CLASSES OF WEIGHTED SYMMETRIC FUNCTIONS\*

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**Abstract.** We generalize the concept of the  $k$ -th symmetric difference in the sense of Stein and Zygmund to that of symmetric difference with respect to a weight system of order  $n$  and the concept of symmetrically continuous functions and symmetric functions to that of functions symmetric with respect to a weight system of order  $n$ . We also study the classes of even symmetry and odd symmetry consisting of functions whose limits to the right and to the left exist at each point; hence, their set of points of discontinuity is countable, and they are in Baire class one. The functions symmetric with respect to a fixed weight system  $W_n$  of order  $n$  form a linear space  $V(W_n)$ , and the subclass  $B(W_n)$  consisting of bounded functions forms a Banach space with the norm  $\|f\| = \sup |f(x)|$ .

### Classes of weighted symmetric functions

*Definitions:* We call a *weight system of order  $n$*  a set of real numbers

$$W_n = \{w_{-n}, \dots, w_{-1}, w_0, w_1, \dots, w_n\}$$

such that  $\sum_{k=-n}^n w_k = 0$  and  $|w_n| + |w_{-n}| > 0$ .

We say that a weight system  $W_n$  is *even* if

$$w_{-k} = w_k, \quad k = 0, 1, \dots, n,$$

with  $\sum_{k=1}^n w_k \neq 0$  and  $w_0 = -2 \sum_{k=1}^n w_k \neq 0$  and a weight system  $W_n$  is *odd* if

$$w_{-k} = -w_k, \quad k = 0, 1, \dots, n,$$

with  $\sum_{k=1}^n w_k \neq 0$  and  $w_0 = 0$ .

We call *symmetric difference with respect to a weight system  $W_n$  of order  $n$  for a finite real function  $f(x)$*  the following expression

$$\Delta f(x; W_n, h) = \sum_{k=-n}^n w_k f(x + kh/2).$$

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This symmetric difference includes the  $k$ -th symmetric difference introduced by Stein and Zygmund [4]:

$$\Delta^k f(x, h) = \sum_{i=0}^k (-1)^i \binom{k}{i} f[x + (k - 2i)h/2].$$

A function  $f(x)$  will be called *symmetric with respect to a weight system*  $W_n$  if

$$\lim_{h \rightarrow 0} \Delta f(x; W_n, h) = 0,$$

and then we write  $f \in S(W_n)$ .

We see clearly that if  $W_1$  is odd, then  $S(W_1)$  is the class of symmetrically continuous functions, that is, the class of functions satisfying the condition

$$\lim_{h \rightarrow 0} (f(x + h) - f(x - h)) = 0,$$

and if  $W_1$  is even,  $S(W_1)$  is the class of symmetric functions satisfying the condition

$$\lim_{h \rightarrow 0} (f(x + h) + f(x - h) - 2f(x)) = 0.$$

*Remark:* Some classes are equal.

*Example 1:* Consider

$$\begin{aligned} W_1 &= \{1, -2, 1\} \\ W_2 &= \{1, 0, -2, 0, 1\} \\ W_3 &= \{1, 0, 0, -2, 0, 0, 1\} \\ W_n &= \{1, 0, \dots, 0, -2, 0, \dots, 0, 1\}, \end{aligned}$$

then

$$S(W_1) = S(W_2) = S(W_3) = \dots = S(W_n).$$

*Example 2:* Let

$$\begin{aligned} W_1 &= \{-1, 0, 1\} \\ W_2 &= \{-1, 0, 0, 0, 1\} \\ W_n &= \{-1, 0, \dots, 0, \dots, 0, 1\}; \end{aligned}$$

then

$$S(W_1) = S(W_2) = \dots = S(W_n).$$

*Example 3:* Let  $W_1 = \{1, -2, 1\}$  and let  $W_2$  be an even weight system

$$W_2 = \{a, b, c, b, a\}$$

such that

$$a \neq 0, \quad c = -2(a + b), \quad |a| < |b| \quad \text{and} \quad w = b/a < 0.$$

Let  $B(W_1)$  be the class of functions bounded and symmetric with respect to  $W_1$ ;  $B(W_2)$  the class of functions bounded and symmetric with respect to  $W_2$ . Then

$$B(W_2) = B(W_1).$$

*Proof:* Prop. 2, which follows this example, shows that

$$f \in B(W_1) \Rightarrow f \in B(W_2).$$

Now we prove

$$f \in B(W_2) \Rightarrow f \in B(W_1).$$

Since  $f$  is bounded, we may assume  $f(0) = 0$ . For if  $f(0) \neq 0$ , the function

$$g(x) = f(x) - f(0)$$

satisfies  $g(0) = 0$  and

$$\begin{aligned} \lim_{h \rightarrow 0} \sum_{k=-2}^2 w_{-k} g(kh/2) &= \lim_{h \rightarrow 0} \sum_{k=-2}^2 w_k [f(kh/2) - f(0)] \\ &= \lim_{h \rightarrow 0} \sum_{k=-2}^2 w_{-k} f(kh/2) - f(0) \sum_{k=-2}^2 w_k = 0. \end{aligned}$$

There exists an  $M > 0$  such that, for every  $x$ ,

$$|f(x)| < M.$$

We show that if  $f$  is not symmetric with respect to  $W_1$  at  $x = 0$ , then there is a contradiction.

Suppose there exists a sequence  $h_n \rightarrow 0$  and

$$f(h_n) + f(-h_n) - 2f(0) > \varepsilon.$$

Since  $f$  is symmetric with respect to  $W_2$ , there exists a  $\delta > 0$  such that for  $|h| < \delta$

$$|a[f(-2h) + f(2h)] + b[f(-h) + f(h)]| < \varepsilon|a|.$$

Let  $w = b/a$  with  $|w| > 1$ . Then

$$|f(-2h) + f(2h) + w[f(-h) + f(h)]| < \varepsilon.$$

Since  $h_n \rightarrow 0$ , we have, for  $n$  sufficiently large,

$$(1) \quad -\varepsilon < f(-2h_n) + f(2h_n) + w[f(-h) + f(h)] < \varepsilon.$$

But  $-w > 1$ , and thus we have

$$(2) \quad -w\varepsilon < -w[f(-h_n) + f(h_n)].$$

From (1) and (2)

$$(3) \quad (-w - 1)\varepsilon < f(-2h_n) + f(2h_n).$$

We also have, for  $n$  sufficiently large,

$$(4) \quad -\varepsilon < f(-4h_n) + f(4h_n) + w[f(-2h_n) + f(2h_n)] < \varepsilon.$$

And by (3) it follows that

$$(5) \quad (w^2 + w)\varepsilon < -w[f(-2h_n) + f(2h_n)].$$

From (4) and (5):

$$(w^2 + w - 1)\varepsilon < f(-4h_n) + f(4h_n).$$

Similarly, for  $n$  sufficiently large:

$$(-w^3 - w^2 + w - 1)\varepsilon < f(-8h_n) + f(8h_n).$$

In general, let

$$a_p = (-1)^p w^p + (-1)^p w^{p-1} + (-1)^{p-1} w^{p-2} + (-1)^{p-2} w^{p-3} + \dots - 1.$$

Then for  $n$  sufficiently large it follows that

$$a_p \varepsilon < f(-2^p h_n) + f(2^p h_n).$$

We can choose  $p$  and  $n$  so that

$$a_p \varepsilon > M \quad \text{and} \quad 2^p |h_n| < \delta.$$

Then

$$f(2^p h_n) + f(2^p h_n) > M.$$

This is a contradiction.

**PROPOSITION 1.** *If  $f$  is continuous, then  $f \in S(W_n)$  for any weight system  $W_n$ .*

*Proof:* Since

$$\lim_{h \rightarrow 0} f(x + kh/2) = f(x), \quad k = -n, \dots, n,$$

we have

$$\lim_{h \rightarrow 0} \sum_{k=-n}^n w_k f(x + kh/2) = f(x) \sum_{k=-n}^n w_k = 0.$$

**PROPOSITION 2.** *If  $f$  is symmetric, then  $f$  is symmetric with respect to any even weight system of any order.*

*Proof:* Let  $W_n$  be any even weight system. Then

$$\begin{aligned} \Delta F(x; W_n, h) &= \sum_{k=-n}^n w_k f(x + kh/2) \\ &= \sum_{k=1}^n w_k [f(x + kh/2) + f(x - kh/2) - 2f(x)]. \end{aligned}$$

Since

$$\lim_{h \rightarrow 0} f(x + h/2) + f(x - h/2) - 2f(x) = 0,$$

we have

$$\lim_{h \rightarrow 0} f(x + kh/2) + f(x - kh/2) - 2f(x) = 0, \quad k = 0, 1, \dots, n.$$

Thus

$$\lim_{h \rightarrow 0} \Delta f(x; W_n, h) = 0.$$

COROLLARY.  $S_{\text{sym}} = \bigcap_{n=1}^{\infty} S(\text{even-}W_n)$  where  $S_{\text{sym}}$  is the class of symmetric functions, and  $S(\text{even-}W_n)$  is the class of functions symmetric with respect to an even system  $W_n$ .

PROPOSITION 3. *If  $f$  is symmetrically continuous, then  $f$  is symmetric with respect to any odd weight system of any order.*

*Proof:* Let  $W_n$  be any odd weight system. Then

$$\Delta f(x; W_n, h) = \sum_{k=1}^n w_k [f(x + kh/2) - f(x - kh/2)].$$

Since

$$\lim_{h \rightarrow 0} f(x + h/2) - f(x - h/2) = 0,$$

we have

$$\lim_{h \rightarrow 0} f(x + kh/2) - f(x - kh/2) = 0.$$

Thus

$$\lim_{h \rightarrow 0} \Delta f(x; W_n, h) = 0.$$

COROLLARY.  $S_{\text{sc}} = \bigcap_{n=1}^{\infty} S(\text{odd-}W_n)$  where  $S_{\text{sc}}$  is the class of symmetrically continuous functions, and  $S(\text{odd-}W_n)$  is the class of functions symmetric with respect to an odd system  $W_n$ .

PROPOSITION 4. *There are functions symmetric with respect to an even weight system of order 2 but not symmetric.*

*Proof:* 1) First we construct a bounded function, which is symmetric with respect to

$$W_2 = \{1, 1, -4, 1, 1\} \quad \text{at } x = 0$$

but not symmetric at  $x = 0$ .

Let  $c$  be a fixed positive number.

We define  $f$  as follows:  $f(0) = 0$ . For  $x \notin (-2c, 2c)$ ,  $f(x) = 0$ .

For  $x > 0$ ,

$$\begin{aligned}
f(x) &= 0 && \text{for } x = 2c, c, c/2, c/2^2, \dots, c/2^n \dots \\
f(x) &= 1 && \text{for } x = 3c/2, 3c^3/2, \dots, 3c/2^{2n+1}, \dots \\
f(x) &= -1 && \text{for } x = 3c/2^2, 3c/2^4, \dots, 3c/2^{2n} \dots
\end{aligned}$$

For  $x < 0$  :

$$\begin{aligned}
f(x) &= 0 && \text{for } x = -2c, -c, -c/2, -c/2^2, \dots, -c/2^n, \dots \\
&&& x = -3c/2, -3c/2^2, 3c/2^3, \dots, 3c/2^n, \dots \\
f(x) &= 1 && \text{for } x = -7c/2^3, -7c/2^5, \dots, -7c/2^{2n+3}, \dots \\
&&& x = -5c/2^3, -5c/2^5, \dots, -5c/2^{2n+3}, \dots \\
f(x) &= -1 && \text{for } x = -7c/2^2, -7c/2^4, \dots, -7c/2^{2n}, \dots \\
&&& x = -5c/2^2, -5c/2^4, \dots, -5c/2^{2n}, \dots
\end{aligned}$$

Between any two consecutive points listed above,  $f(x)$  is linear. Then we see easily that  $f$  is symmetric with respect to  $W_2$  at  $x = 0$  but not symmetric at  $x = 0$ .

2) We also construct an unbounded function, which is symmetric with respect to the system

$$W_2 = \{2, 1, -6, 1, 2\} \quad \text{at } x = 0$$

but not symmetric at  $x = 0$ .

Let  $c > 0$  and let

$$\begin{aligned}
f(x) &= 1 && \text{for } x \in (-2c, 2c) \\
f(x) &= 0 && \text{for } x = 0 \\
f(x) &= 1 && \text{for } x = -2c, -c/2, -c/2^3, \dots, -c/2^{2n+1}, \dots \\
f(x) &= 1/8 && \text{for } x = -c, -c/2^2, -c/2^4, \dots, -c/2^{2n}, \dots \\
f(x) &= 1 && \text{for } x = 2c \\
f(x) &= -33/8 && \text{for } x = c.
\end{aligned}$$

At the points

$$x = c/2, c/2^2, c/2^3, \dots$$

we define  $f(x)$  by the recursion formula

$$2[f(c/2^n) + f(-c/2^n)] + f(c/2^{n+1}) + f(-c/2^{n+1}) = 0$$

and we let  $f(x)$  be linear between any two consecutive points listed above. This function is symmetric with respect to  $W_2$  but not symmetric at  $x = 0$ . To show this, we need the lemma:

LEMMA. For the function  $f$  given above and each  $h = c/2^n$

$$\begin{aligned}
(1) \quad & 2[f(-2h) + f(2h)] + f(-h) + f(h) = 0 \\
(2) \quad & 2[f(-h) + f(h)] + f(-h/2) + f(h/2) = 0
\end{aligned}$$

and, for any  $t$  in  $[h/2, h]$ ,  $f$  satisfies

$$2[f(-2t) + f(2t)] + f(-t) + f(t) = 0.$$

*Proof of the lemma:* Let

$$\begin{aligned} f(-2h) &= a_0, & f(2h) &= a'_0 \\ f(-h) &= a_1, & f(h) &= a'_1 \\ f(-h/2) &= a_2, & f(h/2) &= a'_2. \end{aligned}$$

Thus

$$(1) \text{ holds } \Leftrightarrow 2(a_0 + a'_0) + a_1 + a'_1 = 0,$$

$$(2) \text{ holds } \Leftrightarrow 2(a_1 + a'_1) + a_2 + a'_2 = 0.$$

Let  $L_1$  be the segment joining  $(-2h, a_0)$  and  $(-h, a_1)$ ;  $L_2$  joining  $(2h, a'_0)$  and  $(h, a'_1)$ ;  $L_3$  joining  $(-h, a_1)$  and  $(-h/2, a_2)$ ;  $L_4$  joining  $(h, a'_1)$  and  $(h/2, a'_2)$ .

$$\begin{aligned} L_1: & \quad y - a_0 = (a_1 - a_0)(x + 2h)/h \\ L_2: & \quad y - a'_0 = (a'_1 - a'_0)(x - 2h)/h \\ L_3: & \quad y - a_1 = 2(a_2 - a_1)(x + h)/h \\ L_4: & \quad y - a'_1 = -2(a'_2 - a'_1)(x - h)/h. \end{aligned}$$

Thus,

$$\begin{aligned} f(-2t) &= (a_1 - a_0)(-2t + 2h) + a_0 \\ f(2t) &= -(a'_1 - a'_0)(2t - 2h) + a'_0 \\ f(-t) &= 2(a_2 - a_1)(-t + h)/h + a_1 \\ f(t) &= -2(a'_2 - a'_1)(t - h)/h + a'_1. \end{aligned}$$

So for every  $t$  in  $[h/2, h]$ ,

$$2[f(-2t) + f(2t)] + f(-t) + f(t) = 0.$$

This lemma applied to each step, shows that  $f$  is symmetric with respect to  $W_2$  at  $x = 0$ . But by the recursion formula

$$f(c/2^{n+1}) + f(-c/2^{n+1}) = -2[f(c/2^n) + f(-c/2^n)],$$

we have:

$$\begin{aligned} f(c/2^{n+1}) + f(-c/2^{n+1}) &= (-2)[f(c) + f(-c)] \\ &= (-2)^{n+1}[f(2c) + f(-2c)] = 2(-2)^{n+1}. \end{aligned}$$

Thus  $f$  is not symmetric at  $x = 0$ .

**PROPOSITION 5.** *There are functions symmetric with respect to an odd weight system of order 2 but not symmetrically continuous.*

*Proof.* 1) First we construct a bounded function symmetric with respect to

$$W'_2 = \{-1, -1, 0, 1, 1\}.$$

Let  $f(x)$  be the function of Part 1 in the proof of Prop. 4. Then we define

$$\begin{aligned} g(x) &= f(x) & \text{for } x \geq 0 \\ g(x) &= -f(x) & \text{for } x < 0. \end{aligned}$$

This function  $g(x)$  is symmetric with respect to  $W'_2$  and not symmetrically continuous at  $x = 0$ .

2) We also construct an unbounded function symmetric with respect to

$$W''_2 = \{-2, -1, 0, 1, 2\}.$$

Let  $c > 0$  and let

$$\begin{aligned} f(x) &= -1 & \text{for } x \leq -2c \\ f(x) &= 0 & \text{for } x = 0 \\ f(x) &= -1 & \text{for } x = -2c, -c/2, -c/2^3, \dots, -c/2^{2n+1} \\ f(x) &= -1/8 & \text{for } x = -c, -c/2^2, -c/2^4, \dots, -c/2^{2n} \\ f(x) &= 1 & \text{for } x \geq 2c \\ f(x) &= -33/8 & \text{for } x = c. \end{aligned}$$

At the points

$$x = c/2, c/2^2, c/2^3, \dots, c/2^n, \dots$$

we define  $f(x)$  by the recursion formula

$$2[f(c/2^n) - f(-c/2^n)] + f(c/2^{n+1}) - f(-c/2^{n+1}) = 0$$

and let  $f(x)$  be linear between any two consecutive points listed above. Then by a proof similar to that of Part 2 in Prop. 4, we see that the function  $f(x)$  defined here is symmetric with respect to the odd system  $W''_2$  but not symmetrically continuous at  $x = 0$ .

Thus there is a variety of functions satisfying different weighted symmetric conditions.

**PROPOSITION 6.** *A function  $f$  is continuous iff  $f$  is both symmetric and symmetrically continuous.*

*Proof:* If  $f$  is continuous, by Prop. 1,  $f$  is both symmetric and symmetrically continuous.

Conversely, suppose  $f$  is both symmetric and symmetrically continuous, then

$$(1) \quad \lim_{h \rightarrow 0} (f(x + h/2) - f(x - h/2)) = 0$$



$$(2) \quad \lim_{h \rightarrow 0} (f(x + h/2) + f(x - h/2) - 2f(x)) = 0.$$

From (1) and (2) we have

$$\lim_{h \rightarrow 0} (2f(x + h/2) - 2f(x)) = 0.$$

Thus,  $f(x + 0) = f(x - 0) = f(x)$  and  $f$  is continuous.

**PROPOSITION 7.** *If  $f$  is symmetric with respect to an even (odd) weight system  $W_n$  of order  $n$ , and if at each point the limits to the right and to the left of  $f$  exist, then  $f$  is symmetric with respect to any even (odd) weight system of any order.*

*Proof:* Fix  $n \geq 1$ . Let  $f$  be symmetric with respect to an even or odd system  $W_n$ .

If  $n = 1$ , then the results are true by Prop. 2 and Prop. 3. Suppose  $n > 1$  and  $W_n$  is even.

Since the right limits and left limits exist at each point  $x$ , we have

$$\begin{aligned} \lim_{h \rightarrow 0} \sum_{k=1}^n w_k [f(x + h/2) + f(x - h/2) - 2f(x)] &= 0 \\ [f(x + 0) + f(x - 0) - 2f(x)] \sum_{k=1}^n w_k &= 0. \end{aligned}$$

By the definition of an even weight system, we have

$$\sum_{k=1}^n w_k \neq 0.$$

Thus

$$f(x + 0) + f(x - 0) - 2f(x) = 0.$$

So  $f$  is symmetric, and by Prop. 2,  $f$  is symmetric with respect to any even system of any order.

If  $W_n$  is odd, we have

$$\lim_{h \rightarrow 0} \sum_{k=1}^n w_k [f(x + kh/2) + f(x - kh/2)] = 0$$

and

$$[f(x + 0) + f(x - 0)] \sum_{k=1}^n w_k = 0.$$

Since  $\sum_{k=1}^n w_k \neq 0$ , we have

$$f(x + 0) - f(x - 0) = 0.$$

Thus  $f$  is symmetrically continuous and hence, by Prop. 3,  $f$  is symmetric with respect to any odd system of any order.

By Prop. 7, we see that functions which are symmetric with respect to an even (odd) weight system and whose limits to the right and to the left at each point exist form a special subclass of symmetric functions. There are only two such subclasses which we call even symmetry class and odd symmetry class denoted by  $S_{\text{even}}$  and  $S_{\text{odd}}$ .

1) *A function in the even (odd) symmetry class is symmetric with respect to any even (odd) weight system of any order.*

2) *If  $f$  is measurable and  $f$  is in  $S_{\text{even}}$  or  $S_{\text{odd}}$ , then the set of points of discontinuity of  $f$  is countable; hence,  $f$  is in Baire class one.* For it is well known that, for a function  $f$ , if the right limit and left limit exist at each point, then the set of points of discontinuity is countable, and  $f$  is, therefore, Baire 1 [2, p. 283].

3)  $S_{\text{even}} \subset \bigcap_{n=1}^{\infty} S(\text{even-}W_n) = S_{\text{sym}}$ .

The inclusion is strict; for instance, the function

$$\begin{aligned} f(x) &= 1 + \sin(1/x) & \text{for } x > 0 \\ f(x) &= 0 & \text{for } x = 0 \\ f(x) &= -1 + \sin(1/x) & \text{for } x < 0 \end{aligned}$$

is symmetric, but the right limit and left limit do not exist at  $x = 0$ .

4)  $S_{\text{odd}} \subset \bigcap_{n=1}^{\infty} S(\text{odd-}W_n) = S_{\text{sc}}$ .

The inclusion is strict because the function

$$\begin{aligned} f(x) &= \cos(1/x) & \text{for } x \neq 0 \\ f(x) &= 0 & \text{for } x = 0 \end{aligned}$$

is symmetrically continuous, but the right limit and left limit do not exist at  $x = 0$ .

Now consider the class of functions such that at each  $x$ ,

$$\lim_{h \rightarrow 0^+} \lim_{m \rightarrow 0^+} \sum_{k=1}^n w_k [f(x + kh/2) + f(x - km/2) - 2f(x)] = 0$$

if  $W_n$  is even, and

$$\lim_{h \rightarrow 0^+} \lim_{m \rightarrow 0^+} \sum_{k=1}^n w_k [f(x + kh/2) + f(x - km/2)] = 0$$

if  $W_n$  is odd.

Then it is clear that *the left limit and the right limit of  $f$  exist at each point  $x$ . Hence, the set of points of discontinuity of  $f$  is countable, and  $f$  is Baire 1.*

**PROPOSITION 8.** *If  $\{f_i\}$  is a sequence of functions symmetric with respect to a fixed weight system  $W_n$  ( $W_n$  need not be even or odd), and if  $f$  is the uniform limit of  $\{f_i\}$ , then  $f$  is symmetric with respect to  $W_n$ .*

*Proof.* We have

$$\begin{aligned}\Delta f(x; W_n, h) &= \Delta f(x; W_n, h) - \Delta f_i(x; W_n, h) + \Delta f_i(x; W_n, h) \\ &= \sum_{k=-n}^n w_k [f(x + kh/2) - f_i(x + kh/2)] + \Delta f_i(x; W_n, h).\end{aligned}$$

Then

$$|\Delta f(x; W_n, h)| \leq \sum_{k=-n}^n |w_k| \cdot |f(x + kh/2) - f_i(x + kh/2)| + |\Delta f_i(x; W_n, h)|.$$

Since  $f_i$  is symmetric with respect to  $W_n$ , there is a  $\delta > 0$  such that

$$|\Delta f_i(x; W_n, h)| < \varepsilon/2 \quad \text{for } |h| < \delta.$$

Since  $f_i \rightarrow f$  uniformly, there is an  $N > 0$  such that

$$|f(x + kh/2) - f_i(x + kh/2)| < \varepsilon \cdot \left(2 \sum_{k=-n}^n |w_k|\right)^{-1} \quad \text{for } i > N.$$

Thus

$$|\Delta f(x; W_n, h)| < \varepsilon \quad \text{for } |h| < \delta.$$

**PROPOSITION 9.** *Given a weight system  $W_n$  of order  $n$ , the sum of two functions symmetric with respect to  $W_n$  is symmetric with respect to  $W_n$ .*

*Proof:* Suppose

$$\lim_{h \rightarrow 0} \Delta f(x; W_n, h) = 0, \quad \lim_{h \rightarrow 0} \Delta g(x; W_n, h) = 0;$$

then

$$\lim_{h \rightarrow 0} \Delta(f + g)(x; W_n, h) = \lim_{h \rightarrow 0} \Delta f(x; W_n, h) + \lim_{h \rightarrow 0} \Delta g(x; W_n, h) = 0.$$

**PROPOSITION 10.** *The product of a function symmetric with respect to a weight system  $W_n$  by a scalar is symmetric with respect to  $W_n$ .*

*Proof:* Suppose

$$\lim_{h \rightarrow 0} \Delta f(x; W_n, h) = 0.$$

Then

$$\lim_{h \rightarrow 0} \Delta(cf)(x; W_n, h) = c \lim_{h \rightarrow 0} \Delta f(x; W_n, h) = 0.$$

**THEOREM.** *Given a weight system  $W_n$  of order  $n$ , the functions symmetric with respect to  $W_n$  form a linear space  $V(W_n)$ . The uniform limit of a sequence in  $V(W_n)$  belongs to  $V(W_n)$ . Moreover, if  $B(W_n)$  is the subspace of  $V(W_n)$  consisting of bounded functions, then  $B(W_n)$  is a Banach space with the norm  $\|f\| = \sup |f(x)|$ .*

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