PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 46 (60), 1989, pp. 50–54

STOCHASTIC STRUCTURE OF SOME COMPLETELY MONOTONE FUNCTIONS

P. M. Peruničić

Abstract. We describe the stochastic structure of some completely monotone functions. The presented results are connected with stability in some rarefaction procedures [**3**].

Introduction. Let $\kappa(s) = s^{-1}\big(1 - \exp(-s)\big)$ $(\kappa(0) = 1)$ be the Laplace transform of $\mathcal{U}(0,1)$ measure, and

$$\lambda_1(s) = \exp\left\{-\int_0^s \kappa(u)\,du\right\}. \tag{1}$$

We will show that $\lambda_1(s)$ is the Laplace transform of the probability measure on $\mathbf{R}^+$, and give the precise construction of a random variable with such distribution.

Theorem 1. *Let $\{X_n, n \geq 1\}$ be the markovian sequence of random variables given by*

$$X_1 : \mathcal{U}(0,1) \& \ X_{n+1} | X_n : \mathcal{U}(0, X_n) \quad (n \in \mathbf{N}).$$

Then

$$S = \sum_1^\infty X_n$$

exists in mean square and with probability one.

Proof. By induction on n we get that the density function for X_n is

$$f_n(x) = \big(\Gamma(n)\big)^{-1}\big(-\ln(x)\big)^{n-1}, \quad 0 < x < 1.$$

Therefore $\mathbf{E}X_n = 2^{-n}$, $\mathbf{E}X_n^2 = 3^{-n}$ for $n \geq 1$.

Let $S_n = X_1 + \cdots + X_n$. For $m > n$

$$\mathbf{E}|S_m - S_n|^2 = \sum_{n+1}^m \mathbf{E}(X_k^2) + \sum_{k \neq l} \mathbf{E}(X_k X_l).$$

AMS Subject Classification (1980): Primary 44 A 10, 60 E 10

$\{S_n, n \geq 1\}$ is a Cauchy sequence in L_2 sense. Indeed, as

$$\mathbf{E}(X_k X_l) \leq \big(\mathbf{E}(X_k^2)\big)^{1/2}\big(\mathbf{E}(X_l^2)\big)^{1/2} = 3^{-(k+l)/2}$$

for every k and l, it follows that

$$\mathbf{E}|S_m - S_n|^2 \leq \left(\sum_{n+1}^{m} 3^{-k/2}\right)^2.$$

In this way, $\mathbf{E}|S_m - S_n|^2 \to 0$, $n, m \to \infty$, and we have proved that S exists in mean square.

S exists with probability one, too. That follows from the fact that

$$\sup_{m>n}\left(\sum_{n+1}^{m} X_k\right) \xrightarrow{P} 0, \quad n, m \to \infty.$$

Put $Y_n = X_{n+1}X_1^{-1}$, $n \geq 1$. In the following theorem we will prove that the sequence Y_n has the same stochastic structure as X_n.

Theorem 2. *$\{Y_n,\ n \geq 1\}$ is a markovian sequence, independent of X_1, such that* $Y_1 : \mathcal{U}(0,1)$& $Y_{n+1}|Y_n : \mathcal{U}(0, Y_n)$.

Proof. Let $y \in (0,1)$. Then

$$P\{Y_1 < y\} = \mathbf{E}\big(P\{X_2 < yX_1\}|X_1\big) = \mathbf{E}(y|X_1) = y$$

so Y_1: $\mathcal{U}(0,1)$ is independent of X_1. Furthermore,

$$P\{Y_{n+1} < y|Y_n\} = P\{X_{n+2} < yX_1|X_{n+1}X_1^{-1}\}.$$

As $\mathcal{F}(X_{n+1}X_1^{-1}) \subset \mathcal{F}(X_{n+1}, X_1)$, where $\mathcal{F}(\quad)$ denotes the σ-field generated by the random variable indicated between the brackets, we have

$$\begin{aligned}
P\{Y_{n+1} < y|Y_n\} &= \mathbf{E}\left(\mathbf{E}\big(I\{X_{n+2} < yX_1\}|X_{n+1}, X_1\big)|X_{n+1}X_1^{-1}\right)\\
&= \mathbf{E}\left(\mathbf{E}\big(I\{X_{n+2} < yX_1\}|X_{n+1}\big)|X_{n+1}X_1^{-1}\right)\\
&= \mathbf{E}\left(P\{X_{n+2} < yX_1|X_{n+1}\}|X_{n+1}X_1^{-1}\right)\\
&= \mathbf{E}\big(yX_{n+1}^{-1}X_1|X_{n+1}^{-1}X_1^{-1}\big)\\
&= yY_n^{-1},
\end{aligned}$$

so $Y_{n+1}|Y_n : \mathcal{U}(0, Y_n)$. Let us prove that the sequence $\{Y_n\}$ is markovian:

$$P\{Y_{n+1} < y|Y_1, \dots, Y_n\} = P\{X_{n+2} < yX_1|X_2X_1^{-1}, \dots, X_{n+1}X_1^{-1}\}.$$

Since $\mathcal{F}(X_2X_1^{-1}, \dots, X_{n+1}X_1^{-1}) \subset \mathcal{F}(X_1, \dots, X_{n+1})$ the conditional distribution above is equal to

$$\begin{aligned}
&\mathbf{E}\left(\mathbf{E}\big(I\{X_{n+2} < yX_1\}|X_1, \dots, X_{n+1}\big)|X_2X_1^{-1}, \dots, X_{n+1}X_1^{-1}\right)\\
&= \mathbf{E}\left(\mathbf{E}\big(I\{X_{n+2} < yX_1\}|X_{n+1}\big)|X_2X_1^{-1}, \dots, X_{n+1}X_1^{-1}\right)\\
&= \mathbf{E}\left(yX_1X_{n+1}^{-1}|X_2X_1^{-1}, \dots, X_{n+1}X_1^{-1}\right)\\
&= \mathbf{E}\left(yY_n^{-1}|Y_1, \dots, Y_n\right)\\
&= yY_n^{-1} = P\{Y_{n+1} < y|Y_n\},
\end{aligned}$$

so the statement is proved.

Let us consider the distribution of S. Having in mind the properties of the sequence $\{Y_n\}$, it is easy to prove

THEOREM 3. *The Laplace transform of the distribution of* S *is* $\lambda_1(u)$, *introduced in* (1). *Also,*

$$\lambda(u) = 1 - u\lambda_1(u) \quad \textit{and} \quad \lambda_1(u)\big(\lambda_1(qu)\big)^{-1}, \quad 0 < q < 1,$$

are Laplace transforms of some probability measures on $\mathbf{R}^+$.

Proof. As

$$\mathbf{E}\,\exp(-uS) = \mathbf{E}\mathbf{E}\big(\exp(-uS)|X_1\big) = \mathbf{E}\exp(-uX_1)\mathbf{E}\left(\exp\left(-uX_1\sum_{n=1}^{\infty} Y_n\right)\right)$$

we have $\theta(u) = \mathbf{E}\exp(-uS) = \mathbf{E}\exp(-uX_1)\theta(X_1u)$.

Since $X_1 : \mathcal{U}(0,1)$ it follows that

$$\theta(u) = \int_0^1 \exp(-ux)\theta(ux)\,dx \qquad \text{or} \qquad u\theta(u) = \int_0^u \exp(-y)\theta(y)\,dy.$$

As

$$\mathbf{E}X_n = 2^{-n}, \quad \mathbf{E}S = \sum_1^{\infty} \mathbf{E}X_n < \infty$$

it follows that $\theta(u)$ is differentiable for all $u \geq 0$. Hence

$$\theta(u) + u\theta'(u) = \exp(-u)\theta(u).$$

The solution of that simple differential equation, with the initial condition $\theta(0) = 1$, is

$$\lambda_1(u) = \exp\left(-\int_0^u x^{-1}\big(1-\exp(-x)\big)\,dx\right).$$

Now we prove that $\lambda(s) = 1 - s\lambda_1(s)$ is an L-transform of some probability measure on $\mathbf{R}^+$.

As $\lambda'(s) = -\exp(-s)\lambda_1(s)$ and $\exp(-s)$ is an L-transform of the distribution concentrated in the point $x = 1$, it follows that $-\lambda'(s) = \exp(-s)\lambda_1(s)$, is an L-transform of some probability measure on $\mathbf{R}^+$. In this way, $\big(-\lambda'(s)\big)$ is completely monotone (CM) and for all $n \geq 0$

$$(-1)^n\big(-\lambda'(s)\big)^{(n)} = (-1)^{n+1}\lambda^{n+1}(s) \geq 0$$

or

$$(-1)^n\lambda^{(n)}(s) \geq 0, \quad n \geq 1.$$

Let us show that $\lambda(s) \geq 0$. From 1.4.2. we have

$$s\lambda_1(s) = \int_0^s \exp(-y)\lambda_1(y)\,dy \leq \int_0^s \exp(-y)\,dy = 1 - \exp(-s) \leq 1,$$

which is equivalent to $\lambda(s) \geq 0$.

In this way, $\lambda(s)$ is a CM function with the property $\lambda(0) = 1$.

Finally we prove that $\lambda_1(s)\big(\lambda_1(qs)\big)^{-1}$ is an L-transform of some probability measure on $\mathbf{R}^+$ for every $q \in (0,1)$. Indeed

$$\begin{aligned}
\lambda_1(s)\big(\lambda_1(qs)\big)^{-1} &= \exp\left\{-\int_0^s \kappa(u)\,du + \int_0^{qs} \kappa(u)\,du\right\} \\
&= \exp\left\{-\int_0^s \big(\kappa(u) - q\kappa(qu)\big)\,du\right\} \\
&= \exp\left\{-\int_0^\infty x^{-1}\big(1 - \exp(-sx)\big)d\big(\min\{x,1\} - \min\{x,q\}\big)\right\}.
\end{aligned}$$

It is obvious that $\min\{x,1\} - \min\{x,q\}$ is a measure on $\mathbf{R}^+$. It is so-called canonical measure of some infinitely divisible law [1].

Consider the distribution function with L-transform λ_1. It has been proved that $S \overset{\mathcal{D}}{=} X_1(1 + S')$, where X_1 and S' are independent random variables, $S' \overset{\mathcal{D}}{=} S$ and $X_1 : \mathcal{U}(0,1)$. If $\mathbf{L}_1$ denotes the distribution function for S, then for $z > 0$

$$\mathbf{L}_1(z) = \iint_A dx\, d\mathbf{L}_1(y),$$

where $A = \{(x,y) \mid x(y+1) < z,\ 0 < x < 1,\ y > 0\}$. Therefore,

$$\mathbf{L}_1(z) = \int_0^{1\wedge z} \mathbf{L}_1(zx^{-1} - 1)\,dx,$$

where $1 \wedge z = \min\{1,z\}$. For $0 < z \leq 1$

$$\mathbf{L}_1(z) = cz, \qquad c = \int_0^\infty \mathbf{L}_1(y)(1+y)^{-2}\,dy,$$

and for $z > 1$

$$\mathbf{L}_1(z) = z\int_{z-1}^\infty \mathbf{L}_1(y)(1+y)^{-2}\,dy.$$

If $\mathbf{l}_1$ denotes the density function of this probability law, it follows that $\mathbf{l}_1(z) = c$ for $0 < z \leq 1$ and

$$\mathbf{l}_1(z) = z^{-1}\big(\mathbf{L}_1(z) - \mathbf{L}_1(z-1)\big), \quad z > 1.$$

In this way, the distribution function $\mathbf{L}_1(z)$ can be determined by solving that differential equation over the intervals $(n, n+1]$, $n \in \mathbf{N}$.

Let $\mathbf{L}$ be the distribution function with the Laplace transform λ, introduced in Theorem 3. As $\mathbf{l}_1(z) = 1 - \mathbf{L}(z)$, it follows that $\mathbf{L}(z) = 1 - c$, $z \in (0,1]$. In this way, $\mathbf{L}(z)$ has the jump in zero, i.e. $\mathbf{L}(0+) - \mathbf{L}(0) = 1 - c$. Of course, $\mathbf{L}(z)$ is not the unique distribution on $\mathbf{R}^+$ with stationary distribution $\mathbf{L}_1(z)$ [2]. If we introduce

$\mathbf{F}(z) = c^{-1}\{\mathbf{L}(z) - (1-c)\}$ then $\mathbf{F}(z)$ is also a distribution function on $\mathbf{R}^+$ with the same stationary distribution $\mathbf{L}_1(z)$. At the same time, $\mathbf{F}$ is continuous in zero.

REFERENCES

[**1**] W. Feller, *An Introduction to Probability Theory and its Applications II*, Wiley, 1971.

[**2**] L. Takacs, *Stochastische Prozesse*, R. Oldenbourg Verlag, Munchen-Wien, 1966.

[**3**] P. Peruničić, *On a functional rarefaction procedure*, in: *Probability Theory and Mathematical Statistics with Applications*, D. Reidel, Dordrecht & Akademiai Kiado, Budapest, (1988), 181–189.

Matematički fakultet
11000 Beograd, Yugoslavia

(Received 14 10 1988)
(Revised 03 04 1989)